Buckling Analysis of Symmetrically-Laminated Plates using Refined Theory including Curvature Effects

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ABSTRACT

A refined theory is successfully extended in this study for critical buckling loads of rectangular, symmetrically-laminated plates, including curvature effects. The theory accounts for a quadratic variation of the transverse shear strains across the thickness, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the plate and avoids the need of shear correction factors. The numerical results are presented for critical buckling loads for orthotropic laminates subjected to biaxial inplane loading. Using the Navier solution method, the differential equations have been solved analytically and the critical buckling loads presented in closed-form solutions. The significant effects of curvature terms on buckling loads are studied, with various loading conditions and thickness-side ratio. Some exact buckling solutions for simplified cases with and without the inclusion of curvature terms are obtained and compared with results available elsewhere in literature.

Keywords: *Buckling, Symmetrically-laminated, Refined Theory (RT), Curvature Effects.*

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Nomenclature

x, y, z	Directions of Cartesian coordinate system
u, v, w	In-plane displacements
w_b and w_s	Bending and shear components of transverse displacement, respectively
a, b	Plate length and width, respectively
h	Plate thickness
E_1 and E_2	Young's moduli along and transverse to the fibre, respectively
G_{12} , G_{23} and G_{13}	In-plane and transverse shear moduli
v_{12} and v_{21}	Poisson's ratios along and transverse to the fibre, respectively
k	Number of layers
Q_{ij}	Plane stress reduced elastic constants in the material axes of the plate
$\overline{Q}_{ij}$	Transformed material constants
$A_{ij}, A_{ij}^s, D_{ij}, D_{ij}^s, H_{ij}^s$	Plate stiffness
M_i^b, M_i^s, Q_j ($i=x,y,xy$ & $j=xz, yz$)	Resultants moments, shear forces, respectively
$\sigma_x^0, \sigma_y^0, \tau_{xy}^0$	In-plane stresses
$\varepsilon_x^{Nl}, \varepsilon_y^{Nl}, \gamma_{xy}^{Nl}$	Second order strain
U	Strain energy of the plate
V	Potential energy of the plate
V_1	In-plane force terms
V_2	Curvature terms
N	Force per unit length
ξ	Load parameter
$\overline{N}$	Critical buckling factor
N_{cr}	Critical buckling Load
L_{ij}	Linear operators
c_c	Scalar indicator of curvature terms
m, n	Number of half waves in the x- and y-directions, respectively

Introduction

The advancement of technology in the search of structural materials with high specific strength and stiffness properties has resulted in the application of laminated composites in aerospace and transportation industries. The increasingly wider application to other fields of engineering has necessitated the evolution of adequate analytical tools for the better understanding of the structural behaviour and efficient utilization of the materials.

Recently, Khalili et al [1] studied the buckling analysis of the sandwich plates for the general cases of the problem and the analytical exact solutions using a simple and fast computational code. Moita et al. [2], finite element model is presented for buckling and geometrically nonlinear analysis of multilayer sandwich structures and shells, with a soft core sandwiched between stiff elastic layers. Ruocco [3] examined the influence of the nonlinear Green–Lagrange strain tensor terms on the buckling of orthotropic, moderately thick plates by the Mindlin hypotheses. Raju et al [4] studied the buckling and postbuckling of variable angle tow composite plates under in-plane shear loading. Kazemi [5] proposed a new method for calculating the critical buckling load has been developed based on the polar representation of tensors. This method can help to analyze the influence of anisotropy on the buckling behavior of simply supported rectangular laminated plates subjected to biaxial compression, thus avoiding the complexities associated with the Cartesian formulation. Chalak et al [6] proposed a new plate model is proposed for the stability analysis of laminated sandwich plate having a soft compressible core based on higher-order zig-zag theory. Kumar et al [7] presented the design of a graded fiber-reinforced composite lamina and graded laminates with an objective of reduced inter-laminar stress-discontinuity in composite laminates. Thai et al [8] presented the novel numerical approach using a NURBS-based isogeometric approach associated with third-order shear deformation theory (TSDT) is formulated for static, free vibration, and buckling analysis of laminated composite plate structures. Rachchhet al. [9] studied the Effect of red mud filler on mechanical and buckling characteristics of coir fibre reinforced polymer composite. Bohlooly and Mirzavand [10] studied Buckling and postbuckling behavior of symmetric laminated composite plates with surface mounted and embedded piezoelectric actuators subjected to mechanical, thermal, electrical, and combined loads. Venkatachari et. al. [11] examined buckling characteristics of curvilinear fibre composite laminates exposed to hygrothermal environment. The formulation is based on the transverse shear deformation theory and it accounts for the lamina material properties at elevated moisture concentrations and thermal gradients. Kumar et. Al. [12] proposed a new lamination scheme is through the design of a graded orthotropic fiber-reinforced composite ply for achieving continuous variations of material properties along the thickness direction of laminated composite plates.

In the past three decades, researches on laminated plates have received great attention, and a variety of plate theories has been introduced based on considering the transverse shear deformation effect. The classical plate theory (CPT), which neglects the transverse shear deformation effect, provides reasonable results for thin plate [13-15].The Reissner [16] and Mindlin [17] theories are known as the first-order shear deformation plate theory (FSDT), and account for the transverse shear effect by the way of linear variation of in-plane displacements through the thickness. There are many two dimensional theories that have been proposed to account for the shear deformation of moderately deep structures and highly anisotropic composites. Reddy [18] proposed a parabolic shear deformation plate theory. Touratier [19] proposed a trigonometric shear deformation plate theory where the transverse strain distribution is given as a sine function. Soldatos [20] proposed a hyperbolic shear deformation plate theory. Aydogdu [21] presented a new shear deformation theory for laminated composite plates. Therefore, Lee et al. [22] proposed a higher-order shear deformable theory using the similar approach of representing transverse displacement using two components. Recently, Shimpi [23] has developed a new refined plate theory which is simple to use and extended by Shimpi and Patel [24,25] for orthotropic plates.

