SEMI ANALYTICAL ITERATIVE METHOD FOR SOLVING KLEIN-GORDON EQUATION

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Abstract

In this article, a semi analytical iterative method had been applied to solve a type of partial differential equation namely Klein-Gordon equation. Four examples of linear and nonlinear Klein-Gordon equations were considered to show the performance of the method. The results obtained revealed the effectiveness of this method.

Keyword: Klein-Gordon equation, Partial differential equation, Semi analytical iterative method

Introduction

The Klein-Gordon equation is an important model in mathematical physics such as quantum field theory, soliton and plasma physics. Due to its wide range of applications, Klein-Gordon equation has attracted much attention to find the accurate and efficient method for solving the equation. Some of these methods are Adomian decomposition method (Wazwaz, 2006; El-Sayed, 2003; Rabie, 2015; Agom & Ogunfiditimi, 2018; Kulkarni & Kalyanrao, 2015), homotopy perturbation method (Chowdhury & Hashim, 2009), variational homotopy perturbation method (Yousif & Mahmood, 2017), Laplace decomposition method (Rabie, 2015), Elzaki iterative method (Alderremy et al., 2018), Elzaki transform method (Ige et al., 2019) and homotopy Sumudu transform method (Mahdy et al., 2015).

The purpose of this paper is to solve the Klein-Gordon equation by applying an analytical method called semi analytical iterative method (SAIM). This method was proposed by Temimi and Ansari (2011a; 2011b) in order to solve nonlinear differential equations and nonlinear second order multi-point boundary value problems, respectively. Then, the SAIM has been successfully implemented by many researchers for solving others linear and nonlinear differential equations problem. For example, wave, wave-like, heat and heat-like problems (AL-Jawary & Mohammed, 2015), chemistry problems (AL-Jawary & Raham, 2017), thin flow problems (Al-Jawary, 2017), Blasius equations (Selamat et al., 2019), Newell-Whitehead-Segel equations (Latif et al., 2020), Fokker-Plank's equations (AL-Jawary et al., 2017), nonlinear Burgers and advection-diffusion equations (AL-Jawary et al., 2018), differential algebraic equations (AL-Jawary & Hatif, 2017), duffing equation (Al-Jawary & Al-Razaq, 2016) and some nonlinear differential equations in physics (AL-Jawary et al., 2020).

Materials and Methods

Semi Analytical Iterative Method

In this section, we presented the semi analytical iterative method (SAIM) which is introduced by Temimi and Ansari (2011a, 2011b) and detailed by Al-Jawary and Mohammed (2015): Consider the general differential equation,

$$L\big(u(x,t)\big) + N\big(u(x,t)\big) + g(x,t) = 0, \tag{1}$$

with boundary conditions $B\left(u, \frac{\partial u}{\partial t}\right) = 0$,

where u is unknown function, L is the linear operator, N is the nonlinear operator and g is a known function. Noticed that, it is possible to take some or all parts of the $g(x,t)$ and add them to the nonlinear operator N as needed. The proper choice of these parts depends mainly on trial basis.

By assuming that $u_0(x,t)$ is a solution of equation (1) of the initial condition

$$L\big(u_0(x,t)\big) + g(x,t) = 0, \text{ with } B\left(u_0, \frac{\partial u}{\partial t}\right) = 0. \tag{2}$$

The next iteration found by resolved the following equation:

$$L\big(u_1(x,t)\big) + N\big(u_0(x,t)\big) + g(x,t) = 0, \text{with } B\left(u_1, \frac{\partial u}{\partial t}\right) = 0. \tag{3}$$

Thus, an iterative procedure can be created by solving the following problem,

$$L\big(u_{n+1}(x,t)\big) + N\big(u_n(x,t)\big) + g(x,t) = 0, \text{with } B\left(u_{n+1}, \frac{\partial u}{\partial t}\right) = 0, \tag{4}$$

where, $n = 1, 2, \ldots$. Each of u_n are solution of the equation (1).

Solution of Klein-Gordon Equation Using SAIM

The general equation of Klein-Gordon equation can be written as:

$$u_{tt} - u_{xx} + b_1 u + b_2 g(u) = f(x,t) \tag{5}$$

with boundary condition

$$u(x,0) = c_1 \text{ and } u_t(x,0) = c_2 \tag{6}$$

In view of SAIM, we have $L\big(u(x,t)\big) = u_{tt}$, $N\big(u(x,t)\big) = -u_{xx} + b_1 u + b_2 g(u)$ and $g(x,t) = -f(x,t)$.

