UNIVERSITI TEKNOLOGI MARA

MATHEMATICS LETTERS
https://sites.google.com/tmsk.uitm.edu.my/mathematicsletters

Comparative Analysis of Three Selected Classical Conjugate Gradient Coefficients under the Armijo Inexact Line Search for Unconstrained Optimization

[1]Siti Nur'Ain Binti Hasbullah, [2,*]Nur Hamizah Abdul Ghani & [3]Wan Khadijah Wan Sulaiman

Center of Mathematical Sciences, Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara (UiTM), 40450 Shah Alam, Selangor, Malaysia.
[1]snrainnz@gmail.com
[*2] hamizahghani@uitm.edu.my
[3] wankhadijah@uitm.edu.my
*Corresponding author

Received: 12/10/2026 Revised: 21/10/2026 Accepted: 25/10/2026 Published: 31/10/2026

ABSTRACT

Optimization plays an important role in solving unconstrained optimization problems that arise in various scientific and engineering applications. Among the available optimization techniques, the conjugate gradient (CG) method is widely used due to its low memory requirement and computational efficiency. The performance of the CG method is highly influenced by the selection of the conjugate gradient coefficient and the line search strategy. This study aims to implement selected classical CG methods, namely Hestenes–Stiefel (HS), Polak–Ribière–Polyak (PRP), and Belloufi–Rahali–Benzine (BRB). These methods use the Armijo inexact line search for solving unconstrained optimization problems. The methods were implemented and tested on a collection of 20 benchmark optimization problems with different dimensions and initial points. Their performance was evaluated based on the number of iteration and CPU time. The numerical results were analysed using performance profiles to compare the efficiency of the three methods. The findings indicate that the HS method consistently outperformed the PRP and BRB methods by solving a larger proportion of test problems with fewer iterations and shorter times. Hence, the HS method is considered the most efficient and reliable among the selected methods under the Armijo inexact line search. This study demonstrates the effectiveness of classical CG methods in unconstrained optimization and provides a useful reference for future research on optimization algorithms.

Keywords: Armijo Inexact Line Search, Classical Conjugate Gradient Method, Hestenes-Stiefel Method, Performance Profile, Unconstrained Optimization

INTRODUCTION

Optimization is a very significant area of applied mathematics and numerical analysis since it allows to obtain the minimum or maximum of an objective function in a systematic manner. It is used in resource allocation, scientific computing, engineering and economics. Optimization problems are divided into

constrained and unconstrained problems. In constrained optimization, there are explicit equality or inequality constraints on the decision variables, while in unconstrained optimization, there are no such explicit constraints. Consequently, gradient based methods are largely applied to continuously differentiable functions, because they incorporate information about the derivatives to help direct a search towards an optimum. However, they can be effective only based on their search directions and convergence behaviour. The steepest descent method is easy to implement, but can converge slowly on difficult problems, especially if the objective functions have narrow valleys, or have poor conditioning (Fletcher, 1987; Hestenes & Stiefel, 1952; Nocedal & Wright, 2006). One of the most interesting methods is the nonlinear conjugate gradient (CG) method, which is attractive due to its use of previous search directions and its relatively low memory requirements, as well as combining current gradient information with past search directions (Abubakar et al., 2022; Ghani et al., 2024). Several classical CG methods, such as Hestenes–Stiefel (HS) method (Hestenes & Stiefel, 1952), Polak–Ribière–Polyak (PRP) method (Polyak, 1969), and the Belloufi–Rahali–Benzine (BRB) method (Belloufi et al., 2020), which provides another coefficient formulation for nonlinear optimization. The various coefficient formulas could affect the convergence and computational efficiency. The Armijo inexact line search determines a suitable step size by making sure there is enough decrease in the objective function (Armijo, 1966). But choosing an optimal CG coefficient is not easy, since the methods could behave differently in different optimization problems and line search environments. Comparisons of these methods on a common platform can help to better understand their advantages and disadvantages. Therefore, this study investigates the theoretical properties and numerical performance of HS, PRP, and BRB using the Armijo inexact line search for unconstrained optimization. Twenty benchmark functions with different dimensions and initial points are used to evaluate their performance based on iteration counts, CPU time, and performance profiles. The findings aim to identify the relative computational efficiency of the three methods under the selected conditions.

