

UNIVERSITI TEKNOLOGI MARA

**A MODIFIED TOPSIS AND
GENERALIZED HAMACHER
HERONIAN MEAN AGGREGATION
OPERATOR INCORPORATING
BIPOLAR PYTHAGOREAN
NEUTROSOPHIC SETS AND THEIR
APPLICATIONS**

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MSc

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ABSTRACT

Amongst complex and data-driven systems, Multi-Criteria Decision-Making (MCDM) models like Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) have several limitations, including a heavy reliance on distance-based calculations and conventional aggregation procedures, which may not adequately capture complex relationships among criteria or effectively represent uncertainty, bipolarity, and indeterminacy in expert evaluations. To address these limitations, this study extends the TOPSIS framework by proposing the Adaptive Ranking with Ideal Evaluation (ARIE) method, which retains the ideal-solution concept but introduces a new evaluation and ranking mechanism. To enhance its ability to handle uncertainty, Bipolar Pythagorean Neutrosophic Generalized Hamacher Heronian Mean (BPN-GHHM) and its weighted extension, BPN-GWHHM operators are introduced. These operators are supported by two BPNS linguistic scales to represent positive, negative and indeterminate judgments. The proposed frameworks are applied to evaluate English vocabulary learning strategies in higher education. Results show that gamification-based learning strategies emerge as the most effective approach for improving English language learning outcomes. Thus, the proposed framework produces consistent results and effectively handles uncertainty and complex relationships among evaluation criteria in educational decision-making contexts. Future studies are recommended to apply the framework to larger datasets and other domains.

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LIST OF SYMBOLS

Symbols

μ	Membership Degree
η	Indeterminacy Degree
ν	Non-Membership Degree
Z	Universe of Discourse (Set of All Possible Elements)
z	An Individual Element from the Universe
D	A Set Defined on Z
$\mu_D(z)$	Membership Degree of Element z in Set D
$\eta_D(z)$	Indeterminacy Degree of Element z in Set D
$\nu_D(z)$	Non-Membership Degree of Element z in Set D
$\mu_D^+(z)$	Positive Truth Membership Degree of Element z in Set D
$\eta_D^+(z)$	Positive Indeterminacy Membership Degree of Element z in Set D
$\nu_D^+(z)$	Positive Falsity Membership Degree of Element z in Set D
$\mu_D^-(z)$	Negative Truth Membership Degree of Element z in Set D
$\eta_D^-(z)$	Negative Indeterminacy Membership Degree of Element z in Set D
$\nu_D^-(z)$	Negative Falsity Membership Degree of Element z in Set D
X	Decision Matrix
x_{ij}	Performance Value of Alternative i Against Criteria j
i	Number of Alternatives

j	Number of Criteria
m	Total Number of Alternatives
n	Total Number of Criteria Used to Evaluate the Alternative
n_{ij}	Normalized Value of Alternative i Against Criteria j
$x_j^{\max}$	Maximum Value of Criterion j
$x_j^{\min}$	Minimum Value of Criterion j
x_j^T	Target Value of Criterion j
w_j	Weight of Criterion j
v_{ij}	Weighted Normalized Value of Alternative i Against Criteria j in ARIE Method
$v_j^{\max}$	Maximum Value of Weighted Normalized in ARIE Method
$v_k^{\min}$	Minimum Value of Weighted Normalized in ARIE Method
Sim_i^{best}	Ideal Solution in ARIE Method
Sim_i^{worst}	Anti-Ideal Solution in ARIE Method
γ	Parameter Value in ARIE Method
RC_i	Relative Closeness in ARIE Method
κ	Parameter Value in ARIE Method
T	t-norm Structure in Hamacher Operator
T^*	t-conorm Structure in Hamacher Operator
a_i	Membership Degree for An Element i in Hamacher Operator
a_j	Membership Degree for An Element j in Hamacher Operator
ζ	Scalar in Hamacher Operator

x_i	i -th Element in Non-Negative Real Number Set in Heronian Mean Operator
p	Non-Negative Parameter in Heronian Mean Operator
q	Non-Negative Parameter in Heronian Mean Operator
n	Total Number of Elements in a Set in Heronian Mean Operator
X	Linguistic Variable Name
$T(X)$	Set of Linguistic Variables in Terms of X
U	Universe of Discourse
G	Synthetic Rule of Generating Linguistic Term
M	Semantic rule

LIST OF ABBREVIATIONS

Abbreviations

AHP	Analytic Hierarchy Process
AHP – express	Analytic Hierarchy Process - express
AO	Aggregation Operator
ARAS	Additive Ratio Assessment System
ARIE	Adaptive Ranking with Ideal Evaluation
AROMAN	Alternative Ranking Order Method Accounting for Two-Step Normalization
BFS	Bipolar Fuzzy Set
BM	Bonferroni Mean
BNS	Bipolar Neutrosophic Set
BPN-GHHM	Bipolar Pythagorean Neutrosophic-Generalized Hamacher Heronian Mean
BPN-GWHHM	Bipolar Pythagorean Neutrosophic-Generalized Weighted Hamacher Heronian Mean
BPFS	Bipolar Pythagorean Fuzzy Set
BNNN	Bipolar Pythagorean Neutrosophic Number
BPNS	Bipolar Pythagorean Neutrosophic Set
BWM	Best Worst Method
CoCoSo	COmbined COmpromise SOlution
CODAS	COmbinative Distance-based ASsessment
COPRAS	COmplex PROportional ASsessment

CRADIS	Compromise Ranking of Alternatives from Distance to Ideal Solution
CRITIC	CRiteria Importance Through Inter-criteria Correlation
CRITIC-AROMAN	CRiteria Importance Through Inter-criteria Correlation - Alternative Ranking Order Method Accounting for Two-Step Normalization
DEA	Data Envelopment Analysis
DM	Decision-Making
ELECTRE	ELimination Et Choix Traduisant la REalite
FB-ITS	Fuzzy Bayesian – Intelligent Tutoring System
FD	Fuzzy Decision
FF	Fermatean Fuzzy
FS	Fuzzy Set
FT	Fuzzy TOPSIS
GHM	Generalized Heronian Mean
GIS	Geographic Information Systems
GWHM	Generalized Weighted Heronian Mean
HM	Heronian Mean
HSS	Hyper Soft Set
IF-AHP	Intuitionistic Fuzzy – Analytic Hierarchy Process
IFS	Intuitionistic Fuzzy Set
InnoRank-ARIE	Innovative Ranking – Adaptive Ranking with Ideal Evaluation
INS	Interval Neutrosophic Set
IVNS	Interval-Valued Neutrosophic Set
LNIS	Larger Neutrosophic Ideal Solution

MABAC	Multi-Attributive Border Approximation Area Comparison
MADM	Multi-Attribute Decision-Making
MARCOS	Measurement Alternatives and Ranking according to the COnpromise Solution
MCDM	Multi-Criteria Decision-Making
MCGDM	Multi-Criteria Group Decision-Making
MACONT	Mixed Aggregation by COnprehensive Normalization Technique
MEREC	Method based on the Removal Effects of Criteria
MODM	Multi-Objective Decision-Making
MOORA	Multi-Objective Optimization on the basis of Ratio Analysis
MULTIMOORA	Multi-Objective Optimization by RAtio analysis plus the full multiplicative form
NIS	Negative Ideal Solution
NS	Neutrosophic Set
OWA	Ordered Weighted Average
PARIS	Preference Analysis for Reference Ideal Solution
PFS	Pythagorean Fuzzy Set
PIS	Positive Ideal Solution
PIV	Proximity Indexed Value
PNS	Pythagorean Neutrosophic Set
PPBFSS	Possibility Pythagorean Bipolar Fuzzy Soft Set
PROMETHEE	Preference Ranking Organization METHod for Enrichment Evaluations
PSI	Preference Selection Index
PTS	Progression Teaching System

RC	Relative Closeness
ROV	Range Of Values
SAW	Simple Additive Weighting
SNIS	Smaller Neutrosophic Ideal Solution
SNS	Simplified Neutrosophic Set
SME	Small- and Medium-sized Enterprise
SVN-PIS	Single-Valued Neutrosophic – Negative Ideal Solution
SVN-NIS	Single-Valued Neutrosophic – Positive Ideal Solution
SVNS	Single-Valued Neutrosophic Set
SWARA	Stepwise Weight Assessment Ratio Analysis
TOPSIS	Technique for Order of Preference by Similarity to Ideal Solution
WASPAS	Weighted Aggregated Sum Product ASsessment
WPM	Weighted Product Model
WSM	Weighted Sum Model
VIKOR	Vise Kriterijumska Optimizacija I Kompromisno Resenje

CHAPTER 1

INTRODUCTION

1.1 Research Background

Decision-Making (DM) is defined as the process of selecting the most appropriate solution from a set of alternatives based on multiple criteria. The concept arose in the mid-1950s in response to the growing need for systematic approach to identify optimal solutions. Multi-Criteria Decision-Making (MCDM) developed as a rigorous mathematical framework, theoretical model, and conceptual tool aimed at helping decision-makers in finding the best suitable solution from among multiple alternatives in contemporary decision science applications. MCDM methodologies serve to benchmark, rank, and prioritize available options, enabling decision-makers with insights based on the performance evaluations (Cui et al., 2024). Some of the most notable models of MCDM are Simple Additive Weighting (SAW), Vise Kriterijumska Optimizacija I Kompromisno Resenje (VIKOR), and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS). The rapid development of MCDM methodologies in the early 1970s was driven by the growing demand for methods capable of addressing complex MCDM issues. However, decision-makers are typically face various challenges in real-world applications, scientists, researchers, and scholars have generally acknowledged the quick evolution of advanced MCDM methods. Major issues faced by decision-makers are incomplete information and uncertainty, which caused by confusion due to the inaccuracies and unclear data (Zakeri et al., 2025), leading to difficulties in expecting reliable outcomes.

Uncertainty in the MCDM process occur due to the lack of full knowledge or information on assessing numerous factors that hugely affect the quality and reliability of decision outcomes (Li & Xu, 2024). Various sources and information may lead to the uncertainty, where they can be the data uncertainty, model uncertainty, decision-maker uncertainty, and environmental uncertainty. To begin with, the data uncertainty which defined as the uncertainties concerned with the input data with, which the decision is made may include measurement errors, missing data, and inconsistency. The uncertainty can influence the input parameter of criteria and alternative evaluation. Second, model uncertainty is associated with the complex mathematical or

computational algorithms in MCDM models, assumptions in MCDM models, choice of model parameters or model faithfulness to real life situations. Next, uncertainty in decision-makers refers to the subjective decisions and preferences of decision-makers, which may be different because of diverse risk attitude or incompatible priorities. Also, the external factors, including policy changes and technological advancements bring uncertainty in the DM process and may affect the results or implications of decision alternatives. In the MCDM environment, uncertainty may result in suboptimality or may put individuals into more risks when it is not effectively dealt with in the process. This can undermine the integrity of decision results and undermine trust on the DM processes by the stakeholders. Thus, uncertainty may lead to information delays or indecision because decision-makers may not know what to do with ambiguous or conflicting information.