Thus, the primary problem to solve is,

$$L(u_0(x,t)) = f(x,t) \tag{7}$$

or,

$$(u_0)_{tt}(x,t) = f(x,t) \tag{8}$$

with $u_0(x,0) = c_1$ and $(u_0)_t(x,0) = c_2$.
The general iteration problem can be done through the following

$$L(u_{n+1}(x,t)) + N(u_n(x,t)) + g(x,t) = 0, \tag{9}$$

or,

$$(u_{n+1})_{tt} - (u_n)_{xx} + b_1 u_n + b_2 g(u_n) = f(x,t) \tag{10}$$

with $u_{n+1}(x,0) = c_1$ and $(u_{n+1})_t(x,0) = c_2$, where, $n = 1, 2, \dots$.

Example 1:

Consider the linear homogeneous Klein-Gordon equation (Chowdhury & Hashim, 2009)

$$u_{tt}(x,t) - u_{xx}(x,t) = u(x,t), \tag{11}$$

with initial condition,

$$u(x,0) = 1 + \sin(x),\ u_t(x,0) = 0 \tag{12}$$

where, the exact solution is,

$$u(x,t) = \sin(x) + \cosh(t)\ . \tag{13}$$

The SAIM will be applied as, $L\big(u(x,t)\big) = u_{tt}(x,t)$, $N\big(u(x,t)\big) = -u_{xx}(x,t) - u(x,t)$ and $g(x,t) = 0$.

Thus, the primary problem is

$$L(u_0(x,t)) = u_{tt}(x,t) = 0 \tag{14}$$

with initial conditions,

$$u_0(x,0) = 1 + \sin(x),\ (u_0)_t(x,0) = 0 \tag{15}$$

By solving the primary problem (14), we obtained,

$$u_0(x,t) = \sin(x) + 1. \tag{16}$$

The first repetition can be done through the following

$$(u_1)_{tt}(x,t) - (u_0)_{xx}(x,t) = u_0(x,t), \quad (17)$$

with initial conditions, $u_1(x,0) = 1 + \sin(x)$, $(u_1)_t(x,0) = 0$.
Then, the solution of (17) is,

$$u_1(x,t) = \sin(x) + 1 + \frac{1}{2}t^2. \quad (18)$$

Similarly, we obtained,

$$u_2(x,t) = \sin(x) + 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 \quad (19)$$

$$u_3(x,t) = \sin(x) + 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 \quad (20)$$

$$\vdots$$

Hence, when $n \to \infty$, the solution will lead to the close form

$$u(x,t) = \sin(x) + \cosh(t)\ , \quad (21)$$

which is the exact solution of equation (11).

Example 2:

Consider the linear nonhomogeneous Klein-Gordon equation (Chowdhury & Hashim, 2009)

$$u_{tt}(x,t) - u_{xx}(x,t) - 2u(x,t) = -2\sin(x)\sin(t), \quad (22)$$

with initial condition

$$u(x,0) = 0,\ u_t(x,0) = \sin(x), \quad (23)$$

where the exact solution is

$$u(x,t) = \sin(t)\sin(x). \quad (24)$$

The SAIM will be applied as: $L\big(u(x,t)\big) = u_{tt}(x,t)$, $N\big(u(x,t)\big) = -u_{xx}(x,t) - 2u(x,t) + 2\sin(x)\sin(t)$ and $g(x,t) = 0$

Thus, the primary problem is

$$L(u_0(x,t)) = u_{tt}(x,t) = 0, \quad (25)$$

with initial conditions

$$u_0(x,0) = 0,\ (u_0)_t(x,0) = \sin(x). \quad (26)$$

By solving the primary problem (25), we obtained,

$$u_0(x,t) = t\sin(x). \quad (27)$$

The first repetition can be done through the following

$$(u_1)_{tt}(x,t) - (u_0)_{xx}(x,t) - 2u_0(x,t) = -2\sin(t)\sin(x), \quad (28)$$

with initial conditions, $u_1(x,0) = 0,\ (u_1)_t(x,0) = \sin(x)$.