To the best of authors' knowledge, there are no research works for mechanical buckling analysis of laminated plates based on new refined theory including curvature effects. In this work, the effect of curvature terms on the buckling analysis of symmetrically-laminated rectangular plates subjected to biaxial inplane loading has been investigated using the refined theory and Navier solution. The formulation theory accounts for the shear deformation effects without requiring a shear correction factor. Number of unknown functions involved is only two, as against three in case of simple shear deformation theories of Mindlin and Reissner and common higher-order shear deformation theories. Governing equations have been developed for determining critical buckling loads of rectangular, symmetrically-laminated plates, including transverse shear deformation and curvature effects. Using the Navier solution method, the differential equations have been solved analytically and the critical buckling loads presented in closed-form solutions. The sensitivity of critical buckling loads to the effects of curvature terms and other factors has been examined. The analysis is validated by comparing results with those in the literature. The basic equations of plane problem and the general solution for mechanical buckling of laminated plate including curvature effects are given in Section 2. The numerical examples are given in Section 3 and a summary is given in Section 4.

Theoretical Formulation

Buckling of symmetric, anisotropic laminates plates

The displacement field, which accounts for parabolic variation of transverse shear stress through the thickness, and satisfies the zero traction boundary conditions on the top and bottom faces of the plate, is assumed as follows [22,23]:

$$u(x,y,z) = -z\frac{\partial w_b}{\partial x} + \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s(x,y)}{\partial x}$$
$$v(x,y,z) = -z\frac{\partial w_b}{\partial y} + \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s(x,y)}{\partial y} \quad (1)$$
$$w(x,y,z) = w_b(x,y) + w_s(x,y)$$

where w_b and w_s are the bending and shear components of transverse displacement, respectively; and h is the plate thickness. The kinematic relations can be obtained as follows:

$$\{\varepsilon\} = \{k^b\}z + \{k^s\}f(z)$$
$$\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = \begin{Bmatrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{Bmatrix}g(z) \quad (2)$$

where

$$\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T$$
$$\{k^b\} = \{k_x^b, k_y^b, k_{xy}^b\}^T = \left\{-\frac{\partial^2 w_b}{\partial x^2}, -\frac{\partial^2 w_b}{\partial y^2}, -2\frac{\partial^2 w_b}{\partial x \partial y}\right\}^T$$
$$\{k^s\} = \{k_x^s, k_y^s, k_{xy}^s\}^T = \left\{-\frac{\partial^2 w_s}{\partial x^2}, -\frac{\partial^2 w_s}{\partial y^2}, -2\frac{\partial^2 w_s}{\partial x \partial y}\right\}^T \quad (3)$$
$$\gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \gamma_{yz}^s = \frac{\partial w_s}{\partial y}$$
$$f(z) = -\frac{1}{4}z + \frac{5}{3}z\left(\frac{z}{h}\right)^2, \; f'(z) = \frac{df(z)}{dz}, \; g(z) = 1 - f'(z),$$

Constitutive equations

Under the assumption that each layer possesses a plane of elastic symmetry parallel to the x–y plane, the constitutive equations for a layer can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$
$$Q_{66} = G_{12}, Q_{44} = G_{23}, Q_{55} = G_{13} \tag{5}$$

Transforming the above equations of an arbitrary k layer in local coordinate system into the global coordinate system, the laminate constitutive equations can be expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & Q_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \tag{6}$$

where usual notations for normal and shear stress components are adopted. The relationship of the global reduced stiffness matrix $\overline{Q}_{ij}$ can be referred to any standard texts such as [26, 27].

Governing equation

The strain energy of the plate is calculated by

$$2U = \int_V \sigma_{ij}\varepsilon_{ij} dV = \int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz}\right) dV \tag{7}$$

Substituting Eq. (2) into Eq. (7) and integrating through the thickness of the plate, the strain energy of the plate can be rewritten as

$$2U = \int_A \left[M_x^b \delta k_x^b + M_y^b \delta k_y^b + M_{xy}^b k_{xy}^b + M_x^s k_x^s + M_y^s k_y^s + M_{xy}^s k_{xy}^s + Q_{yz}\gamma_{yz}^s + Q_{xz}\gamma_{xz}^s\right] dxdy \tag{8}$$

where (M_x^b, M_y^b, M_{xy}^b), (M_x^s, M_y^s, M_{xy}^s) denote the total moment resultants and (Q_{xz}, Q_{yz}) are transverse shear stress resultants and they are defined as

$$
\begin{aligned}
(M_x^b, M_y^b, M_{xy}^b) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) z dz \\
(M_x^s, M_y^s, M_{xy}^s) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) f(z) dz \\
(Q_{xz}, Q_{yz}) &= \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz}) g(z) dz
\end{aligned}
\tag{9}
$$

From Eq. (9), one can obtain the following equations:

$$
\begin{bmatrix} M_x^b \\ M_y^b \\ M_{xy}^b \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{bmatrix} + \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{bmatrix} \tag{10}
$$

$$
\begin{bmatrix} M_x^s \\ M_y^s \\ M_z^s \end{bmatrix} = \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{bmatrix} + \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{bmatrix} \tag{11}
$$

$$
\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix} \begin{bmatrix} \partial w_s / \partial y \\ \partial w_s / \partial x \end{bmatrix} \tag{12}
$$

where,

$$
\begin{aligned}
(D_{ij}, D_{ij}^s, H_{ij}^s) &= \int_{-h/2}^{h/2} \overline{Q}_{ij} (z^2, zf(z), (f(z))^2) dz \quad (i, j = 1,2,6) \\
A_{ij}^s &= \int_{-h/2}^{h/2} \overline{Q}_{ij} (g(z))^2 dz \qquad (i, j = 4,5)
\end{aligned}
\tag{13}
$$