The challenges of uncertainty in MCDM have been effectively addressed through Fuzzy Set (FS) theory, introduced by Zadeh (1965), and Intuitionistic Fuzzy Set (IFS) theory, developed by Atanassov (1983). FS and IFS established the foundations for handling uncertainty through the concepts of membership, μ , and non-membership, ν , respectively. Building on these concepts, Smarandache (1998) proposed the Neutrosophic Set (NS), which incorporated an additional indeterminacy membership, η , thereby significantly enhancing the representation of ambiguous and imprecise information. However, due to the lack of bipolarity representation in NS, Bipolar Neutrosophic Set (BNS) was later developed to simultaneously evaluate both positive and negative perspectives. Moreover, Pythagorean structures were developed in both fuzzy and neutrosophic domains to offer a better balance between logical consistency and representational constraints (Zhang & Chen, 2024). A more recent advancement is the Bipolar Pythagorean Neutrosophic Set (BPNS) that allows a more sophisticated model that capable of handling higher degrees of uncertainty and scalability (Ahmad et al., 2024). BPNS extends both BNS and Pythagorean Neutrosophic Set (PNS) to represent uncertainty more comprehensively by integrating positive and negative aspects with Pythagorean structures. However, when multiple complex criteria are involved, BPNS alone remains insufficient for effective aggregation of values. Thus, combining BPNS with Aggregation Operators (AOs) such as Hamacher and Heronian Mean (HM) operators are useful to data-driven DM in MCDM environment.

The divergent criteria could be converted into an integrated and coherent evaluation techniques by integrating BPNS with Hamacher and HM operators. Among them, the Hamacher operator is known to balance the interdependence and complementarity between decision attributes (Hanif & Yaqoob, 2025). Furthermore, the Hamacher operator is flexible due to its parameterised structure, allowing decision makers to consider the degree of compensation among criteria depending on the actual decision needs. It has been successfully applied in various fuzzy environments such as NS and q-Rung Orthopair FS. Similarly, the HM operator effectively measures the correlations among input factors to solve the limitations of the Bonferroni Mean (BM). Its generalized versions, including Generalized Heronian Mean (GHM) and the Generalized Weighted Heronian Mean (GWHM), also enhance the flexibility by allowing prioritization of individual inputs (Fan et al., 2022). These operators can be very useful in cases where there are interactions between attributes, to capture interactions and thereby provide more realistic aggregations. In addition, the use of weighting mechanisms allows the decision-maker to account for the relative importance of the criteria, thus enhancing the representativeness and reliability of the final assessment. These AOs have been widely applied in many areas, such as cybersecurity, healthcare, and transportation, where they have proven vital in improving DM accuracy and reliability.

In the context of educational research, and specifically language learning research, decision-makers often confronted complex challenges in selecting an appropriate teaching technique, measuring the learning success, and testing the learning instruments in the case of incomplete information and subjectivity. Among these challenges, learning English vocabulary remains one of the most critical and corresponding obstacles faced by second language learner' motivations, learning environment, and contexts remains a complex evaluation. Many factors are involved in effective vocabulary learning, that might influence each other, such as learners' motivation, cognitive ability, learning style, level of proficiency, pedagogy, and environment. Therefore, the assessment and ranking of strategies for vocabulary learning should not only be based on quantitative data, but also on qualitative judgments and expert opinions, which are often unclear and approximate. Furthermore, different experts can have varying opinions about the effectiveness of a specific strategy, adding another layer of complexity to the evaluation. Systematic and data-driven models are needed to support reliable DM process in educational sector.

1.2 Motivation for This Work

Ranking alternatives has been shown to be effective for many widely used MCDM methods like TOPSIS, VIKOR and SAW. The impact of aggregation parameters on stability, robustness and decision for ranking however, is not sufficiently studied. Sensitivity parametric have been incorporated in different fuzzy and neutrosophic context but there are only a few MCDM approaches related to parameter variations in the calculation procedure. Therefore, different parameter values offer expressiveness opinion by decision-makers in the rankings obtained. Besides, BPNSs are powerful tools for representing uncertainty, indeterminacy and bipolar information, but the combination of BPNSs with Hamacher and HM operators underexplored. As such, there is a missing aggregation mechanism that can integrate bipolar judgments, interdependent criteria and flexible aggregation behaviours in a single model.

In addition, the use of the MCDM approaches and advanced BPNS models for MCDM in field of educational DM is still limited. Indeed, the assessment of English vocabulary learning strategies has been relatively neglected despite its significance in the field of second language acquisition. Given that such assessments frequently rely on subjectivity from experts, linguistic evaluation and numerous conflicting criteria, there is a need for a comprehensive decision support framework that can effectively cope with these challenges. In view of these deficiencies, this study aims to present a new MCDM approach, integration of BPNS with Hamacher HM operators and apply them in the evaluation of English vocabulary learning strategies.

1.3 Problem Statement

TOPSIS is one of the most popular classical MCDM methods because of its simplicity and capability of ranking the set of alternatives based on their distances from the Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS). There are, however, some methodological issues that found as a problem. In the conventional TOPSIS model, the distance-based measures, such as Euclidean distance, are used to measure the proximity of the alternatives to the ideal solutions. This method assumes that criteria are independent of each other and it does not necessarily reflect the relationships and interactions between the evaluation criteria, which means that other alternatives with significantly different performance characteristics can result in the

same distance, which can make it difficult to get a consistent ranking. In addition, the aggregation used in TOPSIS uses weighted averaging methods which enable a low performance in one criterion to be offset by a high performance in another criterion. This compensatory behaviour can obscure significant limitations and diminish discriminative power of the ranking process. In addition, recent research has found that distance to PIS, distance to NIS and Relative Closeness (RC) measures are not always consistent, which suggests that the traditional TOPSIS approach does not yield consistent and reliable decision results in complex MCDM processes. These limitations establish the need for a new ranking structure and motivate the proposed ARIE method, which replaces the distance-based RC of TOPSIS with a target-based normalization, a similarity measure, and dual adjustable sensitivity parameters to restore ranking consistency.

The conventional TOPSIS has been extended by researchers to overcome the disadvantages by introducing uncertainty, incompleteness and inconsistency in MCDM models through FS and NS theory. However, there are still some problems confronted by the existing AOs. Most AOs merge all the evaluation information into a single aggregated evaluation, but this may not be sufficient to capture the various opinions and conflicting evaluations from decision-makers. Under one criterion, an alternative can have both desirable and undesirable attributes in the real world of MCDM. For example, an alternative might be effective, but it may also require significant amount of effort, cost, or resources. Conventional AOs, however, usually present such assessments in a single assessment scheme, which makes it hard to explicitly differentiate positive perceptions from negative perceptions. This means that valuable evaluative information can be lost during the aggregation process and so the results of the ranking can be inaccurate and difficult to interpret. Most of the current AOs assume that the criterion weights are fixed and that there are only few interactions among criteria, which may not capture all aspects of decision problems in the real world. These restrictions show a need for a more flexible aggregate framework that can capture opposing evaluations, a more complete modelling of uncertainty, and interrelationships between criteria.

The problem of effective English vocabulary learning strategies is a complex MCDM problem with several stakeholders, subjective evaluation, and several conflicting criteria. A single measure of performance is insufficient to evaluate the effectiveness of a specific strategy because of many factors such as learner motivation, learner retention ability, cognitive load, level of engagement, implementation ease and

long-term learning outcomes. Moreover, the learning environment, the experiences of educational experts, and personal preferences of learners can create situations in which education assessments are not certain, complete and consistent. A vocabulary learning strategy can have positive and negative attributes in many situations, such as a strategy could improve retention of words while it takes a lot of time and mental effort from learners. This bipolar assessment is hard to depict with traditional MCDM method and AOs mechanisms. Conventional MCDM approach and the existing AOs structure are hard to depict this bipolar assessment. Furthermore, the evaluation criteria are not necessarily independent, where the improvement of one criterion may depend on the performance of another. These complexities pose great difficulties in consolidating decision information and producing dependable rankings of alternative vocabulary learning strategies. Thus, an advanced DM framework that can simultaneously cope with uncertainty, incompleteness, bipolarity and criterion interrelationships to facilitate systematic evaluation and selection of English vocabulary learning strategies is needed.

1.4 Research Objectives

The objectives of this study are as follows:

- a) To formulate the Adaptive Ranking with Ideal Evaluation (ARIE) method as an advanced MCDM framework that improves the robustness and consistency of the foundational principles of TOPSIS method.
- b) To formulate aggregation approaches by integrating the Bipolar Pythagorean Neutrosophic Set with the Hamacher and Heronian Mean operators (BPN-GHHM) and the weighted BPN-GWHHM operators.
- c) To apply the BPN-GHHM operator for aggregating multiple decision matrices, followed by ARIE framework to rank English vocabulary learning strategies.

1.5 Research Question

- i) What methodologies and principles in TOPSIS that can be employed to design ARIE method for effectively addressing complex MCDM problems?

- ii) How can the integration of BPN-GHHM and BPN-GWHHM operators improve the representation and aggregation of complex decision criteria under uncertainty within the MCDM framework?
- iii) How can be the ARIE method, in combination with the BPN-GHHM operator, be effectively applied and validated to enhance MCDM processes in evaluating English vocabulary learning strategies?

1.6 Significance of Study

A major problem face in MCDM is uncertainty, especially in the education sector, where human opinion and subjective evaluations play a significant role. Most of the factors in the application of English vocabulary learning strategies involves ambiguous or imprecise information, such as learner motivation, cognitive style, learning environment, and resource availability. These uncertainties complicate the process to accurately measuring and ranking the alternative strategies using traditional evaluation methods. To overcome these weaknesses, this study introduces the ARIE method that offer a more stable, robust and flexible approach as compared to TOPSIS.

