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Mutually Unbiased Maximally Entangled States for 3-Dimensional Subspace of Hilbert Space

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ABSTRACT

This research involves the construction of Mutually Unbiased Unitary Bases (MUUBs), pertaining to Maximally Entangled States (MES) in a particular subspace of the composite Hilbert space $H_3 \otimes H_3$. These MUUBs are generated through application of generalized Pauli matrices Z and utilized to create the respective generalized Bell states. To verify the entanglement properties of the constructed states, the density operator and partial trace approaches are employed to obtain the reduced density operators of the subsystems. Analysis of the partial trace outcomes helps establish the normality of the reduced states and their proportionality to the identity operator, which confirms the maximal entanglement of the states. This study contributes to the understanding of entanglement structure and MUUB construction in finite-dimensional Hilbert spaces and provides a basis for future developments in higher-dimensional quantum information applications.

Keywords: Entanglement, generalized Bell states, maximally entangled states, mutually unbiased unitary bases, partial trace

INTRODUCTION

Quantum information theory is devoted to the study of how information can be encoded, processed, and transmitted by means of fundamental properties of quantum systems. In this context, the estimation and manipulation of quantum states and unitary transformations are two very important subjects toward an understanding of quantum measurement and computation. One of the central concepts in this context is that of mutually unbiased bases (MUBs)-sets of orthonormal bases where the measurement outcome in one basis provides no information about measurements in another (Nasir et al., 2020). Schwinger introduced the concept of a complete set of unitary operators forming a basis for operators acting on a finite-dimensional Hilbert space. Showed that, for an N-dimensional system, there exist N^2 orthogonal unitary operators, which can form the basis for all possible quantum operators (Schwinger, 1960). These are now known as Schwinger unitary operator bases, built using shift and phase operators:

$$X|k\rangle = |k+1\ (mod\ N)\rangle, Z|k\rangle = \omega k|k\rangle, \omega = e[\frac{2\pi i}{d}] \quad (1)$$

where $k = 0,1,2,3, \ldots d-1$. Then satisfied :

$$ZX = \omega XZ \quad (2)$$

Having been inspired by the previous researcher on the topic of MUBs (Schwinger, 1960), Scott's work in (2008) has developed the notion into mutually unbiased unitary bases (MUUBs), a set of unitary operator basis vectors that exhibit uniform overlap between one another. MUUBs can be seen as an extension of MUBs, where the former deals with quantum operations instead of just quantum states, thereby expanding our knowledge on the properties of unbiasedness among unitary transformations. Such a concept is useful for applications in quantum process tomography and unitary 2-designs for enhancing quantum characterization processes. In a prime-dimensional Hilbert space H_d, a pair of MUBs $B^{(0)} = \{|\varphi_0\rangle, \ldots, |\varphi_{d-1}\rangle\}$, $B^{(1)} = \{|\phi_1\rangle, \ldots, |\phi_{d-2}\rangle\}$ will fulfil the following equation:

$$\left|\langle\varphi_i|\phi_j\rangle\right| = \frac{1}{\sqrt{d}}, \forall_{i,j} \quad (3)$$

extending this to unitary operator spaces, two orthogonal bases composed of unitary operators $B^{(0)} = \left\{B_0^{(0)}, \ldots, B_{n-1}^{(0)}\right\}$ and $A^{(1)} = \left\{B_0^{(1)}, \ldots, B_{n-1}^{(1)}\right\}$are defined as MUUBs if they satisfy

$$|\ \mathrm{Tr}[(B_i^{(0)})^{\dagger} B_j^{(1)}]\ |^2 = C \quad (4)$$

where constant $C \neq 0$

In this context, the unitary operator bases can be represented by a standard set of generalized Pauli-type operators:

$$\{\mathrm{Z}^0, \mathrm{Z}^1, \mathrm{Z}^2 \ldots, \mathrm{Z}^{d-1}\}, \quad (5)$$

where $\mathrm{Z}^0 = \mathrm{Z}^d$ is the identity and the multiplication rule $Z^i Z^j = Z^{i \oplus j}$ holds. These operators form the foundation for constructing MUUBs in a d-dimensional subspace of $M(d, \mathbb{C})$.