LITERATURE REVIEW

The conjugate gradient method was first used to solve systems of linear equations and later in the field of nonlinear optimization (Hestenes & Stiefel, 1952). In a nonlinear CG scheme, the next iterate is computed by taking a step in the direction of the search along from the current point with a chosen step size. The search direction is a combination of the present gradient information and the previous search direction; the coefficient used in this combination distinguishes the various of the CG methods. In classical methods, one specific coefficient formula is applied at each iteration, while in hybrid methods several coefficient formulas are applied at the same iteration and/or switched to attain greater robustness (Andrei, 2008; Gilbert & Nocedal, 1992). HS and PRP are classic CG methods that have their own distinct coefficient formulas, which can yield different search directions and convergence. Belloufi et al. (2020) have also presented another coefficient formula for accelerating gradient based optimization, known as the BRB method. The various update rules for these methods make it necessary to evaluate the relative performance by conducting numerical experiments under the same conditions; this cannot be done with the definitions of the coefficients alone. The Armijo inexact line search is typically used in optimization procedures to find an appropriate step size to achieve a sufficient-decrease in the objective function. In inexact line search, a condition is used to find a step size that is acceptable along the search direction, as compared to exact line search where the search direction is taken to the minimizing direction. Armijo (1966) introduced a sufficient-decrease condition that requires the objective function to decrease according to the selected step size and the directional derivative. The condition is expressed as

$$f(x_k + \alpha_k d_k) \leq f(x_k) + c\alpha_k g_k^T d_k, \quad (1)$$

where x_k is the current iterate, d_k is the search direction, α_k is the step size, $g_k = \nabla f(x_k)$ is the gradient at iteration k, and c is a positive parameter < 1. In the numerical experiments, the Armijo parameter is set to $c = 10^{-4}$, and the stopping criterion is $\| g_{k+1} \| \leq 10^{-6}$. The sufficient descent condition considered as

$$g_k^T d_k \leq -c \| g_k \|^2, \quad (2)$$

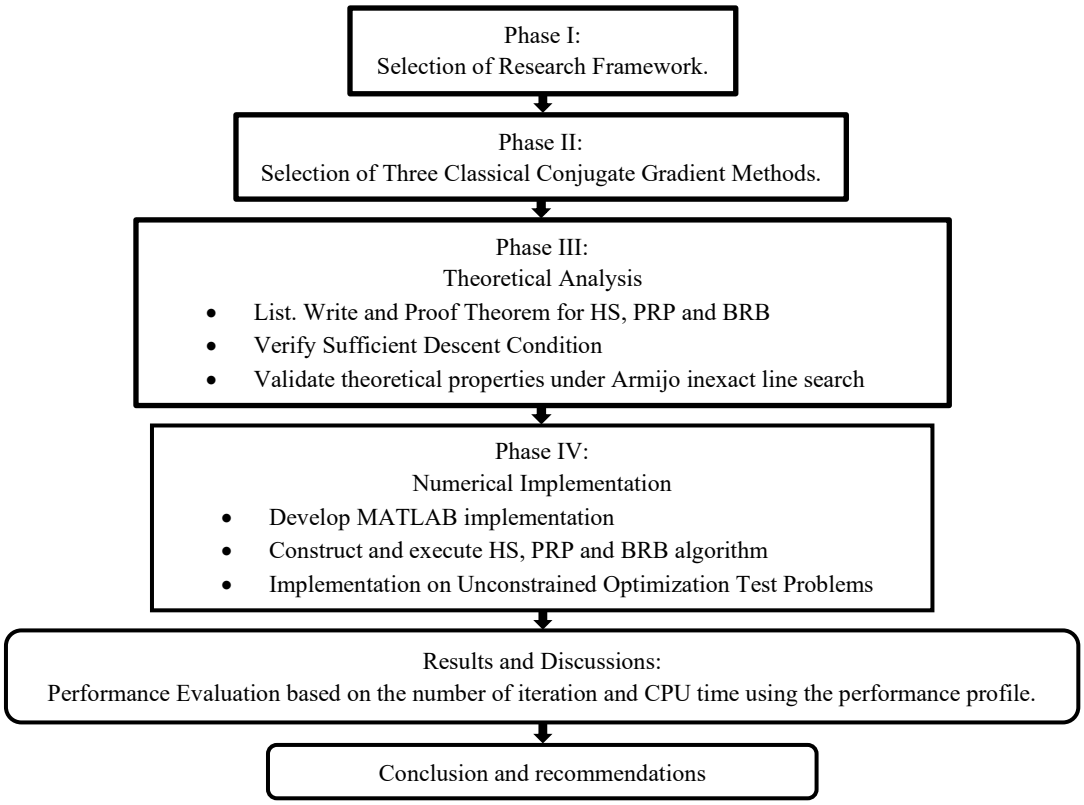
where $c > 0$. This condition enables the analysis of the descent behaviour of conjugate gradient methods. The performance of nonlinear conjugate gradient methods can be evaluated using iteration count and CPU time. Iteration count is the number of iterations needed to meet a stopping criterion, and CPU time is the cost of the calculations by time. To compare the performance of optimization methods with each other on benchmark problems, Dolan and Moré (2004) added performance profiles, which are the percentage of the problems that are solved within a given multiple of the best solution. In the past, some studies have focused on the nonlinear conjugate gradient method and its convergence (Abubakar et al., 2022; Fang et al., 2022; Gilbert & Nocedal, 1992; Hallal et al., 2022; Jiang et al., 2023; Wu et al., 2021). The numerical performance, however, can depend on the choice of coefficient formula and line-search schemes. Therefore, comparing the Hestenes–Stiefel (HS), Polak–Ribière–Polyak (PRP), and Belloufi–Rahali–Benzine (BRB) methods under the Armijo line-search framework provide a consistent basis for evaluating their performance.