ARIE method adds a new type of criterion that effectively treat both quantitative and qualitative data in a nuanced way that make it more responsive to diverse decision environments. By allowing the decision-makers to express nuanced evaluations, the new criteria type in the ARIE method contributes to greater precision and responsiveness in ranking outcomes. Additionally, the introduction of dual sensitivity parameter offers ARIE method with dynamic ability to adapt to variants data and weight distribution that significantly reducing the impact of outliers and ensuring consistent ranking stability. These innovative structures allow the ARIE method to achieve good adaptability and precision in comparison with TOPSIS method. ARIE method offers a more balanced benchmark over classical static reference points by incorporating dual adjustable sensitivity parameters. Its dynamic features ensure that proper evaluations, accurate rankings, and stability within a broad range of complex decision processing.

In the proposed framework, based on the recent findings that emphasize the necessity to model the interrelations between criteria values (Akram et al., 2023; Mahmood et al., 2023), this study introduces the BPN-GHHM and BPN-GWHHM operators as novel aggregation tool for handling bipolar Pythagorean neutrosophic data

in MCDM contexts. The proposed operators improve flexibility and responsiveness of single input assessments that may result in increased stability and accuracy in rankings. Supported by recent advancements in the field (Hanif & Yaqoob, 2025; Sajid et al., 2024), the new operators contribute to a more robust and accurate MCDM structure capable of addressing uncertainty and bipolarity in complex DM problems. This study utilizes the linguistic variables in the form of BPNS into consolidated neutrosophic values. To make these aggregated BPNS elements comparable in terms of their values, a score function is then applied to convert them into similar scalar values, which are the input of the ARIE method to rank the English vocabulary learning strategies. This chronological process is important because as it secures uncertainties, ambiguity, and bipolar information are effectively managed prior to ranking outcomes.

To address these difficulties, this study proposes the ARIE method to assess and find the most efficient methods of learning vocabulary. This is further supported by the BPN–GHHM operator integration to successfully handle uncertainty and bipolarity in the linguistic and educational DM environments. By improving the accuracy and consistency of data aggregation, the BPN-GHHM operator complements the ARIE method, which delivers adaptive and robust rankings of alternatives. By fusing advanced AOs with the ARIE method, this study offers a scalable, adaptable, and reliable MCDM model that will guarantee better accuracy and reliability in any high stake's scenario. Both models are adaptable for real-world applications.

1.7 Ethical Committee

In this study, human participants provide expert evaluations on English vocabulary learning strategies. This study has received ethical approval from the Universiti Teknologi MARA (UiTM) Research Ethics Committee, and all research procedures will be carried out under the ethical guidelines. Before the data collection process, experts will be informed regarding the purposes of the study, its procedures, and voluntary organization. All information will remain confidential to any third party, and it will be used solely for research purposes and published anonymously. The anonymity of experts will be ensured by the fact that no personal information will be placed in any publications or reports. This study will guarantee that no physical, psychological, or social harm is inflicted upon the experts, and all data will be securely kept in concordance with institutional data protection policies.

1.8 Thesis Scope

This study aims to develop and apply a comprehensive DM framework that focuses on the designing, refining, and implementing new MCDM approaches to improve reliability, flexibility, and adaptability in complex and uncertain decision environments. The proposed framework seeks to provide a more robust mechanism for addressing decision problems involving multiple conflicting criteria, diverse expert opinions, and uncertain information, thereby enhancing the quality and consistency of the final decision outcomes. Furthermore, this study also integrates two advanced bipolar Pythagorean neutrosophic AOs, which is the BPN-GHHM and BPN-GWHHM operators, to effectively handle uncertainty, interdependence, and ambiguity while simultaneously capturing the complex relationships and interactions among evaluation criteria in the decision analysis process. By incorporating these operators, the proposed approach is expected to produce more reliable and realistic decision results in situations characterized by vague and bipolar assessments. Additionally, the scope of this study is limited to the education sector, specifically the field of second language learning, where uncertainty, subjectivity, and the inherent bipolar nature of expert judgments frequently arise during the evaluation process. This domain provides an appropriate context for validating the effectiveness and practical applicability of the proposed MCDM framework under real-world DM conditions.

Next, this study involves the design and implementation of a new BPNS linguistic variable, facilitating the modelling of both the inherent vagueness and the bipolar nature of human judgments in educational scenarios. In addition, the scope also includes the development and implementation of the InnoRank–ARIE system (see APPENDIX 4), a custom web-based decision support system that operationalises the ARIE procedure by automating data processing, aggregation, scoring and ranking of alternatives. In short, InnoRank-ARIE is the custom web-based software tool developed in this study to operationalise ARIE method, which allow the full evaluation to be executed automatically and reproducibly rather than by manual calculation. The system provides a proof-of-concept platform for illustrating the feasibility of the proposed framework for DM based on ARIE. The evaluation is based on the expert opinions to rank English vocabulary learning strategies in uncertain conditions. No large-scale empirical validation or experiments based on learners and development of other software is in the study, beyond the InnoRank–ARIE prototype.

1.9 Thesis Outline

The rest of the thesis is structured below with visualisation in Figure 1.1.

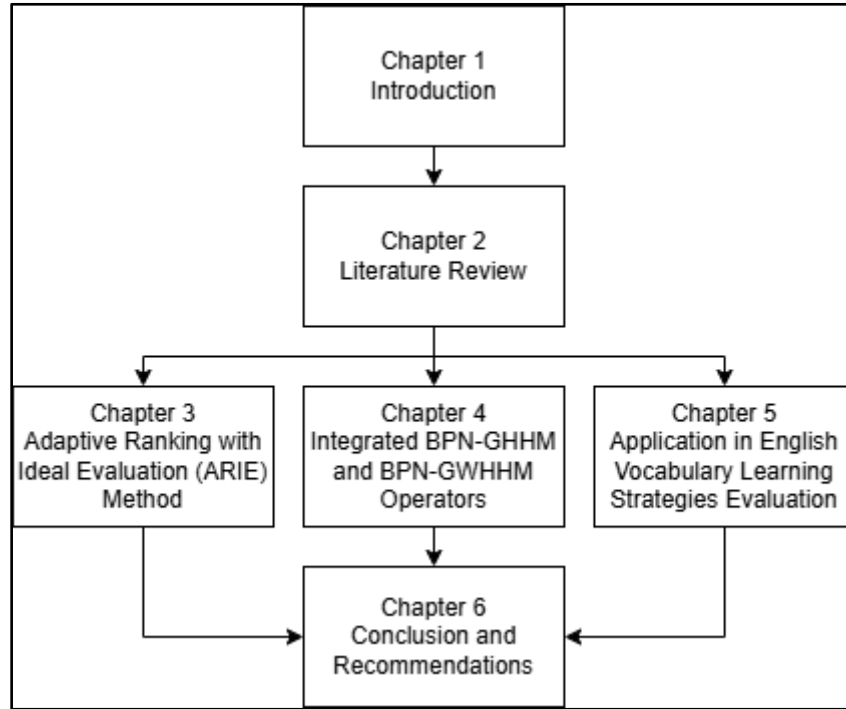


Figure 1.1 Structure of the Thesis

Based on Figure 1.1, CHAPTER 2 presents an extensive review on the related literature, encompassing advances in MCDM, a description of TOPSIS and its extensions, the development of BPNS theories, hybrid models utilizing the Hamacher and HM operators, along with applied criterion explanation in educational research contexts.

CHAPTER 3 describes the proposed methodology, including mathematical formulations of ARIE method, step-by-step calculations, sensitivity and comparative analyses, and the evaluation of Kendall's τ correlation coefficients with permutation testing.

CHAPTER 4 discusses the formulation of the two proposed AOs, known as the BPN-GHHM and BPN-GWHHM operators. The chapter also presents validations of three key properties, demonstrates a real-life application, and comparative and sensitivity analyses.

Two new BPNS linguistic variables are presented in CHAPTER 5. The BPN-GHHM operator and a score function are used to transform aggregated neutrosophic values into comparable scalar scores. ARIE method is then applied to evaluate English vocabulary learning strategies by using decision support system which is InnoRank-ARIE system, including comparative and sensitivity analyses with other MCDM approaches. In addition, validation via Spearman correlation coefficients between the ARIE method and these methods are presented to assess consistency and reliability.

Finally, CHAPTER 6 brings the study to conclusion and provides recommendations on the way the research should be carried out in future.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter provides a detailed review of the literature relevant to this study. Section 2.2 examines the theoretical base of MCDM. Section 2.3 presents both the classical TOPSIS method and its extensions. In Section 2.4, the concept of NS and their correlation with FS and IFS is discussed. Section 2.5 analyses the Hamacher operator, while Section 2.6 focuses on the HM operator and its generalized form. Finally, Section 2.7 emphasizes the application of MCDM methods in the educational context, particularly in assessing English vocabulary learning approaches as well as research methodology emphasized in Section 2.8.

2.2 Multi-Criteria Decision-Making

MCDM is a fundamental branch of operations research and decision sciences, focusing on decision problems with involve multiple, usually conflict criteria. In contrast to single-objective optimization, MCDM precisely recognizes the complexity and trade-offs involved in the actual decision structure problems, encompassing domains such as engineering design, environmental planning, healthcare and public policy. As noted by Balasbaneh et al. (2025), the origins of MCDM can be traced to utility theory and social choice theory, where the preferences of decision-makers are modelled and a method is established for evaluating the suitability of alternatives across conflicting objectives. As centuries pass, the field became developed into a structured methodological approach of both discrete and continuous decision problems, making it possible to make systematic comparisons and rankings, selection or classification of alternatives (Baydaş et al., 2024; Maral & Özdemir, 2025a). Consequently, various MCDM methods have emerged to address different decision contexts (Yasin et al., 2025), depending on the nature of the alternatives, criteria, and decision objectives involved. Figure 2.1 shows the structure of existing MCDM approaches divided specifically by Multi-Attribute Decision-Making (MADM) and Multi-Objective Decision-Making (MODM).