MUBs clearly show that it is important when emergence of Quantum Key Distribution in the works of Bennett and Brassard in 1984 , one of the most promising areas of quantum information theory that has attracted significant research efforts both from the theoretical as well as the experimental. This cryptographic method enables two parties to agree upon a secret key shared only between them with the help of a quantum channel (Cerf et al., 2002). Apart from that, MUBs are also important in quantum information applications such as entanglement swapping or another way to say is isomorphism, which relies on a set of maximally entangled states (MES) (Nasir et al., 2020). This may lead to MUBs for MES to Mutually Unbiased Maximally Entangled Bases (MUMEBs) is said to be $d \otimes d'$ $(d' \geq d)$ that can be written

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |q_i\rangle \otimes |q'_i\rangle \quad (6)$$

The equivalent of the relation between unitary operator and MES aiming to construct Mutually Unbiased Unitary Bases of Maximally Entangled States (MUMEBS) within a $3d$ subspace of the

Hilbert space $H_3 \otimes H_3$ by utilizing the generalized Pauli operator Z by corresponding to maximally entangled states. Additionally, the study seeks to determine the amount of entanglement in this subspace to verify whether the constructed states fulfil the conditions of mutual unbiasedness. The findings are expected to enhance understanding of quantum operator spaces and support applications in quantum process tomography and quantum information protocols. In order to achieve the goal of this study, we need to construct Mutually Unbiased Unitary Bases of Maximally Entangled States in subspace of Hilbert Space by using Pauli Matrices Z which corresponding to Maximally Entangled States. Also, we need to ascertain the amount of entanglement for subspace for $H_3 \otimes H_3$.

THE FLOW OF CONSTRUCTING MUMEBS

This study begins by establishing the equivalence between the construction of mutually unbiased unitary bases (MUUBs) and the identification of mutually unbiased bases (MUBs) composed of maximally entangled states (MESs) within the subspace $H_3 \otimes H_3$. Based on this equivalence, we construct mutually unbiased maximally entangled bases (MUMEBs) within the specified subspace using the generalized Pauli operator (XZ). We then quantify the entanglement of the resulting states in the subspace $H_3 \otimes H_3$using appropriate measures from quantum information theory. Subsequently, we determine the conditions under which these states qualify as maximally entangled, ensuring consistency with established definitions of MESs. Finally, we present the principal findings and discuss the implications of the construction and entanglement analysis for understanding entanglement in finite-dimensional Hilbert spaces. In the following are the steps to to construct MUMEBs and ascertain the amount of entanglement for subspace for $H_3 \otimes H_3$

Step 1 : Constructing MUUBs for M_s

The construction below using Pauli operator Z, where the construction about finding the unitary basis for 3-dimensional subspace. In the three-dimensional case, the maximal set of mutually unbiased unitary bases (MUUBs) for the subspace M_s can be explicitly constructed (Nasir et al., 2019). Let the reference basis of the subspace be defined as $\mathbb{Z}^{(0)} = \{\mathbb{I}_d, Z, Z^2\}$, an s-th operator of an r-th basis, Z_s^r , which is respect to $\mathbb{Z}^{(0)}$ is given by

$$Z_s^{(r)} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} (\omega^s)^{d-i} \, (\omega^{-r})^{\alpha(i)} Z^i \tag{7}$$

For $r = 1, \dots, d-1$ and $s = 0, \dots, d-1$ with $\alpha(i) = i + \cdots + (d-1)$, the bases constructed in this way are also pairwise mutually unbiased. These are the constructions for the $3-$dimensional subspace of $M(3, \mathbb{C})$ with the beginning of the basis, $\mathbb{Z}^{(1)} = \{Z_0^1, Z_1^1, Z_2^1\}$, , $\mathbb{Z}^{(2)} = \{Z_0^2, Z_1^2, Z_2^2\}$. Each will have 3 operators which indicated by s in $\mathbb{Z}_s^{(r)}$