METHODOLOGY

The main phases of the methodology are presented as illustrated in Figure 1.

Figure 1

Conceptual Framework



The methodology adopted in this study is structured into four phases. It begins with the selection of the research framework and three classical conjugate gradient methods, followed by theoretical analysis to establish their convergence properties and verify the sufficient descent condition. The methods are then implemented in MATLAB and evaluated using unconstrained optimization test problems, with their performance assessed based on the number of iterations and CPU time through performance profiles. Finally, the numerical results are discussed to obtain conclusions and provide recommendations for future research.

Phase 1: Selection of Research Framework
This study examines three classical CG methods; HS, PRP, and BRB using the Armijo inexact line search. It involves the theoretical analysis, numerical implementation, and performance analysis to compare their efficiency and descent properties.

Phase 2: Selection of Three Classical CG Methods
Let $g_k = \nabla f(x_k), y_k = g_{k+1} - g_k,\ d_0 = -g_0$ and $d_{k+1} = -g_{k+1} + \beta_k d_k$.

The coefficients chosen are as follows:
1. Belloufi–Rahali–Benzine (BRB)

$$\beta_k^{BRB} = \frac{\| g_{k+1} \|^2}{\| d_k \|^2}.$$

2. Polak–Ribière–Polyak (PRP)

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\| g_k \|^2}.$$

3. Hestenes–Stiefel (HS)

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k}.$$

Phase 3: Theoretical Analysis
Verify the sufficient descent condition is satisfied for each coefficient with common direction update.

$$g_{k+1}{}^T d_{k+1} = -\|g_{k+1}\|^2 + \beta_k g_{k+1}{}^T d_k.$$

Phase 4: Numerical Implementation
The methods are implemented in MATLAB and tested on 20 unconstrained benchmark problems to evaluate the numerical performance. At each iteration, choose α_k by Armijo backtracking and update