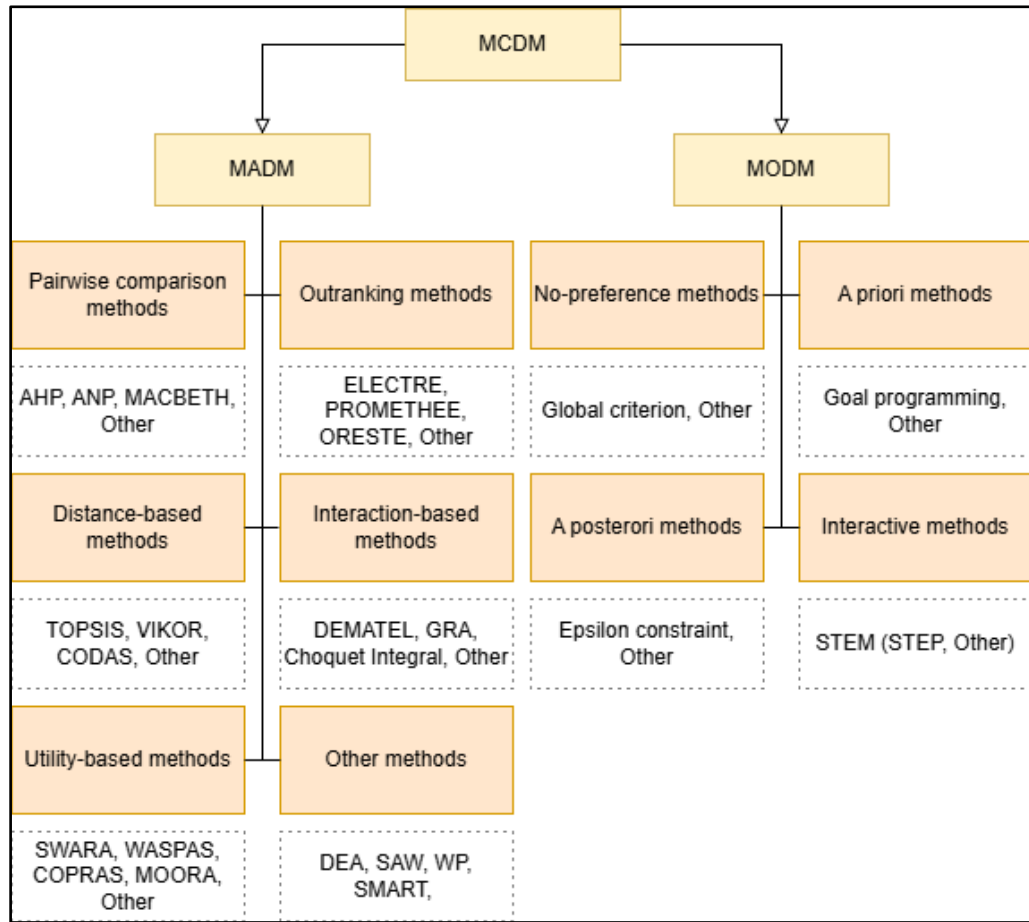


Figure 2.1 Structure of MCDM Approaches (Yalcin et al., 2022)

Figure 2.1 illustrates the traditional classification of MCDM into MADM and MODM. MADM is employed for problems involving a finite set of options, each assessed based on a set of a priori criteria, whereas MODM deals with optimization problems whose decision space are continuous and multiple objective functions (Kumar & Pamucar, 2025). MADM methods, including the Analytic Hierarchy Process (AHP), TOPSIS, ELimination Et Choix Traduisant la REalite (ELECTRE), and Preference Ranking Organization METHod for Enrichment Evaluations (PROMETHEE) have gained widespread acceptance due to their conceptual simplicity and practical applicability. Such methods vary in terms of the logic of aggregation, preference information treatment and compensation across criteria, which therefore provides flexibility in application in a variety of decision situations.

The AHP method continues to be among the most widely referenced MCDM methods in large part by its qualitative hierarchies and the possibility to use both qualitative and quantitative judgements via a series of pair-wise comparisons. According to Tu & Wu (2025), despite its popularity, AHP method has been criticized

on the problem of rank reversal and lack of consistency that led to the development of extensions and hybrid models. By comparison, distance-based techniques like TOPSIS method rely on the static benchmark and sensitive to extreme values (Alghazzawi et al., 2024). Outranking methods such as ELECTRE and PROMETHEE do not require full compensability across criteria and their application is especially appropriate to incomparable and veto thresholds decision problems.

The recent developments in the MCDM studies have also been more inclined towards uncertainty modelling and hybridization. Classical MCDM techniques usually assume that the input data is accurate and deterministic, but realistic DM situations are often associated with ambiguity, vagueness, and lack of complete information. To solve this drawback, scholars have incorporated the use of FS theory, IFS, rough sets, and NS theories into MCDM systems. In specific, fuzzy MCDM techniques enable the user of linguistic expression of preferences which, in turn, increases the reality and strength of decision results (Alhakami, 2023; Rane & Choudhary, 2023). Also, mixed MCDM models that involve the use of weighting schemes, such as entropy, CRITIC and ranking schemes have been suggested to enhance objectivity and method rigor.

MCDM has grown in modern research not only in the development of methodologies, but also in the application to domains and in decision support systems. The combination of machine learning, optimization methods with MCDM as well as machine learning and Geographic Information Systems (GIS) indicates the tendency to the use of data-driven and smart decision support. Moreover, the issues of sustainability and resilience have placed MCDM as an imperative instrument to determine the complex socio-technical systems within an economic, environmental, and social framework (Avramova et al., 2025; Kaya, 2024). Although such progress has been made, there are still concerns about the method selection, result validation, as well as sensitivity analysis, and it is important to consider the systematic frameworks that practitioners can use to select right MCDM techniques based on the problem contexts. Thus, it can be noted that MCDM has developed into a multi-faceted and essential decision-analytic framework. The further development of it can be characterized by the diversification of methods, interdisciplinary fusion, and the focus on uncertainty and sustainability. These tendencies support the importance of MCDM as the conceptual framework and the tool of assisting the complex DM in the dynamic environment.

2.3 TOPSIS Method

The TOPSIS method has attracted considerable attention among other MCDM methods since it provides an intuitive approach for selecting the best alternative that is closest to the ideal solution and the most distant from the negative ideal solution. Also known as one of the most widely used MCDM methods, it was initially proposed by Hwang & Yoon (1981) and subsequently improved by Yoon (1987) and Hwang et al. (1993). The method simultaneously considers a PIS, which represents the optimal performance for each criterion that maximizing benefit criteria and minimizing cost criteria, while the NIS reflects the least desirable performance across all criteria. This dual reference concepts ensure that the maximization of desirable outcomes is achieved, and a minimization of undesirable or disadvantages effect is achieved (Bánhidi & Dobos, 2024; Sulistiani et al., 2024a).

The concept of simplicity, low computing requirements, and the possibility to utilize both qualitative and quantitative criteria are the reason of TOPSIS method's popularity (Aminoroaya et al., 2025). The method open ranking process allows decision-makers to see and understand the evaluation of each alternative with respect to the optimal solution, which contributes to effective communication and justifiable DM environment. According to Kitila et al. (2026), TOPSIS has minimum rank reversal effect among all the MCDM techniques, as it compares the alternatives according to their distances from the ideal and negative ideal solutions. The strength of the method lies in its capability to support trade-offs among competing criteria, provide a direct, and easily interpretable ranking scheme. Accordingly, the TOPSIS method has been widely applied in diverse domains, including supply chain management (Samhoury et al., 2025), logistics (Kütahya et al., 2025), environmental assessment (Han et al., 2025), and sustainability studies (Pandipati, 2025).

For instance, to identify the top suppliers, Alaei et al. (2024) utilised TOPSIS method, which comprises multiple criteria, including cost, quality, and delivery performance, which demonstrates the applicability of the tool in managing the selection of suppliers. In the same way, Elkady et al. (2024) have used TOPSIS method to prioritize a project to allocate resources more efficiently, whereas Yin et al. (2024) have used TOPSIS method to assess the sustainable strategies to reduce environment risks. Tran et al. (2024) have adopted TOPSIS method to determine the best design parameters that can enhance the performance of the gearbox. Bekiryazici & Acar (2024) took the

advantage of the method by identifying the best stormwater management strategies. These different applications highlight the utility of TOPSIS method in dealing with real-world DMMCDM problems which entail multiple and conflicting criteria.

However, the classical TOPSIS method has significant limitations. It normally assumes the independence and linearity of the criteria, based on the crisp values, and deterministic judgments in the definition of best and worst solutions. Such assumptions make it less robust in situations where there exists vagueness, linguistic evaluations, indeterminacy or intercriteria dependencies that usually found in diverse domain especially in medical and educational MCDM problems. Moreover, information between criteria could affect the overall assessment structure since TOPSIS uses Euclidean distance as the main measure of distance without bringing possible correlations among attributes into priority (Madanchian & Taherdoost, 2023). Additionally, the method's sensitivity to extreme values and the subjective inherent in weights assignment (Bánhidi & Dobos, 2024), and ideal value determination can further undermine the result stability and interpretability when decision data are uncertain (Ilieva, 2025; Nazarov, 2025). These critiques have stimulated extensive methodological refinements and comparative evaluations aimed at improving the stability and reliability of TOPSIS outcomes.

2.3.1 Normalization Approaches

Normalization is the process of converting data into a predetermined norm or standard to compare all evaluation criteria in a common scale. This process eliminates differences in units of measurement, optimization direction, and fluctuation range, such as points, percentage, ratios and other measurement scales. According to Trung et al. (2025), while reference-based normalization techniques are thought to be appropriate for cases where decision-maker able to determine ideal or target values for each criterion. In contrast, optimization-based normalization approaches are used if the objective is to choose the best alternative by comparing with the highest criterion value against the lowest criteria value, or vice versa, relying on classification of the criteria as benefit or cost criterion. Nevertheless, non-linear normalizing procedures ought to be employed when the decision-maker feels that the numbers in the criteria do not accurately reflect the monotonically rising or decreasing relationship between the criterion and its corresponding benefit or cost (Mukhametzyanov, 2023). Such

normalization methods can capture complex or non-linear relationships among criteria, which improve the reliability of the decision results. Table 2.1 presents the normalization approach depending on the optimization orientation.