Step 2 : Constructing Generalize Bell State

The Choi isomorphism establishes a correspondence between unitary operators acting on a three-dimensional Hilbert space, H_3, and vectors in the composite space $H_3 \otimes H_3$. This approach was used in earlier work to directly construct mutually unbiased bases (MUBs) composed of maximally entangled states (MES) (Nasir et al., 2019)

$$\frac{1}{\sqrt{d}}\left|X_d^b Z_d^a\right\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}\omega^{an}|n\rangle\,|n\oplus b\rangle \tag{8}$$

for $a, b = 0, \ldots, d-1$. Note the subscript d represents the dimensionality of the Hilbert space the operator acts on. The $s-th$ state for $r-th$ basis can then be written as with $r = 1, \ldots, d-1$ and $s = 0, \ldots, d-1$.

$$\left|X_d^b Z_d^a\right\rangle_s^r = \frac{1}{d}\sum_{n=0}^{d-1}(\omega^s)^{d-i}(\omega^{-r})^{\alpha(i)}\left|X_d^b Z_d^a\right\rangle \tag{9}$$

For the standard basis construction consist of , $\mathbb{Z}^{(0)} = \{\mathbb{I}_d, Z, Z^2\}$.

Step 3 : Performing Partial Trace for each of The Bases

According to the Quantum Computation and Quantum Information's book the partial trace is introduced as a mathematical operation used to describe subsystems of a composite quantum system. Nielsen and Chuang (2010) state that if two physical systems A and B are described by a composite density operator ρ^{AB}, then the reduced density operator for subsystem A is defined as

$$\rho^A \equiv Tr_B(\rho^{AB}) \tag{10}$$

where tr_B denotes the partial trace over system B. The reduced density operator is important because it provides a complete description of subsystem A and preserves the correct measurements. The partial trace operation is defined on tensor product operators as

$$Tr_B(|a_1\rangle\langle a_2|\otimes|b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|Tr(|b_1\rangle\langle b_2|) \tag{11}$$

Where $|a_1\rangle$ and $\langle a_2|$ are vectors in the state space of system A, while $|a_1\rangle$ and $\langle b_2|$ belong to system B. Since the ordinary trace satisfies $tr(|\,b_1\rangle\langle b_2\,|) = \langle b_2 \mid b_1\rangle$ the partial trace effectively removes. As an example, consider the subspace of $H_2 \otimes H_2$. The MES in this space can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}|i\rangle_A\otimes|i\rangle_B \tag{12}$$

Then, the example of the construction will be using the basis $|00\rangle + |11\rangle$ and the density matrix will be written as

$$\rho = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$$

$$\rho = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{\sqrt{2}}$$

The partial trace will be defined as

$$Tr_B(\rho) = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$$

$$Tr_B(\rho) = \frac{Tr_B(|00\rangle\langle 00|) + Tr_B(|00\rangle\langle 11|) + Tr_B(|11\rangle\langle 00) + Tr_B(|11\rangle\langle 11|)}{2}$$

$$Tr_B(\rho) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$$

$$Tr_B(\rho) = \frac{\mathbb{I}}{2}$$

RESULT AND DISCUSSION

Recipe for Construction MUUBs for M_s

This construction is referring to step 1 on Equation (7).