$$x_{k+1} = x_k + \alpha_k d_k.$$

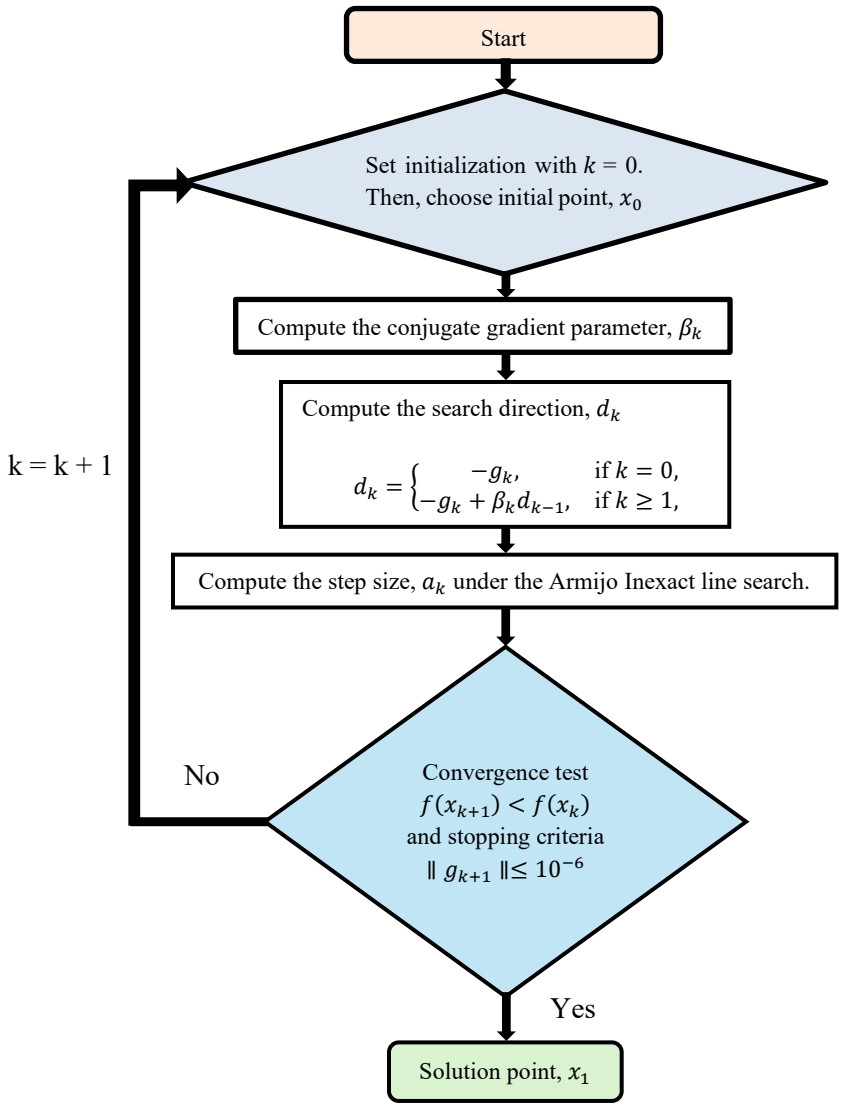
Armijo condition: $f(x_k + \alpha_k d_k) \leq f(x_k) + c_1 \alpha_k g_k{}^T d_k$, with $c_1 = 10^{-4}$. Stop when

$$\|g_{k+1}\| \leq 10^{-6}.$$

The algorithm of classical conjugate gradient method is given in Figure 2 as follow:

Figure 2

Classical CG Algorithm.



RESULTS AND DISCUSSIONS

The results and discussion are presented in two main parts: theoretical analysis and numerical analysis. The theoretical results comprise three theorems that establish the convergence properties of the proposed methods and verify the sufficient descent condition. The numerical analysis evaluates the performance of the three classical conjugate gradient methods on 20 unconstrained optimization test

functions, including a manual calculation for the Extended Maratos function to illustrate the computational procedure. Finally, performance profiles based on the number of iterations and CPU time are employed to compare the numerical performance of the methods.

Theoretical Analysis

Let

$$d_k = -g_k + \beta_k d_{k-1}.$$

Then

$$g_k^T d_k \leq -c \parallel g_k \parallel^2,\ c > 0.$$

Theorem 4.1 (Sufficient Descent Condition of the HS Method)

The HS search direction is defined as

$$d_k = -g_k + \beta_k^{HS} d_{k-1},$$

where

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k}, y_k = g_{k+1} - g_k.$$

Proof

Taking the inner product of the search direction with gradient vector, g_k yields

$$g_k^T d_k = -\parallel g_k \parallel^2 + \beta_k^{HS} g_k^T d_{k-1}.$$

Assume

$$g_k^T d_{k-1} < 0.$$

However, this does not guarantee that the second term is non-positive because the sign of β_k is not yet been established yet. Therefore, the HS method satisfies the sufficient descent condition under the Armijo inexact line search.