Table 2.1
Summary of Normalization Approaches

Approach	Benefit criteria	Cost criteria	Advantages and limitations	Academic sources
Westendorf (n_1)	$\frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}}$	$\frac{\max x_{ij} - x_{ij}}{\max x_{ij} - \min x_{ij}}$	Simple; highly sensitive to outliers	Brauers & Zavadskas (2006)
Linear (n_2)	$\frac{x_{ij}}{\max x_{ij}}$	$\frac{\min x_{ij}}{x_{ij}}$	Reduces skewness; needs positive data	Brauers & Zavadskas (2006); Jahan & Edwards (2015)
Jüttler-Körth (n_3)	$1 - \left \frac{\max x_{ij} - x_{ij}}{\max x_{ij}} \right $	$1 - \left \frac{\min x_{ij} - x_{ij}}{\min x_{ij}} \right $	Preserves order consistency; limited empirical validation	Brauers & Zavadskas (2006); Gardziejczyk & Zabicki (2017); Zavadskas & Turskis (2008)
Peldschus (n_4)	$\left(\frac{x_{ij}}{\max x_{ij}} \right)^2$	$\left(\frac{x_{ij}}{\max x_{ij}} \right)^3$	Smooth extreme values better than linear methods; less commonly used and tested	Brauers & Zavadskas (2006); Gardziejczyk & Zabicki (2017); Zavadskas & Turskis (2008)
Stopp (n_5)	$\frac{100x_{ij}}{\max x_{ij}}$	$\frac{100 \min x_{ij}}{\max x_{ij}}$	Adapts well to mixed criteria; rarely applied	Brauers & Zavadskas (2006); Gardziejczyk & Zabicki (2017)
Log (n_6)	$\frac{\ln x_{ij}}{\ln \left(\prod_{i=1}^n x_{ij} \right)}$	$1 - \frac{\ln x_{ij}}{\ln \left(\prod_{i=1}^n x_{ij} \right)}$	Reduces skewness; cannot handle zero or negative values	Zavadskas & Turskis (2008); Gardziejczyk & Zabicki (2017)
Z-score (n_7)	$\frac{x_{ij} - \sum_{i=1}^n x_{ij} / n}{\sqrt{\sum_{i=1}^n (x_{ij} - \mu_j)^2 / n}}$	$-\frac{x_{ij} - \sum_{i=1}^n x_{ij} / n}{\sqrt{\sum_{i=1}^n (x_{ij} - \mu_j)^2 / n}}$	Suitable for normally distributed data; produce negative values	Fayazbakhsh et al. (2009); Gardziejczyk & Zabicki (2017)

Approach	Benefit criteria	Cost criteria	Advantages and limitations	Academic sources
Enhanced accuracy (n_8)	$1 - \frac{\max x_{ij} - x_{ij}}{\sum_{i=1}^n (\max x_{ij} - x_{ij})}$	$1 - \frac{\min x_{ij} - x_{ij}}{\sum_{i=1}^n (\min x_{ij} - x_{ij})}$	Stable for data magnitudes	Zeng et al. (2013); Jahan & Edwards (2015)
Max linear (n_9)	$\frac{x_{ij}}{\max x_{ij}}$	$1 - \frac{x_{ij}}{\max x_{ij}}$	Easy to apply; distorted by extreme values	Jahan & Edwards (2015)
Min linear (n_{10})	$1 - \frac{\min x_{ij}}{x_{ij}}$	$\frac{\min x_{ij}}{x_{ij}}$	Convert cost criteria to benefit; exaggerate zero values division	Jahan & Edwards (2015)
Sum-based linear (n_{11})	$\frac{x_{ij}}{\sum_{i=1}^n x_{ij}}$	$\frac{\left(\frac{1}{x_{ij}}\right)}{\sum_{i=1}^n \frac{1}{x_{ij}}}$	Simple; sensitive to outliers	Gardziejczyk & Zabicki (2017)
Vector (n_{12})	$\frac{x_{ij}}{\sqrt{\sum_{i=1}^n x_{ij} ^2}}$	$1 - \frac{x_{ij}}{\sqrt{\sum_{i=1}^n x_{ij} ^2}}$	Normalizes scale; suitable for vector models	Gardziejczyk & Zabicki (2017); Zavadskas & Turskis (2008)

Source: (Aytekin, 2021; Trung et al., 2023)

Based on Table 2.1, each normalization approaches have its own advantages and disadvantages according to the nature of the problem in MCDM. The Westendorf normalization is one of the earliest normalizations and it offers a straightforward normalization for benefit and cost criteria. However, it is also very sensitive to the maximum and minimum criterion values, with one outlier having an important impact on the normalized values of all alternatives. Likewise, the linear normalization is easy to compute and is less affected by data skewness than are ratio-based methods. Nevertheless, it is not as well suited for data sets that include a high amount of variability or unusual observations.

Based on Krishnan et al. (2023), Jüttler–Körth normalization maintains the proportional consistency, as it allows to transfer the data into a bounded interval range, but still preserves the relative proportion of alternatives. Despite this benefit, the method has been relatively few studies empirically validated with reference to more widely used normalization methods, and most studies have focused on its robustness in different decision environments. The Peldschus normalization applies a quadratic normalisation to generate smoother normalised values and has acceptable behaviour in some

circumstances. But it has not been widely used, and there are few complete studies which assess its effectiveness in various MCDM methods.

Stopp normalization was found infrequently applied in application (Trung et al., 2025b), but it guarantees consistent bounded outputs and is mentioned in comparative studies. Similarly, logarithmic normalization is a useful technique for reducing skewness in very distributed data, by compressing very large values with a logarithmic transformation. For many practical decision problems, logarithmic unable to directly be used for zero or negative values of the criterion, and so all criterion values must be strictly positive. Additionally, the Z-score normalization is well suited for data sets that are approximately normally distributed as it normalizes observations using the mean and standard deviation. Normally, these benefits do not exist, since the transformation results in positive and negative normalized values, which is not compatible with many of the distance-based MCDM methods that are based on a non-negative bounded interval of normalized values.

In the optimization-based normalisation methods, enhanced accuracy normalization was introduced to increase the stability of the system for different magnitudes of data considering the information from the entire range of the criterion. However, the normalisation still requires extreme values and can suffer from distortion if the data set has substantial outliers. The Max-linear normalization is widely used, due to its simplicity and easy implementation. However, division of each criterion value by the maximum criterion value can lead to the suppression of the discrimination between the alternatives if there are extreme values. On the other hand, Min-linear normalization converts cost criteria to benefit-type ones with minimum value as reference point. While mathematically simple, the method can tend to overemphasize the effect of very small observations and can be numerically unstable if criterion values tend to be near zero (Yadav et al., 2026).

The Sum based linear normalization, which expresses all values of the criteria as a fraction of the sum of the alternatives, preserves the proportional contribution of each alternative. The normalization is sensitive to extreme observations though, since once an extremely high value is added to the denominator, it will result in a lower normalized value of the other alternatives. Finally, vector normalization is one of the most popular normalization methods that is compatible with Euclidean distance-based normalization methods, such as TOPSIS. The method normalizes each value of the criterion to produce dimensionless values which can be used for distance measures,

while maintaining proportional relationships. However, some studies have found that vector normalization is still sensitive to the length of the criterion vector and may reduce the norm of moderate alternatives if large observations occur, thus affecting the results of the ranking process. Table 2.2 emphasizes existing MADM methods that adopted normalization approaches based on the classification presented in Table 2.1.

Table 2.2
Normalization Approches Used in Different MADM

MADM	Normalization approaches											
	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{11}	n_{12}	
ARAS										✓		
CRADIS		✓										
CODAS		✓										
COPRAS										✓		
CoCoSo	✓											
MABAC	✓											
MACONT	✓	✓								✓		
MARCOS		✓										
MOORA		✓										
MULTIMOORA											✓	
PARIS	✓	✓									✓	
PIV											✓	
PSI		✓										
ROV	✓											
SAW		✓										
TOPSIS											✓	
VIKOR	✓											
WASPAS		✓										
WPM						✓						
WSM										✓		

Source: (Trung, 2022)

Table 2.2 presents the normalization methods that are used by various MADM methods and shows that different decision models implement different normalization methods. In general, most MADM methods do not use a universal normalization procedure but use normalization procedures that are consistent and match with the mathematical principles and ranking mechanism in the methods. This observation confirms that normalization process plays crucial role of the ranking process where it affects the final ranking and should not be treated as a preprocessing operation (Malefaki et al., 2025).

The normalization using optimization method is one of the most common normalization methods, such as Westendorf (n_1) and linear (n_2). These normalisation methods are generally most frequently used in methods like Compromise Ranking of Alternatives from Distance to Ideal Solution (CRADIS), COMbined COMpromise Solution (CoCoSo), COMbinative Distance-based ASsessment (CODAS), Multi-Attributive Border Approximation Area Comparison (MABAC), Multi-Objective Optimization on the basis of Ratio Analysis (MOORA), Weighted Aggregated Sum Product ASsessment (WASPAS), Preference Selection Index (PSI), Range Of Value (ROV), SAW and Measurement Alternatives and Ranking according to the COMpromise Solution (MARCOS) due to their computational simplicity and easy interpretation. Based on their wide use, the ratio-based normalization has been seen as the preferred choice of many modern MCDM techniques because of its efficiency and suitability for distance-based or additive aggregation models (San & Senyegit, 2025).

There are also some unique preferences in terms of normalisation among other MCDM methods. All three methods, Additive Ratio Assessment System (ARAS), COMplex PROportional ASsessment (COPRAS) and Weighted Sum Model (WSM), use sum-based linear (n_{11}) which is well known for being simple to calculate and easy to interpret. The fact that Proximity Indexed Value (PIV), TOPSIS and Multi-Objective Optimization by RAtio analysis plus the full multiplicative form (MULTIMOORA) adopt the same Vector normalization (n_{12}), indicates that the vector normalization is compatible with Euclidean distance based ranking mechanisms and geometric distance calculation. Mixed Aggregation by COMprehensive Normalization Technique (MACONT) and Preference Analysis for Reference Ideal Solution (PARIS) combined linear, Max-linear and Vector normalizations with sensitivity parameter to provide

flexibility in dealing with the characteristics of the criterion and the aggregation procedures.

Weighted Product Model (WPM) also differs from most of the MADM methods by incorporating Stopp normalization (n_5) which was more appropriate for multiplicative aggregation models than other norms since it ensures the conversion of the criteria into comparable ratio scales. None of the MADM approaches utilized logarithmic (n_6) and Z-score normalization (n_7) since the normalized values may be negative and thus cannot be used in the calculation process that require normalized values to be in the range $[0,1]$. Most current MADM methods use only a few of the available normalization methods, while several of the mathematically proven normalization methods have been not used or barely used. Since there is a lack of deep analysis on normalization approaches (Kalibatienė & Simanavičienė, 2026), this shows that normalization is still an open research field, and there are still possibilities to build more alternative normalization methods for specific DM needs.

It is noted that no current MADM method exists that explicitly and simultaneously includes multiple normalization philosophies, and accommodates benefit, cost and target criteria under one single framework. Current normalization methods are mostly aimed at optimizing benefit criteria, while cost criteria are much less represented and criteria with predetermined desired target values are even less represented. In practice, problems involving target-oriented evaluation often involve extra transformation processes or special expressions.