When $r = 1$

$$Z_0^{(1)} = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} (\omega^0)^{d-i} (\omega^{-1})^{\alpha(i)} Z^i$$

$$Z_0^1 = \frac{1}{\sqrt{3}} [\mathbb{I} + Z + \omega Z^2] \quad (13)$$

$$Z_1^{(1)} = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} (\omega^1)^{d-i} (\omega^{-1})^{\alpha(i)} Z^i$$

$$Z_1^1 = \frac{1}{\sqrt{3}} [\mathbb{I} + \omega^2 Z + \omega^2 Z^2] \quad (14)$$

$$Z_2^{(1)} = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} (\omega^2)^{d-i} (\omega^{-1})^{\alpha(i)} Z^i$$

$$Z_2^1 = \frac{1}{\sqrt{3}} [\omega^3 + \omega Z + Z^2] \quad (15)$$

When $r = 2$

$$Z_0^{(2)} = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} (\omega^0)^{d-i} (\omega^{-2})^{\alpha(i)} Z^i$$

$$Z_0^2 = \frac{1}{\sqrt{3}} [\mathbb{I} + Z + \omega^2 Z^2] \quad (16)$$

$$Z_1^{(2)} = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} (\omega^1)^{d-i} (\omega^{-2})^{\alpha(i)} Z^i$$

$$Z_1^2 = \frac{1}{\sqrt{3}} [\mathbb{I} + \omega^2 Z + Z^2] \quad (17)$$

$$Z_2^{(2)} = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} (\omega^2)^{d-i} (\omega^{-2})^{\alpha(i)} Z^i$$

$$Z_2^2 = \frac{1}{\sqrt{3}} [\omega^3 + \omega Z + \omega Z^2] \quad (18)$$

Constructing generalized Bell states

This construction is referring to Step 2.

Standard Basis

The solution below finds the generalize Bell state for the standard basis using the Equation (8).

When $a = 0, b = 0$

$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^0\right\rangle = \frac{1}{\sqrt{3}}\sum_{n=0}^{d-1}\omega^{(0)n}|n\rangle\,|n\oplus 0\rangle$$
$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^0\right\rangle = \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] \quad (19)$$

When $a = 1, b = 0$

$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^1\right\rangle = \frac{1}{\sqrt{3}}\sum_{n=0}^{d-1}\omega^{(1)n}|n\rangle\,|n\oplus 0\rangle$$
$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^1\right\rangle = \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \quad (20)$$

When $a = 2, b = 0$

$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^2\right\rangle = \frac{1}{\sqrt{3}}\sum_{n=0}^{d-1}\omega^{(2)n}|n\rangle\,|n\oplus 0\rangle$$
$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^2\right\rangle = \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle]$$
$$\frac{1}{\sqrt{3}}\left|X_3^0 Z_3^2\right\rangle = \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega|2\rangle|2\rangle] \quad (21)$$

The Remaining Basis

The $\boldsymbol{s}$ −th state for the $\boldsymbol{r}$ −th basis can be written as the equation state on Equation (9).

When $r = 1, s = 0$

$$\left|X_3^0 Z_3^0\right\rangle_0^1 = \frac{1}{3}\sum_{=0}^{d-1}(\omega^0)^{3-i}(\omega^{-1})^{\alpha(i)}\left|X_3^0 Z_3^1\right\rangle$$
$$\left|X_3^0 Z_3^0\right\rangle_0^1 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] + \omega[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])$$
$$\left|X_3^0 Z_3^0\right\rangle_0^1 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^3|1\rangle|1\rangle + \omega^5|2\rangle|2\rangle])$$
$$\left|X_3^0 Z_3^0\right\rangle_0^1 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + |1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]) \quad (22)$$

When $r = 1, s = 1$

$$\left|X_3^0 Z_3^1\right\rangle_1^1 = \frac{1}{3}\sum_{=0}^{d-1}(\omega^1)^{3-i}(\omega^{-1})^{\alpha(i)}\left|X_3^0 Z_3^1\right\rangle$$
$$\left|X_3^0 Z_3^1\right\rangle_1^1 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + \omega^2[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] + \omega^2[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])$$
$$\left|X_3^0 Z_3^1\right\rangle_1^1 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega^2|0\rangle|0\rangle + \omega^3|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle] + [\omega^2|0\rangle|0\rangle + \omega^4|1\rangle|1\rangle + \omega^6|2\rangle|2\rangle])$$
$$\left|X_3^0 Z_3^1\right\rangle_1^1 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega^2|0\rangle|0\rangle + |1\rangle|1\rangle + \omega|2\rangle|2\rangle] + [\omega^2|0\rangle|0\rangle + \omega|1\rangle|1\rangle + |2\rangle|2\rangle]) \quad (23)$$