Theorem 4.2 (Sufficient Descent Condition of the PRP Method)

The PRP search direction is defined as

$$d_k = -g_k + \beta_k^{PRP} d_{k-1},$$

where

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\parallel g_k \parallel^2}, y_k = g_{k+1} - g_k.$$

Proof

Taking the inner product with the gradient vector, g_k yields

$$g_k^T d_k = -\| g_k \|^2 + \beta_k^{PRP} g_k^T d_{k-1}.$$

The previous direction is assumed to satisfy

$$g_k^T d_{k-1} < 0.$$

Under the Armijo inexact line search, the generated search direction maintains sufficient reduction of the objective function during the iteration process. Therefore,

$$g_k^T d_k \leq -\| g_k \|^2.$$

Hence,

$$g_k^T d_k \leq -c \| g_k \|^2, c > 0.$$

Therefore, the PRP method satisfies the sufficient descent condition under the Armijo inexact line search.

Theorem 4.3 (Sufficient Descent Condition of the BRB Method)

The BRB search direction is defined as

$$d_k = -g_k + \beta_k^{BRB} d_{k-1},$$

where

$$\beta_k^{BRB} = \frac{\|g_{k+1}\|^2}{\|d_k\|^2}.$$

Proof

Taking the inner product with the gradient vector, g_k yields

$$g_k^T d_k = g_k^T(-g_k + \beta_k^{BRB} d_{k-1}).$$

Since β_k^{BRB} whenever it is defined, and

$$g_k^T d_{k-1} < 0,$$

It follows that

$$g_k^T d_k \leq -c \| g_k \|^2, c = 1.$$

Therefore, the BRB method satisfies the sufficient descent condition under the Armijo inexact line search.

Numerical Analysis

The results and discussion are presented in two main parts: theoretical analysis and numerical analysis. The theoretical results comprise three theorems that establish the convergence properties of the proposed methods and verify the sufficient descent condition. The numerical analysis evaluates the performance of the three classical conjugate gradient methods on 20 unconstrained optimization test functions (Andrei, 2008) as listed in Table 1, including a manual calculation for the Extended Maratos function to illustrate the computational procedure. Finally, performance profiles based on the number of iterations and CPU time are employed to compare the numerical performance of the methods.

List of Test Functions

Table 1

List of the Selected Test Functions Used in the Numerical Analysis.

No.	Test function	Objective function / formula	Dimension(s)	Initial point(s)
1	Hager	$f(x) = \sum_{i=1}^{n} (exp(x_i) - \sqrt{(i)}x_i$	2, 10, 100	3, 7, 9
2	Extended Quadratic Penalty QP2	$f(x) = \sum_{i=1}^{n} (x_i - 1)^2 + (\sum_{i=1}^{n} x_i^2 - ¼)^2$	50, 500	1, 5, 10
3	Extended Powell	$f(x) = \sum_{i=1}^{n} / 4 [(x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4]$	2, 10, 100	1, 10, 15
4	Fletcher (CUTE)	$f(x) = \sum_{i=1}^{n-1} 100(x_{i+1} - x_i + 1 - x_i^2)$	2, 10, 100	1, 10, 15
5	Extended Penalty	$f(x) = \sum_{i=1}^{n-1}(x_i - 1)^2 + (\sum_{i=1}^{n} x_i^2 - ¼)^2$	2, 10, 100	1, 10, 15
6	Extended White & Holst	$f(x) = \sum_{i=1}^{n} / 2 [100(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2]$	2, 4, 10	0.5, 3, 5
7	Generalized Quartic	$f(x) = \sum_{i=1}^{n} x_i^4$	2, 100, 1000	1, 5, 10
8	Leon	$f(x,y) = 100(y - x^3)^2 + (1 - x)^2$	2	0.5, 3, 5
9	Treccani	$f(x,y) = x^4 + 4x^3 + 4x^2 + y^2$	2	10, 20, 30
10	Booth	$f(x,y) = (x + 2y - 7)^2 + (2x + y - 5)^2$	2	1, 10, 20
11	Three Hump Camel	$f(x,y) = 2x^2 - 1.05x^4 + x^6/6 + xy + y^2$	2	0.5, 5, 15
12	Extended Himmelblau	$f(x) = \sum_{i=1}^{n} / 2 [(x_{2i-1}^2 + x_{2i} - 11)^2 + (x_{2i-1} + x_{2i}^2 - 7)^2]$	2, 100, 1000	1, 10, 15