2.3.2 Distance and Similarity Measure

One of the basic elements of TOPSIS is the separation between the alternatives and the ideal ones. In classical TOPSIS, Euclidean distance is used to calculate the similarity of each alternative to the PIS and NIS. The logic behind this is that the most desirable alternative should be the closest to the PIS and the furthest away from the NIS. Although popular, the Euclidean distance measure has been pointed out to have a few drawbacks. According to Chang & Zou (2026), Euclidean distance assumes that the criteria are independent and equal. As such, the measure may not be a true reflection of the complex relationships between attributes, especially if there are interdependencies among attributes or different levels of influence. Also, Euclidean distance is based on squared deviations, meaning that it allows for larger deviations to have a

disproportionately greater effect on the overall value of the distance. This is a property that can cause instability in ranking in the presence of extreme values or outliers in the decision matrix.

To solve these problems, many alternative distance measures have been suggested in the literature. For instance, the Hamming distance is the absolute difference between the criterion values and is less influenced by extreme deviations than the Euclidean distance. In cases where the data for the decision have a large variation or irregular distribution, as Saeed et al. (2023) said, Hamming distance may give more robust results. The Minkowski distance (Thompson, 1996) has also been generalized as a distance function which includes both Euclidean and Hamming distances. Decision makers can vary the order parameters to give the deviation in the larger scale weight. This flexibility allows the distance measure to be suitable for a variety of preferences in making decisions and problem types.

Similarity measures are also widely considered in TOPSIS related studies, except for distance-based approaches. Similarity measures assess the resemblance between alternatives and ideal solutions, whereas distance measures are about the size of separation. Similarity measures have been discussed as a better alternative to dissimilarity measures, especially in uncertain and fuzzy settings (Kirişci, 2023a). Therefore, several similarity indices, correlation coefficient, cosine similarity measures and projection-based methods have been incorporated into TOPSIS frameworks. But similarity-based methods have their drawbacks. Some similarity measures can calculate the same similarity for two alternatives that have significantly different attribute distributions, which will decrease the discrimination power. Further, some similarity indices are more concerned with the direction of change than with the size of the difference in their alternatives. This can lead to looking at some alternatives that seem very similar but with huge number differences.

For uncertain environments like FS, IFS, PFS, NS, and BNS contexts, researchers have created special distance and similarity measures to handle the information of uncertainty, hesitation, and indeterminacy. However, many of the existing measures are direct extensions of classical distance measures and thus have some of the problems associated with sensitivity to data distributions and interaction of the attributes (Li et al., 2025). Therefore, the results in the literature show that there does not exist a universally best distance or similarity measure for all the DM problems. These approaches focus on different aspects of the information used for a decision,

which results in different rankings of outcomes. This observation has led to constant development of new measures with the goal of enhancing discrimination capability, robustness, flexibility and adaptability to varying decision settings.

2.3.3 Relative Closeness Measures

The RC coefficient is the ultimate ranking tool used in TOPSIS, and it is very important in the ranking of the alternatives. In the classical TOPSIS approach, the distance from the PIS and the distance from the NIS are considered simultaneously in the calculation of RC value. Alternatives that have higher RC values are more preferred since they are closer to the ideal alternative and further away from the anti-ideal alternative. There are some drawbacks to the classical RC coefficient as has been pointed out by the literature, although the theoretical basis is very simple and computation is relatively straightforward. In the traditional formulation the distances to the PIS and NIS are assumed to be equally weighted in the evaluation process (Güler, 2026). But in practice, decision makers might pay different attention to the ideal solution rather than the anti-ideal solution. This equal treatment assumption could limit the flexibility of the decision model. In certain risk sensitive applications, like health, environmental management, or financial investment, decision-makers might prefer to avoid undesirable outcomes rather than just approaching the optimum outcome. On the other hand, in decision situations where opportunities exist, proximity to the ideal solution could be more important than distance from the anti-ideal solution (Roszkowska & Filipowicz-Chomko, 2024). In order to solve this problem, a few researchers have suggested modifications to the RC measures that incorporate some adjustable coefficients or preference parameters. These changes enable decision-makers to control the impact of the positive and negative ideal distances as per their preference and decision situation.

Another one is the interpretability of the resulting closeness values. The traditional RC coefficient is readily interpretable in the range $[0,1]$ but some modified versions can return values with less intuitive and less easily explainable and justifiable results for practitioners (Lin et al., 2023). Moreover, from the studies conducted, it is found that the RC coefficient is also dependent on the distance or similarity measure used. The coefficient is calculated based on separation values from previous stages, therefore any restrictions that may apply for the distance measure will carry forward to

the ranking results (Li et al., 2024). Thus, the improvements to the RC formulations may not be sufficient to alleviate ranking inconsistencies when the distance measurement process is not adequate.

2.3.4 Extensions of TOPSIS Method

Furthermore, many variations of TOPSIS were introduced to facilitate more types of imprecisions to be accommodated when dealing with uncertainty and to overcome the drawbacks associated with the classical TOPSIS. One of the earliest extensions was made by Hwang et al. (1993) with their simplification of a k-objective problem and the development of TOPSIS for MODM. This was accomplished by computing the distances to the PIS, NIS, and using the max–min operator to obtain a compromise solution. The extension made TOPSIS much more general in its application than the traditional MADM and provided the basis for many subsequent methodological developments, such as optimization problems with multiple conflicting objectives.

The second line of research involved generalizing TOPSIS to account for uncertainty in numerical evaluations. Jahanshahloo et al. (2006) introduced an interval-valued TOPSIS in which performance measures are given as intervals instead of using fixed values to capture the uncertainty in the decision information. Based on this idea, Yue (2011) proposed an interval number to calculate the weights of decision makers according to the RC to the ideal solutions. The interval-valued methods are more flexible than the crisp numerical ones, but, at the same time, they are fundamentally numerical. Kaczyńska et al. (2023) have pointed out that the use of a single crisp values, or even deterministic numerical value, might restrict the possibilities of expressing the uncertainty and the variability involved in real life DM. Interval methods expand the range of evaluations but also need to quantify the uncertainty, which may be less appropriate if the decision makers make their judgments in qualitative or linguistic terms.

In addition to uncertainty modelling, a significant path in the work of TOPSIS has been the development of the method to determine the weights of the criteria, which can have a substantial impact on the rank of the alternatives. For instance, Deng et al. (2000) developed an early approach of modified TOPSIS method the weighted Euclidean distance measures were directly added to the distance calculations between

the alternatives and the ideal solutions without the weighted normalized decision matrix. Shih et al. (2007) combined the arithmetic and geometric mean operators in one common framework for normalization and distance measurement in order to aggregate individual preferences. Later, Hu et al. (2016) added a variant of the dynamic weighting method with different centrality measures. Similarly, Li et al. (2018) introduced an inter-criterion correlation approach in the TOPSIS Informative Weighting and Ranking (TIWR) method, and Sulistiani et al. (2024b) developed an alternative ranking method called Objective Weighting-TOPSIS (OWH-TOPSIS) that used an objective weight based on data variation and also took into account the weights of decision makers, thereby increasing the accuracy and consistency of the alternative rankings.

Others have focused on enhancing the distance measures used in TOPSIS. The conventional weighted Euclidean distance assumes that the evaluation criteria are independent and equally contribute to the distance from the ideal solutions, which can decrease the reliability of the ranking if the evaluation criteria are correlated or if greater flexibility in measuring similarity is needed. Huang et al. (2009) solved this problem by using the Euler distance instead of Euclidean distance. Later, Wang & Wang (2014) added the Mahalanobis distance to TOPSIS, which made it possible to consider the correlations between criteria and kept the TOPSIS unaffected by non-singular linear transformations. Similarly, Radulescu & Radulescu (2017) generalized distance calculation by adding a Minkowski distance parameter (p), which enabled the decision makers to choose the distance function that best suits for the decision problem. All these studies show that adjusting the distance measure increases the robustness and flexibility of TOPSIS.

Researchers have also suggested improvements to the ranking system, to fix methodological problems that impact the consistency of alternative rankings. However, Wu et al. (2011) countered the methodological shortcomings found in cross-efficiency assessments by relaxing the averageness assumption to enhance the ability to distinguish options. In the same way, Dymova et al. (2013) removed the heuristic assumptions when defining the ideal solutions to get a more consistent ranking of interval decision data. As another step, Krohling & Pacheco (2015) introduced A-TOPSIS to rank and to compare evolutionary algorithms. In contrast with the classical TOPSIS which uses deterministic values for evaluating alternatives, A-TOPSIS reflects

the average performance of the algorithms and also the stability of the solutions, based on mean and standard deviation of the algorithm performances from multiple runs.

These studies show that TOPSIS is the subject of significant methodological improvements in the optimization processes, weighting schemes, distance measures, ranking process and numerical representation of uncertainty. Although such progress has been made, most of the current extensions are based on numerical modelling and computational detail. The ability to explicitly model human judgment, hesitation, ambiguity and qualitative uncertainty is still limited, especially in complex DM situations where evaluators' assessments are naturally coded in terms of language rather than numerical values. Consequently, there remains a need for TOPSIS extensions capable of modelling uncertainty beyond purely numerical representations.

2.3.5 A Review of Fuzzy and Neutrosophic TOPSIS Models

To cope the limitations of classical TOPSIS in representing linguistic uncertainty, researchers combine FS theory with TOPSIS that led to the developments of various Fuzzy TOPSIS (FT) variants. A pioneering contribution by Lai et al. (1994) was the incorporation of fuzzy membership functions in the optimization process, which led to one-objective optimization model that was more representative of imprecise preference information. Later, Chen (2000) added the vertex method in the calculations of distance vectors and the triangular fuzzy numbers, fuzzy PIS, and fuzzy NIS on to the classical TOPSIS approach. The development of these allowed the decision-makers to give subjective assessments in linguistic terms (high, medium, low) and then convert this into fuzzy membership functions. With its systematic treatment of human judgment and linguistic assessments, FT has become one of the most widely adopted fuzzy MCDM technique (Rajput et al., 2024).

Even with its popularity there remain a few methodological drawbacks with FT. The choice of membership functions is subjective, the weights are typically based on expert knowledge, and uncertainty is only considered to be sufficiently represented by the degree of membership. As a result, conventional fuzzy models cannot fully represent partial knowledge, incomplete information, inconsistency and indeterminacy often encountered in real-world DM. The limitations have inspired researchers to create more expressive extensions of FS theory with more comprehensive representations of uncertainty and maintain the computational benefits of the TOPSIS framework.