Substituting Eqs. (10) - (12) and (3) to Eq. (8), the strain energy per unit area, U, due to the buckling deformation is of the form

$$2U = D_{11}\frac{\partial^4 w_b}{\partial x^4} + D_{22}\frac{\partial^4 w_b}{\partial y^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_b}{\partial x^2 \partial y^2} + 4D_{16}\frac{\partial^4 w_b}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w_b}{\partial x \partial y^3}$$
$$+ 2D_{11}^s\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x^2} + 2D_{22}^s\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial y^2} + 2D_{12}^s\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial y^2} + 2D_{12}^s\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x^2}$$
$$+ 4D_{16}^s\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x \partial y} + 4D_{16}^s\frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x^2} + 4D_{26}^s\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x \partial y} + 4D_{26}^s\frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial y^2} \tag{14}$$
$$+ 8D_{66}^s\frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y} H_{11}^s\frac{\partial^4 w_s}{\partial x^4} + H_{22}^s\frac{\partial^4 w_s}{\partial y^4} + 2\left(H_{12}^s + 2H_{66}^s\right)\frac{\partial^4 w_s}{\partial x^2 \partial y^2} + 4H_{16}^s\frac{\partial^4 w_s}{\partial x^3 \partial y}$$
$$+ 4H_{26}^s\frac{\partial^4 w_s}{\partial x \partial y^3} + A_{44}^s\frac{\partial^2 w_s}{\partial y^2} + A_{55}^s\frac{\partial^2 w_s}{\partial x^2} + 2A_{45}^s\frac{\partial^2 w_s}{\partial x \partial y}$$

The potential energy of the applied in-plane stresses σ_x^0, σ_y^0 and τ_{xy}^0 arises from the action of the applied d stresses on the corresponding second order strain $\varepsilon_x^{Nl}, \varepsilon_y^{Nl}$ and γ_{xy}^{Nl}. Following the usual procedure [28, 29], after taking into account the displacement field given by Equation (1)

$$\varepsilon_x^{Nl} = \frac{z^2}{2}\left[\frac{\partial^4 w_b}{\partial x^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right] + zf(z)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x^2} + \frac{\partial w_b^2}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right]$$
$$+ \frac{f(z)^2}{2}\left[\frac{\partial^4 w_s}{\partial x^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right] + \frac{1}{2}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2} + 2\frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial x}\right) \tag{15}$$

$$\varepsilon_y^{Nl} = \frac{z^2}{2}\left[\frac{\partial^4 w_b}{\partial y^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right] + zf(z)\left[\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial y^2} + \frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right]$$
$$+ \frac{f(z)^2}{2}\left[\frac{\partial^4 w_s}{\partial y^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right] + \frac{1}{2}\left(\frac{\partial^2 w_b}{\partial y^2} + \frac{\partial^2 w_s}{\partial y^2} + 2\frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial y}\right) \tag{16}$$

$$\gamma_{xy}^{Nl} = z^2\left[\frac{\partial^4 w_b}{\partial x^3 \partial y} + \frac{\partial^4 w_b}{\partial x \partial y^3}\right] + zf(z)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_b}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial y^2}\frac{\partial^2 w_b}{\partial x \partial y}\right]$$
$$+ f(z)^2\left[\frac{\partial^4 w_s}{\partial x^3 \partial y} + \frac{\partial^4 w_s}{\partial x \partial y^3}\right] + \left(\frac{\partial^2 w_b}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial y} + \frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial x}\right) \tag{17}$$

The potential energy of the plate fiat of volume is

$$V = \int_{-h/2}^{h/2}\left(\sigma_x^0 \varepsilon_x^{Nl} + \sigma_y^0 \varepsilon_y^{Nl} + \tau_{xy}^0 \gamma_{xy}^{Nl}\right)dz \tag{18}$$

Denoting conventional inplane force terms by V_1 and "curvature" terms by V_2, then after combining Equations (15)-(17) and (18) we find that

$$V = V_1 + V_2 \tag{19}$$

where

$$2V_1 = N_x^0\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_s}{\partial x^2}+2\frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial x}\right)+N_y^0\left(\frac{\partial^2 w_b}{\partial y^2}+\frac{\partial^2 w_s}{\partial y^2}+2\frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial y}\right) \tag{20}$$
$$+2N_{xy}^0\left(\frac{\partial^2 w_b}{\partial x\partial y}+\frac{\partial^2 w_s}{\partial x\partial y}+\frac{\partial w_b}{\partial x}\frac{\partial w_s}{\partial y}+\frac{\partial w_b}{\partial y}\frac{\partial w_s}{\partial x}\right)$$