13	Generalized Tridagonal 2	$f(x) = ((5 - 3x_1 - x_1^2)x_1 - 3x_2 + 1)^2 + \sum_{i=1}^{n-1}((5 - 3x_i - x_i^2)x_i - x_{i-1} - 3x_{i+1} + 1)^2 + ((5 - 3x_n - x_n^2)x_n - x_{n-1} + 1)^2$	10, 500, 1000	1, 5, 10
14	Extended Beale	$f(x) = \sum_{i=1}^{n-1}[(1.5 - x_i + x_i x_{i+1})^2 + (2.25 - x_i + x_i x_{i+1}^2)^2 + (2.625 - x_i + x_i x_{i+1}^3)^2]$	2, 100, 1000	1, 5, 10
15	POWER (CUTE)	$f(x) = \sum_{i=1}^{n} x_i^{(i+1)}$	2, 100, 1000	1, 5, 10
16	Extended Rosenbrock	$f(x) = \sum_{i=1}^{n-1}[100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2]$	2, 100, 1000	1, 5, 7
17	Extended Maratos	$f(x) = \sum_{i=1}^{n/2}[x_{2i-1} + 100(x_{2i-1}^2 + x_{2i}^2 - 1)^2$	2	1, 5, 10
18	Diagonal 4	$f(x) = \sum_{i=1}^{n/2}½(x_{2i-1}^2 + 100x_{2i}^2)$	100, 1000, 10000	1, 5, 10
19	Matyas	$f(x,y) = 0.26(x^2 + y^2) - 0.48xy$	2, 100, 1000	1, 5, 10
20	Sum Square	$f(x) = \sum_{i=1}^{n} x_i^2$	2, 100, 1000	1, 5, 10

From the Table 1, it is observed that the methods vary in the performance depending on the benchmark problem, dimension and initial point. Hence, the methods are compared numerically, as well as by their performance profiles, to give a more general idea of their numerical efficiency.

Manual Calculation using the Extended Maratos Function

The first iteration of the CG algorithm is illustrated using the Extended Maratos Function:

$$f(x) = \sum_{i=1}^{n/2}[x_{2i-1} + c\left(x_{2i-1}^2 + x_{2i}^2 - 1\right)^2]$$

with dimension, $n = 2$, positive constant, $c = 100$, this becomes

$$f(x_1, x_2) = x_1 + 100(x_1^2 + x_2^2 - 1)^2.$$

At the initial point,

$$f(1,1) = 101.$$

The gradient is

$$\nabla f(x) = \begin{bmatrix} 1 + 400x_1(x_1^2 + x_2^2 - 1) \\ 400x_2(x_1^2 + x_2^2 - 1) \end{bmatrix}, (g_0) = \begin{bmatrix} 401 \\ 400 \end{bmatrix}, d_0 = -g_0 = \begin{bmatrix} -401 \\ -400 \end{bmatrix}.$$

The directional derivative is

$$g_0^T d_0 = \begin{bmatrix} 401 & 400 \end{bmatrix}\begin{bmatrix} -401 \\ -400 \end{bmatrix} = -320801.$$

Using the Armijo inexact line search, the step size is determined as $\alpha = 0.001$. The next new point is

$$x_1 = x_0 + \alpha_0 d_0$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.001\begin{bmatrix} -401 \\ -400 \end{bmatrix} = \begin{bmatrix} 0.599 \\ 0.600 \end{bmatrix}.$$

Then

$$f(0.599,0.600) = 0.599 + 100(0.599^2 + 0.600^2 - 1)^2 = 8.50629.$$

And

$$f(x_0) + c\alpha_0 g_0^T d_0 = 101 + (10^{-4})(0.001)(-320801) = 100.96792.$$

Therefore, the selected step satisfies the Armijo condition.