A research line of focus has been on improving FT with the presence of incomplete and interval information. Yong (2006) proposed a FT approach under for linguistic decision environments in which the weights of the criteria and the ratings of the alternatives are formulated as triangular fuzzy numbers. Wang & Lee (2007) generalized TOPSIS to fuzzy Multi-Criteria Group Decision-Making (MCGDM) with the use of fuzzy ordering operators instead of conventional max–min operations. The idea was expanded by Ashtiani et al. (2009) who incorporated interval-valued fuzzy numbers to TOPSIS and Ye (2010) who presented an interval-valued intuitionistic FT approach for group DM under incomplete information. Later, Baky & Abo-Sinna (2013) proposed a FT algorithm for bi-level MODM problems with nonlinear objective functions, and Ye & Li (2014) extended possibilistic mean and standard deviation measures and combined them with possibility theory and triangular fuzzy numbers in the evaluation process.

One area of research has been in building more expressive fuzzy representations that can model more complex degrees of uncertainty. Zhang & Xu (2014) proposed a Pythagorean fuzzy TOPSIS method by defining the Pythagorean fuzzy PIS and NIS using a score function and employing a distance measure to compute the separation of each alternative from these ideal reference points. In order to address the drawbacks mentioned above, Shen et al. (2018) proposed an extended intuitionistic FT approach to overcome the shortcomings of the previous ones by using a new distance measure with desirable mathematical properties. Later, Yu et al. (2019) extended TOPSIS in interval-valued Pythagorean fuzzy sets environments by using both the distance and similarity measures, and Kacprzak (2020) included ordered fuzzy numbers for ranking the alternatives without fusion of the individual decision matrices. Kirişci (2023b) recently proposed the cosine-based similarity and distance measures in a Fermatean fuzzy environment, and Bozyiğit & Ünver (2024) extended TOPSIS to circular IFSs by introducing parametric information, distance measures, and similarity measures. Collectively, these developments have considerably improved the capability of FT to represent vagueness and complex uncertainty.

The effectiveness of FT has also been shown in various application areas. For example, Buasri & Sangpradid (2025) developed a spatial tourism recommendation system for tourism based on FT, which leverages on the preferences data of 250 tourists in Thailand. Das (2024) applied FT to identify the pollution level in surface water of nineteen locations and showed its usefulness in environmental diagnosis in uncertain

conditions. Similarly, Tabor (2024) utilized FT to compare the strategies for occupational risk management in manufacturing companies, highlighting its capacity to account for all three aspects of safety, productivity, and economics. These applications demonstrate the versatility of FT for addressing complex DM situations that incorporate human judgements.

Although fuzzy logic can deal with vagueness, indeterminacy cannot be fully represented as the information is not true or false but uncertain (Alghazzawi et al., 2024). In many practical DM situations, however, information can be not just vague, but incomplete, inconsistent or indeterminate. To address this limitation, the NS theory was developed as a more complete theory that describes information as a combination of three independent components, truth, indeterminacy and falsity. This richer representation offers a more solid theoretical basis for modelling uncertainty in complex DM scenario and, thus, has inspired the development of neutrosophic TOPSIS variants.

In this vein, Chi & Liu (2013) adopted INSs in TOPSIS of MADM with unknown attribute weights, using the maximizing deviation approach to determine the objective weights. Sahin & Yiğider (2014) then presented the neutrosophic TOPSIS model using Single-Valued Neutrosophic Sets (SVNSs) which define the Single-Valued Neutrosophic Positive Ideal Solutions (SVN-PIS) and Single-Valued Neutrosophic Negative Ideal Solutions (SVN-NIS) that incorporate the truth, indeterminacy, and falsity. Ye (2015) also extended TOPSIS method by introducing single-valued neutrosophic linguistic numbers together with the generalized distance measures to effectively handle uncertain, indeterminate, and linguistic information in decision environments.

New developments have been ongoing, by adding more complex neutrosophic representations to neutrosophic TOPSIS. Broumi et al. (2015) extended TOPSIS to interval neutrosophic uncertain linguistic environments, where the basic operational rules and distance measures for interval neutrosophic variables, while determining unknown attribute weights via the maximizing deviation method. Later, Elhassouny & Smarandache (2016) further introduced simplified TOPSIS and neutrosophic simplified TOPSIS models, with the aim of making the algorithm simpler without compromising neutrosophic expressiveness through the concepts of Larger Neutrosophic Ideal Solution (LNIS) and Smaller Neutrosophic Ideal Solution (SNIS). Additionally, Abdel-Basset et al. (2019) proposed bipolar neutrosophic TOPSIS with the bipolar numbers to evaluate smart medical devices by incorporating positive and negative performance

aspects. These studies collectively showed that neutrosophic TOPSIS has significantly contributed to the capability of the TOPSIS family to model uncertainty beyond the traditional fuzzy representation and to incorporate it into other MCDM methods.

2.3.6 Hybrid TOPSIS Models

Along with uncertainty modelling, there has been a growing amount of research in recent years that focuses on the development of hybrid TOPSIS frameworks, merging TOPSIS with subjective and objective weighting methods to enhance the robustness, reliability and credibility of the DM output. The idea of these hybrid methods is to address the drawbacks of using expert judgement or data-driven weighting methods exclusively. Ju & Wang (2012) integrated Dempster-Shafer theory with AHP and extended TOPSIS to construct belief interval decision matrices to evaluate emergency alternatives by using ideal solutions represented by plausibility and belief measures. The integration adds more detail to the uncertainty and enhances consistency in the ranking results. The same way as Yadav et al. (2019) that proposed hybrid TOPSIS model by using the preference selection index before TOPSIS is applied to rank the alternatives and to determine the relative importance of criteria. The proposed method minimizes the subjectivity in the weight assignment and retains the simplicity of computation of the classical TOPSIS method.

Recently, many weighting methods have been introduced into TOPSIS for enhancing the importance of the criteria and obtaining more reliable ranking results. The combination of objective weighting methods such as, entropy and CRITIC, subjective weighting methods, AHP and BWM, with TOPSIS has been widely used, using the expert knowledge along with the information obtained from the decision information (Lu et al., 2022; Shrivastava & Shaikh, 2022; Trivedi et al., 2024). Similarly, comparative studies have shown that hybrid approaches, like TOPSIS with other MCDM methods such as ARAS, VIKOR, and PROMETHEE can offer more flexibility and consistency in ranking alternatives across various DM scenarios (Pisal et al., 2025; Radulescu & Radulescu, 2024; Saragi et al., 2024). In addition to the traditional weighting methods, Rahman et al. (2024) expanded TOPSIS by adding a random hypergraph-based framework to capture higher-order interactions among the criteria being evaluated and to determine the dynamic weights that reflect the characteristics of these interactions. More recently, Kampalasiri et al. (2026) introduced

an advanced DM framework that integrates the BWM with a modified TOPSIS model, further demonstrating the growing trend towards hybrid methodologies. All these advances TOPSIS approach is a solution to improve weight determination, ranking accuracy, and adaptability of TOPSIS in various DM applications.

Consequently, TOPSIS method remains a central method within the MCDM domain, offering both theoretical significance and practical relevance in addressing complex and critical multi-criteria decision problems. The existing literature demonstrates that TOPSIS has evolved from a relatively simple ranking technique into a versatile and adaptable DM support tool that incorporates uncertainty, fuzziness, interval information. Its continued prominence is driven by methodological flexibility, ease of implementation, and strong interpretability. Nonetheless, ongoing research underscores the importance of careful method selection, sensitivity analysis, and hybridization to mitigate inherent limitations and ensure robust decision support. However, the developing literature also shows that the performance of TOPSIS is very sensitive to the use of suitable normalization methods, weight methods, and distance measurement. Such constraints inspire the need for an alternative distance-based ranking method that can utilize more flexible benchmark definitions while retaining the simple concept and computation of TOPSIS.

2.4 Bipolar Pythagorean Neutrosophic Set

Ahmad et al. (2024) proposed the BPNS model which represents a sophisticated extension and mathematical formulization of prevailing fuzzy and neutrosophic models. This model was developed to better reflect human reasoning under several conditions of uncertainty, contradiction, and bipolarity, where information may not be strictly positive or negative. The BPNS model is a combination of the ideas of bipolarity and Pythagorean structures, since it provides more expressive capabilities when working with complex, vague and uncertain data. Such duality enables BPNS to depict the presence of positive and negative membership degree at the same time, followed by the representation of satisfaction and dissatisfaction, as well as acceptance and rejection, within the same framework. As a result, BPNS model offers a more flexible and comprehensive mathematical toll for modelling real-life DM problems that involve conflicting and ambiguous information.

Table 2.3
Author's Contribution to Comparison of BPNS with Related FS and NS

Type of FS and NS	Proposed by	Includes μ	Includes η	Includes ν	Bipolarity	Restricted under condition
FS	Zadeh (1965)	✓				$0 \leq \mu \leq 1$
IFS	Atanassov (1983)	✓		✓		$0 \leq \mu_A(x) + \nu_A(x) \leq 1$
BFS	Zhang (1994)	✓			✓	$0 \leq \mu_A^+(x)^2 + \nu_A^+(x)^2 \leq 1$
NS	Smarandache (1998)	✓	✓	✓		$0^- \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3^+$
INS	Wang et al. (2004)	✓	✓	✓		$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3$
IVNS	Wang et al. (2005)	✓	✓	✓		$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3$
SVNS	Wang et al. (2010)	✓	✓	✓		$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3$
PFS	Yager & Abbasov (2013)	✓		✓		$0 \leq \mu_A(x)^2 + \nu_A(x)^2 \leq 1$
SNS	Ye (2014)	✓	✓	✓		$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, 0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3$
BNS	Deli et al. (2015)	✓	✓	✓	✓	$0^- \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3^+, -3^+ \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 0^-$
BPFS	Mohana & Jansi (2018)	✓		✓	✓	$0 \leq \mu_A^+(x)^2 + \nu_A^+(x)^2 \leq 1, 0 \leq \mu_A^-(x)^2 + \nu_A^-(x)^2 \leq 1$

Type of FS and NS	Proposed by	Includes μ	Includes η	Includes ν	Bipolarity	Restricted under condition
PPBFSS	Palanikumar & Arulmozhi (2021)	✓		✓	✓	$0 \leq \mu_A^+(x)^2 + \nu_A^+(x)^2 \leq 1, -1 \leq -[\mu_A^-(x)^2 + \nu_A^-(x)^2] \leq 0$
PNS	Radha & Stanis (2021)	✓	✓	✓		$0 \leq \mu_A(x)^2 + \nu_A(x)^2 \leq 1, 0 \leq \mu_A(x)^2 + \eta_A(x)^2 + \nu_A(x)^2 \leq 2$
BPFS	Mandal (2023)	✓		✓	✓	N/A
PNS	Ismail et al. (2023)	✓	✓	✓		$0 \leq \mu_A(x)^2 + \eta_A(x)^2 + \nu_A(x)^2 \leq 1$
BPNS	Ahmad, et al. (2024)	✓	✓	✓	✓	$0 \leq \mu_A^+(x)^2 + \nu_A^+(x)^2 \leq 1, 0 \leq \mu_A^-(x)^2 + \nu_A^-(x)^2 \leq 1,$ $0 \leq \mu_A^+(x)^2 + \eta_A^+(x)^2 + \nu_A^+(x)^2 \leq 2, 0 \leq \mu_A^-(x)^2 + \eta_A^-(x)^2 + \nu_A^-(x)^2 \leq 2$