$$2V_2 = \int_{-h/2}^{h/2}\left(\begin{array}{l}\left\{\sigma_x^0\left[\frac{\partial^4 w_b}{\partial x^4}+\frac{\partial^4 w_b}{\partial x^2\partial y^2}\right]+\sigma_y^0\left[\frac{\partial^4 w_b}{\partial y^4}+\frac{\partial^4 w_b}{\partial x^2\partial y^2}\right]+2\tau_{xy}^0\left[\frac{\partial^4 w_b}{\partial x^3\partial y}+\frac{\partial^4 w_b}{\partial x\partial y^3}\right]\right\}z^2\\ \left\{\begin{array}{l}2\sigma_x^0\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x^2}+\frac{\partial w_b^2}{\partial x\partial y}\frac{\partial^2 w_s}{\partial x\partial y}\right]+2\sigma_y^0\left[\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial y^2}+\frac{\partial^2 w_b}{\partial x\partial y}\frac{\partial^2 w_s}{\partial x\partial y}\right]\\ 2\tau_{xy}^0\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x\partial y}+\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x\partial y}+\frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_b}{\partial x\partial y}+\frac{\partial^2 w_s}{\partial y^2}\frac{\partial^2 w_b}{\partial x\partial y}\right]\end{array}\right\}zf(z)\\ \left\{\sigma_x^0\left[\frac{\partial^4 w_s}{\partial x^4}+\frac{\partial^4 w_s}{\partial x^2\partial y^2}\right]+\sigma_y^0\left[\frac{\partial^4 w_s}{\partial y^4}+\frac{\partial^4 w_s}{\partial x^2\partial y^2}\right]+2\tau_{xy}^0\left[\frac{\partial^4 w_s}{\partial x^3\partial y}+\frac{\partial^4 w_s}{\partial x\partial y^3}\right]\right\}f(z)^2\end{array}\right)dz \tag{21}$$

In addition

$$\left(N_x^0, N_y^0, N_{xy}^0\right) = \int_{-h/2}^{h/2}\left\{\sigma_x^0, \sigma_y^0, \tau_{xy}^0\right\}dz \tag{22}$$

In order to put the integral in Equation (21) in a useful form for heterogeneous plates, we utilize the constitutive relations for the inplane loading of a symmetrically-laminated plate [30, 31]

$$\begin{bmatrix} N_x^0 \\ N_y^0 \\ N_{xy}^0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \tag{23}$$

where

$$A_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij} dz \qquad (i, j = 1,2,6) \tag{24}$$

Equation (23) can now be written in the form

$$\varepsilon_{i,j}^{0} = A_{jk}^{*} N_{k}^{0} \qquad (j, k = 1,2,6) \tag{25}$$

where the repeated index denotes summation, and A_{jk}^{*} represents elements of the inverse matrix of A_{jk}. Denoting σ_x^0, σ_y^0 and τ_{xy}^0 by σ_1^0, σ_2^0 and σ_6^0, respectively, we can write the inplane ply constitutive relations in the form:

$$\sigma_i^0 = \overline{Q}_{ij} \, \varepsilon_j^0 \qquad (i, j = 1,2,6) \tag{26}$$

Thus,

$$\begin{aligned}
&\int_{-h/2}^{h/2} \sigma_i^0 z^2 dz = \int_{-h/2}^{h/2} \overline{Q}_{ij} \, \varepsilon_j^0 z^2 dz = D_{ij} \varepsilon_j^0 \\
&\int_{-h/2}^{h/2} \sigma_i^0 z f(z) dz = \int_{-h/2}^{h/2} \overline{Q}_{ij} \, \varepsilon_j^0 z f(z) dz = D_{ij}^s \varepsilon_j^0 \\
&\int_{-h/2}^{h/2} \sigma_i^0 f(z)^2 dz = \int_{-h/2}^{h/2} \overline{Q}_{ij} \, \varepsilon_j^0 f(z)^2 dz = H_{ij}^s \varepsilon_j^0
\end{aligned} \tag{27}$$

Combining Equations (25) and (27), we find that

$$\begin{aligned}
&\int_{-h/2}^{h/2} \sigma_x^0 z^2 dz = D_{ij} A_{jk}^{*} N_k^0 = F_{jk} N_k^0 \\
&\int_{-h/2}^{h/2} \sigma_{i,j}^0 z f(z) dz = D_{ij}^s A_{jk}^{*} N_k^0 = F_{jk}^s N_k^0 \\
&\int_{-h/2}^{h/2} \sigma_{i,j}^0 f(z)^2 dz = H_{ij}^s A_{jk}^{*} N_k^0 = F_{jk}^r N_k^0
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
F_{jk} &= D_{ij} A_{jk}^{*} \\
F_{jk}^{s} &= D_{ij}^{s} A_{jk}^{*} \\
F_{jk}^{r} &= H_{ij}^{s} A_{jk}^{*}
\end{aligned} \tag{29}$$

Taking into account Equations (28) and (29), the “curvature” terms, Equation (21), are of the form

$$
\begin{aligned}
2V_2 = {} & \left(F_{11}N_x^0 + F_{12}N_y^0 + F_{16}N_{xy}^0\right)\left[\frac{\partial^4 w_b}{\partial x^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right] + \left(F_{12}N_x^0 + F_{22}N_y^0 + F_{26}N_{xy}^0\right)\left[\frac{\partial^4 w_b}{\partial y^4} + \frac{\partial^4 w_b}{\partial x^2 \partial y^2}\right] \\
& + 2\left(F_{16}N_x^o + F_{26}N_y^0 + F_{66}N_{xy}^0\right)\left[\frac{\partial^4 w_b}{\partial x^3 \partial y} + \frac{\partial^4 w_b}{\partial x \partial y^3}\right] + 2\left(F_{11}^s N_x^0 + F_{12}^s N_y^0 + F_{16}^s N_{xy}^0\right)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x^2} + \frac{\partial w_b^2}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right] \\
& + 2\left(F_{12}^s N_x^0 + F_{22}^s N_y^0 + F_{26}^s N_{xy}^0\right)\left[\frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial y^2} + \frac{\partial^2 w_b}{\partial x \partial y}\frac{\partial^2 w_s}{\partial x \partial y}\right] \\
& + 4\left(F_{16}^s N_x^o + F_{26}^s N_y^0 + F_{66}^s N_{xy}^0\right)\left[\frac{\partial^2 w_b}{\partial x^2}\frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial^2 w_b}{\partial y^2}\frac{\partial^2 w_s}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial x^2}\frac{\partial^2 w_b}{\partial x \partial y} + \frac{\partial^2 w_s}{\partial y^2}\frac{\partial^2 w_b}{\partial x \partial y}\right] \\
& \left(F_{11}^r N_x^0 + F_{12}^r N_y^0 + F_{16}^r N_{xy}^0\right)\left[\frac{\partial^4 w_s}{\partial x^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right] + \left(F_{12}^r N_x^0 + F_{22}^r N_y^0 + F_{26}^r N_{xy}^0\right)\left[\frac{\partial^4 w_s}{\partial y^4} + \frac{\partial^4 w_s}{\partial x^2 \partial y^2}\right] \\
& + 2\left(F_{16}^r N_x^o + F_{26}^r N_y^0 + F_{66}^r N_{xy}^0\right)\left[\frac{\partial^4 w_s}{\partial x^3 \partial y} + \frac{\partial^4 w_s}{\partial x \partial y^3}\right]
\end{aligned}
\tag{30}
$$