Then, the solution of (28) is,

$$u_1(x,t) = \left(\frac{t^3}{6} - t + 2\sin(t)\right)\sin(x) \quad (29)$$

Similarly, we obtained

$$u_2(x,t) = \left(\frac{t^5}{120} - \frac{t^3}{6} + t\right)\sin(x) \quad (30)$$

$$u_3(x,t) = \left(\frac{t^7}{5040} - \frac{t^5}{120} + \frac{t^3}{6} - t + 2\sin(t)\right)\sin(x) \quad (31)$$

$$\vdots$$

Hence, when $n \to \infty$, the solution will lead to the close form

$$u(x,t) = \sin(t)\sin(x), \quad (32)$$

which is the exact solution.

Example 3:

Consider the nonlinear nonhomogeneous Klein-Gordon equation (Chowdhury & Hashim, 2009)

$$u_{tt}(x,t) - u_{xx}(x,t) - u(x,t)^2 = -x\cos(t) + x^2\cos^2(t), \quad (33)$$

with initial conditions

$$u(x,0) = x,\ u_t(x,0) = 0, \quad (34)$$

where the exact solution is

$$u(x,t) = x\cos(t). \quad (33)$$

The SAIM will be applied as: $L\big(u(x,t)\big) = u_{tt}(x,t),\ N\big(u(x,t)\big) = -u_{xx}(x,t) - u(x,t)^2 - x^2\cos^2(t)$ and $g(x,t) = -x\cos(t)$.

Thus, the primary problem is

$$L(u_0(x,t)) = u_{tt}(x,t) = -x\cos(t), \tag{35}$$

with initial conditions

$$u_0(x,0) = x,\ \ (u_0)_t(x,0) = 0. \tag{36}$$

By solving the primary problem (35), we obtained,

$$u_0(x,t) = x\cos(t). \tag{37}$$

The first repetition can be done through the following

$$(u_1)_{tt}(x,t) - (u_0)_{xx}(x,t) - u_0(x,t)^2 = -x\cos(t)(1 - x\cos(t)), \tag{38}$$

with initial conditions, $u_1(x,0) = x,\ \ (u_1)_t(x,0) = 0.$

Then, the solution of (38) is

$$u_1(x,t) = x\cos(t), \tag{37}$$

Similarly, the higher order solutions are $x\cos t$ which is, the exact solution (Wazwaz, 2006)

$$u(x,t) = x\cos(t). \tag{38}$$

Example 4:

Consider the nonlinear nonhomogeneous Klein-Gordon equation (Chowdhury & Hashim, 2009)

$$u_{tt}(x,t) - u_{xx}(x,t) - u(x,t)^2 = 6xt(x^2 - t^2) + x^6t^6, \tag{39}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = 0, \tag{40}$$

where the exact solution is

$$u(x,0) = 0, u_t(x,0) = 0, \tag{41}$$

The SAIM will be applied as: $L\big(u(x,t)\big) = u_{tt}(x,t),\ N\big(u(x,t)\big) = -u_{xx}(x,t) - u(x,t)^2 - 6xt^3 + x^6t^6$ and $g(x,t) = 6x^3t.$

Thus, the primary problem is

$$L(u_0(x,t)) = u_{tt}(x,t) = 6x^3t, \tag{42}$$

with initial conditions

$$u_0(x,0) = 0,\ (u_0)_t(x,0) = 0. \quad (43)$$

By solving the primary problem (42), we obtained,

$$u_0(x,t) = x^3t^3, \quad (44)$$

The first repetition can be done through the following

$$(u_1)_{tt}(x,t) - (u_0)_{xx}(x,t) - u_0(x,t)^2 = 6xt(x^2 - t^2) + x^6t^6, \quad (45)$$

with initial conditions, $u_1(x,0) = 0,\ (u_1)_t(x,0) = 0.$

Then, the solution of (45) is

$$u_1(x,t) = x^3t^3, \quad (46)$$

Similarly, the higher order solutions are x^3t^3 which is the exact solution (Wazwaz, 2006)

$$u(x,t) = x^3t^3. \quad (47)$$

Conclusion

In this paper, the SAIM has been successfully applied to obtain the exact solution of some examples of linear and nonlinear Klein-Gordon equations. It is appeared that the SAIM is very efficient to yield the solution without required any restricted assumption to deal with nonlinear terms in differential equations.

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Conflict of interests

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