The gradient at the new point is approximately

$$g_1 = \begin{bmatrix} -66.375 \\ -67.488 \end{bmatrix},$$

and

$$y_0 = g_1 - g_0 = \begin{bmatrix} -467.375 \\ -467.488 \end{bmatrix}.$$

Using these vectors, the coefficients are approximately

$$\beta_0^{HS} = 0.16712,\ \beta_0^{PRP} = 0.1951\ ,\beta_0^{BRB} = 0.02793.$$

The next search directions are calculated using $(d_1 = -g_1 + \beta_0 d_0)$. The resulting directions are approximately

$$d_1^{HS} = \begin{bmatrix} -0.64 \\ -0.36 \end{bmatrix},\ d_1^{PRP} = \begin{bmatrix} -11.86 \\ -10.552 \end{bmatrix},\ d_1^{BRB} = \begin{bmatrix} 55.176 \\ 56.312 \end{bmatrix}.$$

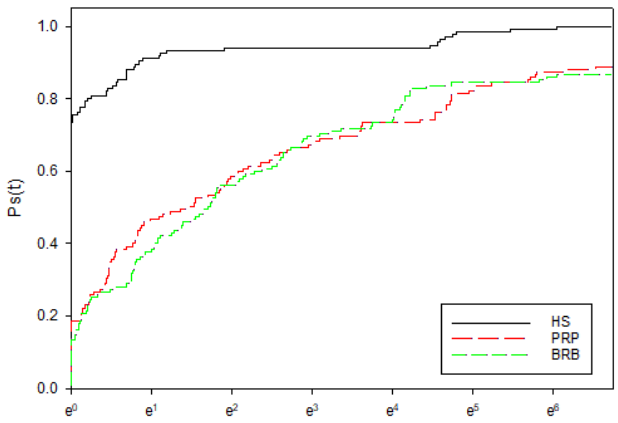
This illustrates the objective-function evaluation, calculation of the gradient, and selection of the step size and the updates of the coefficients for the three methods. This is repeated until the stopping criterion is satisfied.

Performance of three selected classical CG methods

The performance of HS, PRP and BRB methods are analyzed using performance profile by Dolan and Moré (2004) based on iterations number and CPU time.

Figure 3

Performance Profile of Number of Iterations.



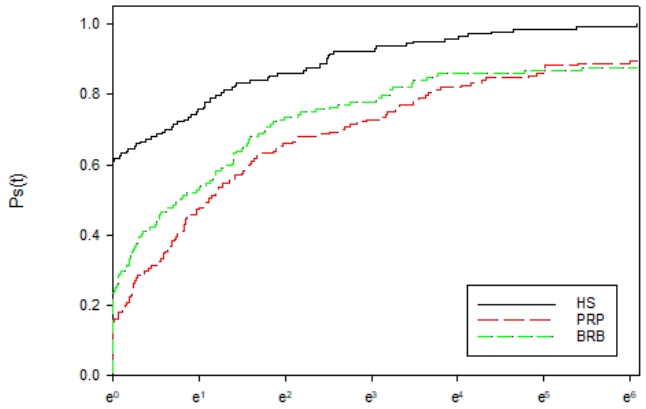
The number of iterations needed for HS, PRP and BRB to solve the benchmark problems is illustrated in Figure 3. It is observed that the HS curve is above the PRP and BRB curves over a larger percentage

of the range of performance-ratio, which means that HS can solve more problems near the optimal number of iterations seen. The HS performance at $\tau=1$ is around 0.75, meaning that it performs best on around 75% of the benchmark cases.

PRP and BRB have a smaller probability at lower performance ratios, but larger probabilities at higher performance ratios. The above results show that the computational setting of this study does not need many iterations in HS. It also turns out that it is not enough to be able to descend in a theoretically optimal manner to decide which numerical method will be most efficient.

Figure 4

Performance Profile of CPU Time.



The performance profile of CPU Time for the three algorithms is given in Figure 4, which indicates that HS is better than PRP and BRB for most of the performance-ratio range. This means that HS can solve more problems from the benchmark in a shorter amount of time. In some areas, BRB is a little better than PRP, but neither can be reliably said to be better than HS.

The number of iterations and CPU Time profiles suggest that HS is best overall numerically for the problems tested in the Armijo inexact line search. In some cases, the performance of BRB is better than the performance of PRP in CPU Time - the performance changes depending on the benchmark problem and the initial point.

CONCLUSION AND RECOMMENDATIONS

This section summarizes the main findings of the study based on the theoretical and numerical results. Recommendations for future research are also provided to suggest possible directions for further development of conjugate gradient methods.

Conclusions

This study compared the HS, PRP and BRB conjugate gradient methods were compared for solving unconstrained optimization problems applied Armijo inexact line search. All the methods have been

applied for 20 benchmark problems and analyzed by iteration, CPU time consumption, and performance profile. The results show that HS can achieve the best numerical performance overall, and in some CPU-time comparisons, it performs better than PRP. Considering the three methods, HS is the most efficient under the computational settings used. The sufficient descent and convergence properties need to be further analyzed theoretically with suitable assumptions.