Governing equations can be obtained by applying the variational relationship

$$
\delta U + \delta V_1 + \delta V_2 = 0 \tag{31}
$$

Substituting Equations (14), (20) and (30) into Equation (31), we obtain the following governing equations in operator form

$$
\begin{aligned}
L_{11} w_b + L_{12} w_s &= 0 \\
L_{12} w_b + L_{22} w_s &= 0
\end{aligned}
\tag{32}
$$

The linear operators L_{ij} are defined as follows:

$$
\begin{aligned}
L_{11} = {} & \left(D_{11} + F_{11}N_x^0 + F_{12}N_y^0 + F_{16}N_{xy}^0\right)(\ )_{,xxxx} + 2\left(2D_{16} + F_{16}N_x^0 + F_{26}N_y^0 + F_{66}N_{xy}^0\right)(\ )_{,xxxy} \\
& + \left(2D_{12} + 4D_{66} + F_{11}N_x^0 + F_{12}N_y^0 + F_{16}N_{xy}^0 + F_{12}N_x^0 + F_{22}N_y^0 + F_{26}N_{xy}^0\right)(\ )_{,xxyy} \\
& + \left(D_{22} + F_{12}N_x^0 + F_{22}N_y^0 + F_{26}N_{xy}^0\right)(\ )_{,yyyy} + 2\left(2D_{26} + F_{16}N_x^0 + F_{26}N_y^0 + F_{66}N_{xy}^0\right)(\ )_{,xyyy} \\
& + N_x^0(\ )_{,xx} + 2N_{xy}^0(\ )_{,xy} + N_y^0(\ )_{,yy}
\end{aligned}
\tag{33}
$$

$$L_{12}=\left(D_{11}^{s}+F_{11}^{s}N_{x}^{0}+F_{12}^{s}N_{y}^{0}+F_{16}^{s}N_{xy}^{0}\right)\left(\ \right)_{,xxxx}+2\left(2D_{16}^{s}+2\left(F_{16}^{s}N_{x}^{0}+F_{26}^{s}N_{y}^{0}+F_{66}^{s}N_{xy}^{0}\right)\right)\left(\ \right)_{,xxxy}$$
$$+\left(2D_{12}^{s}+4D_{66}^{s}+F_{11}^{s}N_{x}^{0}+F_{12}^{s}N_{y}^{0}+F_{16}^{s}N_{xy}^{0}+F_{12}^{s}N_{x}^{0}+F_{22}^{s}N_{y}^{0}+F_{26}^{s}N_{xy}^{0}\right)\left(\ \right)_{,xxyy}$$
$$+\left(D_{22}^{s}+F_{12}^{s}N_{x}^{0}+F_{22}^{s}N_{y}^{0}+F_{26}^{s}N_{xy}^{0}\right)\left(\ \right)_{,yyyy}+2\left(2D_{26}^{s}+2\left(F_{16}^{s}N_{x}^{0}+F_{26}^{s}N_{y}^{0}+F_{66}^{s}N_{xy}^{0}\right)\right)\left(\ \right)_{,xyyy}$$
$$+N_{x}^{0}\left(\ \right)_{,xx}+2N_{xy}^{0}\left(\ \right)_{,xy}+N_{y}^{0}\left(\ \right)_{,yy}$$

$$L_{22}=\left(H_{11}^{s}+F_{11}^{r}N_{x}^{0}+F_{12}^{r}N_{y}^{0}+F_{16}^{r}N_{xy}^{0}\right)\left(\ \right)_{,xxxx}+\left(4H_{16}^{s}+2\left(F_{16}^{r}N_{x}^{0}+F_{26}^{r}N_{y}^{0}+F_{66}^{r}N_{xy}^{0}\right)\right)\left(\ \right)_{,xxxy}$$
$$+\left(2H_{12}^{s}+4H_{66}^{s}+F_{11}^{r}N_{x}^{0}+F_{12}^{r}N_{y}^{0}+F_{16}^{r}N_{xy}^{0}+F_{12}^{r}N_{x}^{0}+F_{22}^{r}N_{y}^{0}+F_{26}^{r}N_{xy}^{0}\right)\left(\ \right)_{,xxyy}$$
$$+\left(H_{22}^{s}+F_{12}^{r}N_{x}^{0}+F_{22}^{r}N_{y}^{0}+F_{26}^{r}N_{xy}^{0}\right)\left(\ \right)_{,yyyy}+\left(4H_{26}^{s}+2\left(F_{16}^{r}N_{x}^{0}+F_{26}^{r}N_{y}^{0}+F_{66}^{r}N_{xy}^{0}\right)\right)\left(\ \right)_{,xyyy}$$
$$+A_{55}^{s}\left(\ \right)_{,xx}+A_{44}^{s}\left(\ \right)_{,yy}+2A_{45}^{s}\left(\ \right)_{,xy}+N_{x}^{0}\left(\ \right)_{,xx}+2N_{xy}^{0}\left(\ \right)_{,xy}+N_{y}^{0}\left(\ \right)_{,yy}$$