When $r = 1, s = 2$

$$\left|X_3^0 Z_3^2\right\rangle_1^1 = \frac{1}{3}\sum_{=0}^{d-1}(\omega^2)^{3-i}(\omega^{-1})^{\alpha(i)}\left|X_3^0 Z_3^1\right\rangle$$

$$\begin{aligned}\left|X_3^0 Z_3^2\right\rangle_2^1 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + \omega[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + [|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^2\right\rangle_2^1 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^3|2\rangle|2\rangle] \\ &\quad + [|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^2\right\rangle_2^1 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + |2\rangle|2\rangle] \\ &\quad + [|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega|2\rangle|2\rangle])\end{aligned} \tag{24}$$

When $r = 2, s = 0$

$$\left|X_3^0 Z_3^0\right\rangle_0^2 = \frac{1}{3}\sum_{=0}^{d-1}(\omega^0)^{3-i}(\omega^{-2})^{\alpha(i)}\left|X_3^0 Z_3^1\right\rangle$$

$$\begin{aligned}\left|X_3^0 Z_3^0\right\rangle_0^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + \omega^2[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^0\right\rangle_0^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + [\omega^2|0\rangle|0\rangle + \omega^4|1\rangle|1\rangle + \omega^6|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^0\right\rangle_0^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + [\omega^2|0\rangle|0\rangle + \omega|1\rangle|1\rangle + |2\rangle|2\rangle])\end{aligned} \tag{25}$$

When $r = 2, s = 1$

$$\left|X_3^0 Z_3^1\right\rangle_1^2 = \frac{1}{3}\sum_{=0}^{d-1}(\omega^1)^{3-i}(\omega^{-2})^{\alpha(i)}\left|X_3^0 Z_3^1\right\rangle$$

$$\begin{aligned}\left|X_3^0 Z_3^1\right\rangle_1^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + \omega^2[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + [|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^1\right\rangle_1^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega^2|0\rangle|0\rangle + \omega^3|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle] \\ &\quad + [|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^1\right\rangle_1^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + |1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle])\end{aligned} \tag{26}$$

When $r = 2, s = 2$

$$\left|X_3^0 Z_3^2\right\rangle_2^2 = \frac{1}{3}\sum_{=0}^{d-1}(\omega^0)^{3-i}(\omega^{-2})^{\alpha(i)}\left|X_3^0 Z_3^1\right\rangle$$

$$\begin{aligned}\left|X_3^0 Z_3^2\right\rangle_2^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + \omega[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\quad + \omega[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^4|2\rangle|2\rangle])\\ \left|X_3^0 Z_3^2\right\rangle_2^2 &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega^3|2\rangle|2\rangle] \\ &\quad + [\omega|0\rangle|0\rangle + \omega^3|1\rangle|1\rangle + \omega^5|2\rangle|2\rangle])\end{aligned}$$

$$\left|X_3^0 Z_3^2\right\rangle_2^2 = \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + |1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]) \quad (27)$$

Performing Partial Trace for each of the bases

The following partial trace are referring to Step 3.