Recommendations

It would be interesting to study the three methods with other line-search techniques, with bigger benchmark collections and with higher dimensional problems in the future. Other studies might include adjustments to the BRB coefficient and extensions to constrained optimization and optimization on non-smooth sets. Further theoretical study is suggested to develop the convergence properties of the methods under suitable conditions.

ACKNOWLEDGEMENT

The author has no conflicts of interest to declare.

REFERENCES

Abubakar, A. B., Malik, M., Kumam, P., Mohammad, H., Sun, M., Ibrahim, A. H., & Kiri, A. I. (2022). A liu-storey-type conjugate gradient method for unconstrained minimization problem with application in motion control. Journal of King Saud University - Science, 34(4), 101923. https://doi.org/10.1016/j.jksus.2022.101923

Andrei, N. (2008). An unconstrained optimization test functions collection. Advanced Modeling and Optimization, 10(1).

Armijo, L. (1966). Minimization of functions having Lipschitz continuous first partial derivatives. Pacific Journal of Mathematics, 16(1), 1–3. https://doi.org/10.2140/pjm.1966.16.1

Belloufi, M., Rahali, N., & Benzine, R. (2020). A new conjugate gradient method for acceleration of gradient descent algorithms. Moroccan Journal of Pure and Applied Analysis, 7(1), 1–11. https://doi.org/https://doi.org/10.2478/mjpaa-2021-0001

Dolan, E., & Moré, J. (2004). Benchmarking optimization software with performance profiles.

Fang, M., Wang, M., & Ding, D. (2022). A new hybrid conjugate gradient projection method for solving nonlinear monotone equations. Journal of Mathematics, 2022(1). https://doi.org/10.1155/2022/5540504

Fletcher, R. I. (1987). Three optimization problems of year-class analysis. ICES Journal of Marine Science, 43(2), 169–176. https://doi.org/https://doi.org/10.1093/icesjms/43.2.169

Ghani, N. H. A., Mohamed Ramli, & W. K., W. S. (2024). A hybrid PRP–dy conjugate gradient with strong Wolfe–powell line search for solving unconstrained optimization problem. Journal of Advanced Research in Applied Sciences and Engineering Technology, 62(2), 27–38. https://doi.org/10.37934/araset.62.2.2738

Gilbert, J. C., & Nocedal, J. (1992). Global convergence properties of conjugate gradient methods for optimization. SIAM Journal on Optimization, 2(1), 21–42. https://doi.org/10.1137/0802003

Goldstein, A. A. (1965). On steepest descent. Journal of the Society for Industrial and Applied Mathematics: Series A, Control, 3(1), 147–151. https://doi.org/10.1137/0303013

Hallal, A., Belloufi, M., & Sellami, B. (2022). An efficient new hybrid CG-method as convex combination of DY and CD and HS algorithms. RAIRO - Operations Research, 56(6), 4047–4056. https://doi.org/10.1051/ro/2022200

Hestenes, M. R., & Stiefel, E. (1952). Methods of conjugate gradients for solving linear systems. Journal of Research of the National Bureau of Standards, 49(6), 409. https://doi.org/10.6028/jres.049.044

Jiang, X., Yang, H., Jian, J., & Wu, X. (2023). Two families of hybrid conjugate gradient methods with restart procedures and their applications. Optimization Methods & Software, 38(5), 947–974. https://doi.org/10.1080/10556788.2023.2189718

Nocedal, J., & Wright, S. (2006). Numerical optimization second edition. https://www.math.uci.edu/~qnie/Publications/NumericalOptimization.pdf

Polyak, B. T. (1969). The conjugate gradient method in extremal problems. USSR Computational Mathematics and Mathematical Physics, 9(4), 94–112. https://doi.org/10.1016/0041-5553(69)90035-4

Wu, X., Zhu, Y., & Yin, J. (2021). A HS-PRP-type hybrid conjugate gradient method with sufficient descent property. Computational Intelligence and Neuroscience, 2021(1). https://doi.org/10.1155/2021/2087438

This is an open access article under the CC BY-SA license (https://creativecommons.org/licenses/by-sa/4.0/).