Exact solutions of mechanical buckling for symmetric cross-ply plates

Consider a rectangular plate with the length a, and width b which is subjected to in-plane loads. Therefore, the pre-buckling forces can be obtained using the equilibrium conditions as [32, 33]

$$N_{x}^{0}=-N,\ \ N_{y}^{0}=\xi N,\ \ N_{xy}^{0}=0 \qquad \left(N>0\right) \tag{34}$$

Where N the force per unit length is, ξ is the load parameter which indicate the loading conditions. Negative value for ξ indicate that plate is subjected to biaxial compressive loads while positive values are used for tensile loads. Also, zero value for ξ show uniaxial loading in x directions, respectively.

The exact solutions of equations (32) and (33) for simply supported, symmetric cross-ply rectangular plates may be obtained by recognizing the following plate stiffness to have zero values

$$\begin{aligned} &A_{16}=A_{26}=D_{16}=D_{26}=D_{16}^{s}=D_{26}^{s}=0\\ &H_{16}^{s}=H_{26}^{s}=A_{45}^{s}=0 \end{aligned} \tag{35}$$

By following the Navier solution procedure, the solutions to the problem are assumed to take the following forms

$$\begin{aligned} w_b(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{b\,mn}\sin\lambda x\sin\mu y\\ w_s(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{s\,mn}\sin\lambda x\sin\mu y \end{aligned} \tag{36}$$

where

$$\lambda = \frac{m\pi x}{a}, \mu = \frac{m\pi x}{a} \tag{37}$$

Substituting Equation (36) into Equation (32) for a symmetric cross-ply laminate, we obtain the following equations

$$\begin{aligned}
&\left[D_{11} - c_c N(F_{11} - \xi F_{12})\right](w_b)_{,xxxx} + \left[2D_{12} + 4D_{66} - c_c N\left((F_{11} + F_{12}) - \xi (F_{12} + F_{22})\right)\right](w_b)_{,xxyy} \\
&+ \left[D_{22} - c_c N(F_{12} - \xi F_{22})\right](w_b)_{,yyyy} + \left[D_{11}^s - c_c N(F_{11}^s - \xi F_{12}^s)\right](w_s)_{,xxxx} \\
&+ \left[2D_{12}^s + 4D_{66}^s - c_c N\left((F_{11}^s + F_{12}^s) - \xi (F_{12}^s + F_{22}^s)\right)\right](w_s)_{,xxyy} + \left[D_{22}^s - c_c N(F_{12}^s - \xi F_{22}^s)\right](w_s)_{,yyyy} \\
&+ N_x^0 (w_b)_{,xx} + N_y^0 (w_b)_{,yy} + N_x^0 (w_s)_{,xx} + N_y^0 (w_s)_{,yy} = 0
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\left[D_{11}^s - c_c N(F_{11}^s - \xi F_{12}^s)\right](w_b)_{,xxxx} + \left[2D_{12}^s + 4D_{66}^s - c_c N\left((F_{11}^s + F_{12}^s) - \xi (F_{12}^s + F_{22}^s)\right)\right](w_b)_{,xxyy} \\
&+ \left[D_{22}^s - c_c N(F_{12}^s - \xi F_{22}^s)\right](w_b)_{,yyyy} + \left[H_{11}^r - c_c N(F_{11}^r - \xi F_{12}^r)\right](w_s)_{,xxxx} \\
&+ \left[2H_{12}^s + 4H_{66}^s - c_c N\left((F_{11}^r + F_{12}^r) - \xi (F_{12}^r + F_{22}^r)\right)\right](w_s)_{,xxyy} + \left[H_{22}^s - c_c N(F_{12}^r - \xi F_{22}^r)\right](w_s)_{,yyyy} \\
&+ N_x^0 (w_b)_{,xx} + N_y^0 (w_b)_{,yy} + \left[A_{55}^s + N_x^0\right](w_s)_{,xx} + \left[A_{44}^s + N_y^0\right](w_s)_{,yy} = 0
\end{aligned} \tag{39}$$

where c_c takes on the value 1 when the "curvature" terms are included in the analysis and is 0 when these terms are neglected.

After substituting the Eq. (36) into Eqs. (38) and (39) we get a systems of two equations for finding the W_{bmn} and W_{smn}. By equaling the determinant of coefficient to zero we have:

$$\begin{bmatrix} \left(a_1 - N(c_c b_1 + c_1)\right) & \left(a_2 - N(c_c b_2 + c_1)\right) \\ \left(a_2 - N(c_c b_2 + c_1)\right) & \left(a_3 - N(c_c b_3 + c_1)\right) \end{bmatrix} \begin{bmatrix} W_{bmn} \\ W_{smn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{40}$$

where

$$\begin{aligned}
a_1 &= D_{11} \frac{m^4\pi^4}{a^4} + D_{22} \frac{n^4\pi^4}{b^4} + \left(2D_{12} + 4D_{66}\right) \frac{m^2 n^2 \pi^4}{a^2 b^2} \\
a_2 &= D_{11}^s \frac{m^4\pi^4}{a^4} + D_{22}^s \frac{n^4\pi^4}{b^4} + \left(2D_{12}^s + 4D_{66}^s\right) \frac{m^2 n^2 \pi^4}{a^2 b^2} \\
a_3 &= H_{11}^s \frac{m^4\pi^4}{a^4} + H_{22}^s \frac{n^4\pi^4}{b^4} + \left(2H_{12}^s + 4H_{66}^s\right) \frac{m^2 n^2 \pi^4}{a^2 b^2} + A_{55}^s \frac{m^2\pi^2}{a^2} + A_{44}^s \frac{n^2\pi^2}{b^2}
\end{aligned} \tag{41}$$