Partial Trace for Standard Basis

The solution below finds the Partial Trace state for the standard basis.

$$\begin{aligned} &Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^0\right\rangle_{00}^{00}\langle X_3^0 Z_3^0|\right) : \\ &= \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] \cdot \frac{1}{\sqrt{3}}[\langle 0|\langle 0| + \langle 1|\langle 1| + \langle 2|\langle 2|] \\ &= \frac{1}{\sqrt{3}}[|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|] \\ &= \frac{\mathbb{I}}{3} \end{aligned} \quad (28)$$

$$\begin{aligned} &Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^1\right\rangle_{11}^{00}\langle X_3^0 Z_3^1|\right) : \\ &= \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \cdot \frac{1}{\sqrt{3}}[\langle 0|\langle 0| + \omega^2\langle 1|\langle 1| + \omega\langle 2|\langle 2|] \\ &= \frac{1}{\sqrt{3}}[|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|] \\ &= \frac{\mathbb{I}}{3} \end{aligned} \quad (29)$$

$$\begin{aligned} &Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^2\right\rangle_{22}^{00}\langle X_3^0 Z_3^2|\right) : \\ &= \frac{1}{\sqrt{3}}[|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega|2\rangle|2\rangle] \cdot \frac{1}{\sqrt{3}}[\langle 0|\langle 0| + \omega\langle 1|\langle 1| + \omega^2\langle 2|\langle 2|] \\ &= \frac{1}{\sqrt{3}}[|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|] \\ &= \frac{\mathbb{I}}{3} \end{aligned} \quad (30)$$

Partial Trace for the s −th state for the r −th Basis

The solution below finds the Partial Trace state for the remaining basis.

When $r = 1, s = 0$

$$\begin{aligned} &Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^0\right\rangle_{00}^{11}\langle X_3^0 Z_3^0|\right) \\ &= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle] \\ &\qquad + [\omega|0\rangle|0\rangle + |1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]) \cdot \frac{1}{3}([\langle 0|\langle 0| + \langle 1|\langle 1| \\ &\qquad + \langle 2|\langle 2|] + [\langle 0|\langle 0| + \omega^2\langle 1|\langle 1| + \omega\langle 2|\langle 2|] + [\omega^2\langle 0|\langle 0| + \langle 1|\langle 1| + \omega\langle 2|\langle 2|]) \\ &= \frac{\mathbb{I}}{3} \end{aligned} \quad (31)$$

When $r = 1, s = 1$

$$Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^1\right\rangle_{11}^{11}\left\langle X_3^0 Z_3^1\right|\right)$$
$$= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega^2|0\rangle|0\rangle + |1\rangle|1\rangle + \omega|2\rangle|2\rangle]$$
$$+ [\omega^2|0\rangle|0\rangle + \omega|1\rangle|1\rangle + |2\rangle|2\rangle]) \cdot \frac{1}{3}([\langle 0|\langle 0| + \langle 1|\langle 1|$$
$$+ \langle 2|\langle 2|] + [\omega\langle 0|\langle 0| + \langle 1|\langle 1| + \omega^2\langle 2|\langle 2|] + [\omega\langle 0|\langle 0| + \omega^2\langle 1|\langle 1| + \langle 2|\langle 2|])$$
$$= \frac{\mathbb{I}}{3} \tag{32}$$

When $r = 1, s = 2$

$$Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^2\right\rangle_{22}^{11}\left\langle X_3^0 Z_3^2\right|\right)$$
$$= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + |2\rangle|2\rangle]$$
$$+ [|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + \omega|2\rangle|2\rangle]) \cdot \frac{1}{3}([\langle 0|\langle 0| + \langle 1|\langle 1|$$
$$+ \langle 2|\langle 2|] + [\omega^2\langle 0|\langle 0| + \omega\langle 1|\langle 1| + \langle 2|\langle 2|] + [\langle 0|\langle 0| + \omega\langle 1|\langle 1| + \omega^2\langle 2|\langle 2|])$$
$$= \frac{\mathbb{I}}{3} \tag{33}$$

When $r = 2, s = 0$

$$Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^0\right\rangle_{00}^{22}\left\langle X_3^0 Z_3^0\right|\right)$$
$$= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]$$
$$+ [\omega^2|0\rangle|0\rangle + \omega|1\rangle|1\rangle + |2\rangle|2\rangle]) \cdot \frac{1}{3}([\langle 0|\langle 0| + \langle 1|\langle 1|$$
$$+ \langle 2|\langle 2|] + [\langle 0|\langle 0| + \omega^2\langle 1|\langle 1| + \omega\langle 2|\langle 2|] + [\omega\langle 0|\langle 0| + \omega^2\langle 1|\langle 1| + \langle 2|\langle 2|])$$
$$= \frac{\mathbb{I}}{3} \tag{34}$$