$$b_1 = \left(F_{11} - \xi F_{12}\right)\frac{m^4\pi^4}{a^4} + \left(\left(F_{11} + F_{12}\right) - \xi\left(F_{12} + F_{22}\right)\right)\frac{m^2n^2\pi^4}{a^2b^2} + \left(F_{12} - \xi F_{22}\right)\frac{n^4\pi^4}{b^4}$$

$$b_2 = \left(F_{11}^s - \xi F_{12}^s\right)\frac{m^4\pi^4}{a^4} + \left(\left(F_{11}^s + F_{12}^s\right) - \xi\left(F_{12}^s + F_{22}^s\right)\right)\frac{m^2n^2\pi^4}{a^2b^2} + \left(F_{12} - \xi F_{22}\right)\frac{n^4\pi^4}{b^4}$$

$$b_3 = (F_{11}^r - \xi F_{12}^r)\frac{m^4\pi^4}{a^4} + \left(\left(F_{11}^r + F_{12}^r\right) - \xi\left(F_{12}^r + F_{22}^r\right)\right)\frac{m^2n^2\pi^4}{a^2b^2} + \left(F_{12} - \xi F_{22}\right)\frac{n^4\pi^4}{b^4}$$

$$c_1 = \left(-\frac{m^2\pi^2}{a^2} + \xi\frac{n^2\pi^2}{b^2}\right)$$

Numerical Examples

Several examples are solved to demonstrate the accuracy and efficiency of the method. In the examples considered, symmetric cross-ply, angle-ply and quasi-isotropic thick rectangular laminates are considered and the following material properties are assumed [29]:

$$E_1/E_2 = 14, G_{12}/E_2 = 0.533, G_{23}/E_2 = 0.323, \nu_{12} = 0.3, \nu_{12} = 0.55$$

Two different cases have been considered in the numerical study: (1) without the effects of curvature terms and (2) with the effect of curvature terms. Note that Case 1 is the conventional consideration of thick plate buckling, which forms the basis of comparison for the case (2). Algorithm used to do the numerical analysis:

A general iterative procedure for obtaining the buckling load N, is as follows:

- The effects of curvature terms are ignored, thus, $c_c = 0$; c_1 and $a_i, i = 1-3$, are calculated. Substitute c_1 and a_i, into matrix in Equation (40). For nontrivial solution of the critical buckling load N_{cr}, the determinant of the matrix in Equation (40) must be equal to zero.
- The effects of curvature terms are included, thus, $c_c = 1$; c_1 and $a_i, b_i, i = 1-3$, are calculated. Substitute c_1 and a_i, b_i, into matrix in Equation (40). For nontrivial solution of the critical buckling load N_{cr}, the determinant of the matrix in Equation (40) must be equal to zero.

- Note that since m, n = 1, 2, . . . ,∞, there is an infinite number of buckling loads N. The critical buckling load N_{cr} is the minimum positive real solution with respect to m and n.

In order to verify the present code, the buckling problem of a simply isotropic square plate (ν = 0.30) under uniaxial compression is studied in Table 1. The numerical results are compared with analytic results of Reddy [34] and strain finite element formulation incorporating a third-order polynomial displacement model results of Nayak [35]. It shows that the present results are compared well with those of the previous works.

Table 1: Comparisons of critical buckling factor $\overline{N} = \overline{N}_{xx} a^2 / \pi^2 D$ for simply supported square isotropic plates under uniaxial compression.

Source	**a/h**				
	5	**10**	**20**	**50**	**100**
Nayak [35]	3.2656	3.7867	3.9445	3.9901	3.9939
Reddy [34]	3.2653	3.7865	3.9443	3.9909	3.9977
Present	3.2653	3.7866	3.9444	3.9910	3.9977

The results of critical buckling load of simply supported square cross-ply laminated composite plates [0°/90°/90°/0°] plate are presented in Tables 2 and 3 and Figs. 1 and 2. In Tables 2 and 3, the critical buckling factor $\left(N_{cr} a^2 / h^3 E_2\right)$ for simply supported square cross-ply laminated composite plates [0°/90°/90°/0°], under biaxial buckling and under in-plane combined tension and compression, respectively for different values of thickness-side ratio (a/h= 5, 10, 15, 20, 25,30). The material and geometry of the square plate considered here are [18]. These results are compared with the results found by Whitney [29] using first-order shear deformation theory. As seen a very good agreement has been achieved between them. Tables 2 and 3 also show that, the critical buckling factor increases with increase in the thickness-side ratio a/h. A comparison of Table 2 with Table 3 shows that the critical buckling load for the plate subjected to compression along x-direction and tension along y-direction, is greater than the corresponding values for the plate under biaxial compression. On the other hand, if the effect of curvature terms is included (Case 2), the buckling factors are always lower than those in Case 1. This appears to be academic, however, as the results in Tables 2 and 3show that the curvature terms have little practical effect on the critical buckling factor for the laminate geometries considered.

Table 2: Comparisons of critical buckling factor $\left(N_{cr}a^2/h^3E_2\right)$ for simply supported square cross-ply laminated composite plates [0°/90°/90°/0°], under biaxial buckling.

a/h	Source			
	FSDT [29] C_c=1	**Present C_c=1**	**FSDT[29] C_c=0**	**Present C_c=0**
5	3.5837	3.9629	3.6706	4.0417
10	5.7459	6.0188	5.8112	6.0853
15	6.5213	6.6801	6.5605	6.7201
20	6.8509	6.9500	6.8758	6.9752
25	7.0163	7.0830	7.0332	7.1000
30	7.1100	7.1576	7.1221	7.1697
CPT	--------	--------	7.3335	7.3335

Table 3: Comparisons of critical buckling factor $\left(N_{cr}a^2/h^3E_2\right)$ for simply supported square cross-ply laminated composite plates [0°/90°/90°/0°], under in-plane combined tension and compression.

a/h	Source			
	FSDT [29] C_c=1	**Present C_c=1**	**FSDT[29] C_c=0**	**Present C_c=0**
5	10.4425	11.5465	10.9050	11.7515
10	26.7192	28.2716	27.4733	28.9051
15	38.0402	39.3715	38.8042	40.1032
20	44.7934	45.8088	45.4372	46.4447
25	48.8479	49.6117	49.3608	50.1229
30	51.3908	51.9739	51.7962	52.8792
CPT	---------	---------	58.3586	58.3586

aMode (1,2)

It should be noted that the present theory involves only two independent variables as against three in the case of first-order shear deformation plate theory [29]. Also, the present theory does not required shear correction factors as in the case of first-order shear deformation plate theory. It can be concluded that the present theory is not only accurate but also efficient in predicting critical buckling load of laminated composite plates.

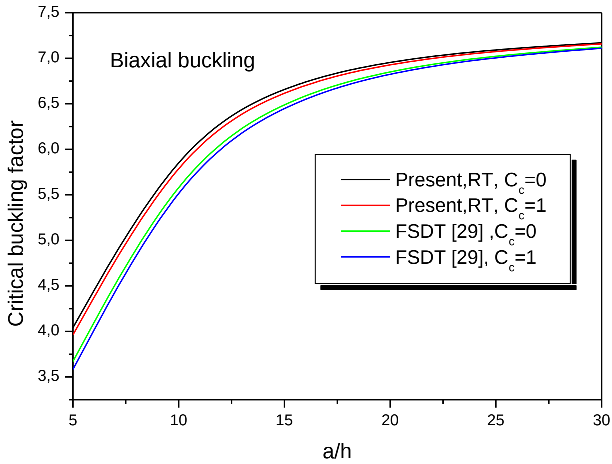

Figure 1: A comparison on buckling responses including curvature effects and effect of shear deformation of simply supported square cross-ply laminated composite plates $[0°/90°]_s$ subjected biaxial compression with those of reported in [29].

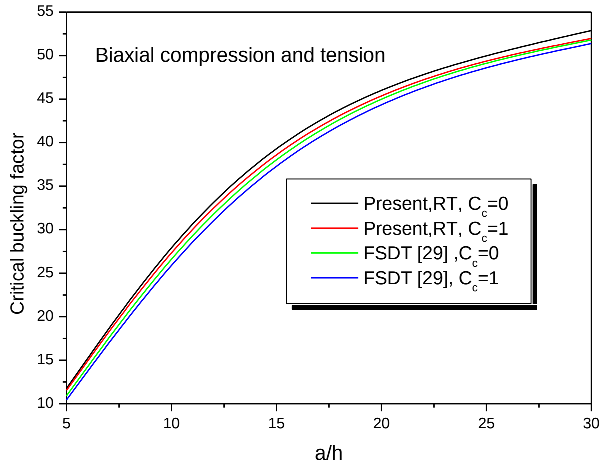

Figure 2: A comparison on buckling responses including curvature effects and effect of shear deformation of simply supported square cross-ply laminated composite plates $[0°/90°]_s$ subjected Biaxial compression and tension with those of reported in [29].

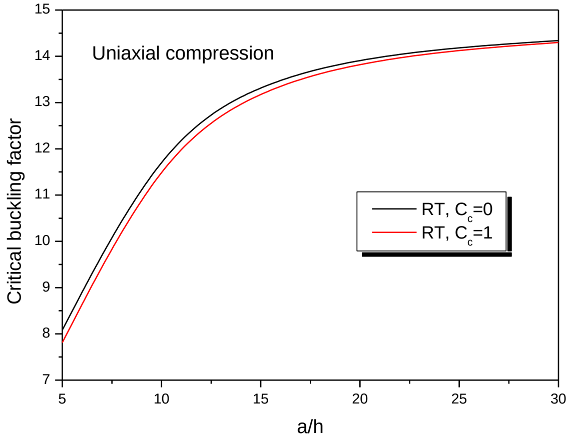

Figure 3: The effect of curvature terms on critical buckling factor of simply supported square plate under uniaxial compression.

Buckling factors are plotted against aspect ratio for plates under uniaxial compression in Figure 3. If only the effect of curvature terms (Case 2) is included, the buckling factors are always lower than those in Case 1.Comparing Figs 1,2 and 3, the responses are very similar, however, the nondimensional critical buckling load of plate under uniaxial compression is greater than that under biaxial compression and less than that under biaxial compression and tension. In addition, the inclusion of curvature terms decreases the buckling factor no matter what loading condition is applied.

Conclusions

In this work, buckling analysis of symmetrically-laminated rectangular plates is investigated. In order to consider the curvature effects, refined two-parameter theory and Navier solution method are used. The present theory has only two unknowns, but it accounts for a parabolic variation of transverse shear strains through the thickness of the plate, without using shear correction factor. Buckling of orthotropic laminates subjected to biaxial inplane loading is investigated. Based on the numerical and graphical results it is concluded that the theory is in good agreement with other higher-order shear deformation theories while predicting the critical buckling response of laminated composite plates. Also, it is observed that, the inclusion of curvature terms decreases the buckling factor no matter what loading condition is applied. In conclusion, it can be said that the proposed theory is accurate and efficient in predicting the buckling responses of symmetrically-laminated rectangular plates with allowance for the effects of higher-order strain terms (curvature terms).

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