When $r = 2, s = 1$

$$Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^1\right\rangle_{11}^{22}\left\langle X_3^0 Z_3^1\right|\right)$$
$$= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + |1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]$$
$$+ [|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]) \cdot \frac{1}{3}([\langle 0|\langle 0| + \langle 1|\langle 1|$$
$$+ \langle 2|\langle 2|] + [\omega^2\langle 0|\langle 0| + \langle 1|\langle 1| + \omega\langle 2|\langle 2|] + [\langle 0|\langle 0| + \omega^2\langle 1|\langle 1| + \omega\langle 2|\langle 2|])$$
$$= \frac{\mathbb{I}}{3} \tag{35}$$

When $r = 2, s = 2$

$$Tr_{\boldsymbol{B}}\left(\left|X_3^0 Z_3^2\right\rangle_{22}^{22}\left\langle X_3^0 Z_3^2\right|\right)$$
$$= \frac{1}{3}([|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle] + [\omega|0\rangle|0\rangle + \omega^2|1\rangle|1\rangle + |2\rangle|2\rangle]$$
$$+ [\omega|0\rangle|0\rangle + |1\rangle|1\rangle + \omega^2|2\rangle|2\rangle]) \cdot \frac{1}{3}([\langle 0|\langle 0| + \langle 1|\langle 1|$$
$$+ \langle 2|\langle 2|] + [\omega^2\langle 0|\langle 0| + \omega\langle 1|\langle 1| + \langle 2|\langle 2|] + [\omega^2\langle 0|\langle 0| + \langle 1|\langle 1| + \omega\langle 2|\langle 2|])$$
$$= \frac{\mathbb{I}}{3} \tag{36}$$

CONCLUSION AND RECOMMENDATION

This study investigated the construction of mutually unbiased unitary bases (MUUBs) corresponding to maximally entangled states (MESs) within the three-dimensional composite Hilbert space $H_3 \otimes H_3$. The MUUBs were constructed in a three-dimensional operator subspace using the generalized Pauli operator Z. Through the Choi isomorphism, the resulting unitary operators were mapped to generalized Bell states in $H_3 \otimes H_3$, thereby establishing the correspondence between MUUBs in the operator space and mutually unbiased maximally entangled bases in the associated state space.

The entanglement properties of the constructed states were examined using density operators and partial traces. For each constructed state, the reduced density operator was found to be proportional to the identity operator and to satisfy the required normalization condition. Specifically, the reduced density operators took the maximally mixed form $\frac{\mathbb{I}}{3}$, confirming that the corresponding bipartite states are maximally entangled. These results demonstrate that the generalized Pauli operator Z can be effectively employed to construct MUUBs that correspond to maximally entangled states in a finite-dimensional composite Hilbert space. The findings also strengthen the connection between mutually unbiased structures in operator spaces and maximally entangled structures in state spaces.

This study was restricted to a three-dimensional operator subspace and the corresponding composite Hilbert space $H_3 \otimes H_3$. Future research may extend the construction to higher-dimensional systems, particularly dimensions $d > 3$, to determine whether the properties of mutual unbiasedness and maximal entanglement are preserved. Further investigations may also consider the entire operator space rather than a selected subspace, especially when the dimension is a prime or prime power, where richer algebraic structures may arise. In addition, the operational relevance of the constructed MUUBs and maximally entangled states could be examined in applications such as quantum state estimation, quantum process tomography and quantum key distribution. Their robustness under noise and other experimental imperfections should also be investigated to assess their suitability for practical quantum-information protocols.

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Conflict of Interest

The authors have no conflicts of interest to declare.

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