Table 2.3 provides a brief history of the development of FS and NS theories that have developed to increasingly represent uncertainty in DM contexts. This development is based on the FS theory introduced by Zadeh (1965) that characterized vagueness using a membership function to represent the degree of membership of an element in a set. This idea was later improved by Atanassov (1983) through the development of the Intuitionistic Fuzzy Set (IFS) theory, which incorporates membership (μ) and non-membership degrees (ν), with the constraint that their sum is bounded above by one. These initial models gave a fundamental mathematical basis of describing uncertainty and imprecision information, but they somehow failed to capture the bipolarity or competing information at the same time. To rectify this limitation, Zhang (1994) proposed Bipolar Fuzzy Set (BFS) that brought the concept of bipolarity emphasizing the simultaneous evaluation of positive and negative information. This development allows the decision-makers to depict the situation in which an alternative can have both admirable and unadmirable factors or features simultaneously. Although this has been improved, BFS and its variations continued to have problems with handling indeterminate or sporadic information present in real-world decisions.

The third independent component, the indeterminacy degree, (η) was introduced in NS theory by Smarandache (1998) to complete this conceptual gap. In contrast to FS or IFS, a model is based on three independent elements which are truth, indeterminacy, and falsity defines NS, as opposed to relying on the membership and non-membership degrees only. This three-part element representation enables the representation of uncertain, incomplete, and inconsistent information where it can be recognized as a valuable generalization of classical fuzzy frameworks. After its initial introduction, a number of subclasses of NS theory have been formulated to increase mathematical interpretability and flexibility. These are Interval Neutrosophic Set (INS), Interval-Valued Neutrosophic Set (IVNS), SVNS, and Simplified Neutrosophic Set (SNS). All of these subclasses narrow the mathematical requirements of the original NS theory to allow computational implementation whilst maintaining the possibility of expressing indeterminacy.

The other important development in uncertainty modelling was the advent of the Pythagorean Fuzzy Set (PFS). PFS is a superset of IFS as it loosens the membership constraints, and the squared sum of the membership and non-membership degrees that is limited to one. This is more flexible and realistic change especially useful in providing

flexibility and realistic nature to models of human judgment that are subject to ambiguity or hesitation. Raising on this basis, there were some frameworks of hybrid character which integrated the ideas of bipolarity and Pythagorean structures. To illustrate, Bipolar Pythagorean Fuzzy Set (BPFS) and Possibility Pythagorean Bipolar Fuzzy Soft Set (PPBFSS) are the extension models of BFS theory and concept that admit the positive and negative membership values subject to Pythagorean constraints to give a more expressive framework of decision situations that involve counterbalancing judgments.

Subsequent developments attempted to combine the expressiveness of neutrosophic logic and the flexibility of bipolar Pythagorean concept. Specifically, the Bipolar Neutrosophic Set (BNS) was introduced by Deli et al. (2015), whereas the Pythagorean Neutrosophic Set (PNS) was proposed by Radha & Stanis (2021) and later Ismail et al. (2023) improved the Pythagorean constraints. The combination of these two schemes resulted into the development of BPNS which is a very generalized model integrating the main strengths of the two models. Incorporating the positive and negative elements of membership degrees, BPNS use the neutrosophic triad of truth, indeterminacy, and falsity, subject to the Pythagorean squared sum condition, to ensure internal consistency. The result of this integration is a strong mathematical framework that can model multidimensional uncertainty more flexibly and interpretably.

As a DM instrument, BPNS offer the decision-makers to solve bipolarity, indeterminacy, and fuzziness in complex multi-criteria evaluation environments. Some researchers have shown a deep interest in integrating advanced mathematical tool, BPNS model with other powerful tools. Yin (2025) proposed an assessment model of agricultural supply chain finance credit risk, which employs BPNS model with an algebraic lattice structure along with a hybrid entropy weighting approach, which is effective in managing uncertainty, contradiction, and bipolarity in agricultural Small- and Medium-Sized Enterprise (SME) data. Since such approaches as TOPSIS or Stepwise Weight Assessment Ratio Analysis (SWARA) technique might result in the loss of information due to forced normalization, the lattice-based methodology can maintain the depth of the data and does not have to depend on the strict ranking models.

The model's applicability and high performance with regards to the determination of the credit risk of agricultural SMEs are consistent with a numerical case study, demonstrating its superiority over traditional models. Ahmad et al. (2024) integrated the BPNS model with AHP-express framework with the novel of linguistic

variable to enhance the DM effectiveness in various applications. Due to the limitations occur in the traditional AHP model that always face difficulties in handling uncertainty and indeterminacy, this new combination model successfully addresses. These studies demonstrate that BPNS can be effectively integrated with various MCDM tools. However, the majority of existing hybrid models primarily emphasize methodological advancement, while the suitability of BPNS for specific application domains, particularly medical DM environments has not been explicitly discussed.

The BPNS model is particularly suitable for medical DM problems due to the inherent presence of simultaneously positive and negative clinical evidence, along with uncertainty and indeterminacy. Medical problems often involve treatments or diagnostic tools that exhibit beneficial effects while concurrently posing risks or adverse outcomes. In medical contexts, a single criterion cannot be adequately represented using unipolar or crisp values because clinical outcomes frequently demonstrate dual semantic polarity. For instance, a treatment may show high effectiveness, which can be considered a positive truth membership degree in BPNS, while also inducing severe side effects represented as a negative truth membership degree. The positive and negative of truth membership degree are semantically opposite yet independent elements representing supportive and counter evidence respectively. Regarding indeterminacy membership degrees, the positive indeterminacy may denote an unknown patient response, whereas the negative indeterminacy represents unknown side effects. The treatment failure corresponds to the positive falsity, while the absence of side effects corresponds to the negative falsity. The Pythagorean structure helps in balancing the certainty of effectiveness and uncertainty of side effects to ensure that both positive and negative evidence are appropriately accounted for in DM.

For medical DM contexts, the dual representation of both positive and negative aspects is crucial as it reflects the reality that a treatment can be effective but also carry risks. BPNS effectively models this duality by allowing both positive and negative memberships to coexist, each accompanied by degrees of truth, indeterminacy, and falsity. Furthermore, patient heterogeneity, variability in clinical response, and incomplete medical evidence introduce uncertainty that cannot be captured by classical or FS alone. BPNS extends NS theory by incorporating Pythagorean constraints to offer a more flexible and realistic representation of complex medical information. However, despite the advancement of this model, there is still a lack of comprehensive

methodology that systematically apply BPNS model with advanced AOs, leaving a significant room for further exploration and model development.

2.5 Hamacher Operator in MCDM

AOs play a central role in FS theory and MCDM by enabling the synthesis of multiple pieces of information into a single representative value. Hamacher operator is one of powerful AO within the category of mathematical functions that are extensively used in fuzzy, neutrosophic logic, and MCDM. It was introduced by Hamacher (1978) provides more flexibility of dealing with complex relationships and uncertainty than the classical operators in the Archimedean group including algebraic and Einstein operators (Kavitha et al., 2024). Its defining characteristic is the inclusion of a tunable parameter that offers decision-makers to adjust the degree of compensation and interaction among aggregated criteria, thereby offering greater modelling flexibility compared to fixed-form operators.

This advanced operator extends conventional AOs by defining t-norms, the generalized intersection functions, and t-conorms, the generalized union functions, which offers both additive and multiplicative relationships to be presented in a single flexible structure. In contrast to classical AOs such as minimum, maximum, or algebraic product, the Hamacher operator captures nuanced interaction effects, reflecting realistic decision-maker attitudes toward risk, trade-offs, and partial satisfaction (Zhang et al., 2023). As a result, the Hamacher operator has received increasing attention in fuzzy logic systems and advanced MCDM models.

Hamacher operator is preferable especially in the fuzzy, neutrosophic and MCDM environments where uncertainty, inconsistency, and partial truth prevail. Decision-makers can choose the level of interdependence and compensating interaction between decision criteria by tuning ζ parameter to a different level. It follows that it is currently one of the most popular and generalizable operators of aggregation in the current MCDM frameworks, particularly in hybrid systems that combine Pythagorean, spherical, and NS theories. Table 2.4 highlight an overview of the Hamacher operator recent developments based on fuzzy and neutrosophic environment.

Table 2.4
Author's Contribution to Hamacher Operator in MCDM

Author(s)	Year	Description	Set theory
Asif et al.	2025	Developed Pythagorean fuzzy Hamacher for MADM	PFS
Thilagavathy & Mohanaselvi	2024	Introduced T-spherical fuzzy Hamacher HM operator for group DM	Spherical fuzzy set
Ali & Yang	2024	Introduced circular Pythagorean fuzzy operators using Hamacher operator	Circular PFS
Luqman & Shahzadi	2023	Developed interval-valued fermatean fuzzy Hamacher operator	Interval-valued fuzzy set
Paul et al.	2023	Proposed Pythagorean fuzzy Hamacher AOs for MCDM	PFS
Mei et al.	2023	Developed single-valued neutrosophic Hamacher weighted averaging operator	SVNS
Ali	2023	Proposed probabilistic hesitant bipolar fuzzy Hamacher	Hesitant BFS
Jamil et al.	2019, 2022	Introduced bipolar neutrosophic Hamacher operators	BNS

Table 2.4 summarizes significant contribution on the use of Hamacher operator in MCDM between 2019 and 2025. Various key studies have influenced the development of Hamacher operator in MCDM models by increasingly extending its applicability by hybridizing it with different fuzzy and neutrosophic models. According to Jamil et al. (2019, 2022) proposed the Hamacher operations under BNS settings. This synthesis formed a basis of integrating both positive and negative judgment in decision models to increase accuracy in the subjective and conflicting perspectives modelling. Based on Ali (2023) introduced a probabilistic hesitant bipolar fuzzy Hamacher model which utilized the information of probabilistic and hesitation to better model uncertainty in the situation when the exact information was not available.

Hamacher operator been further enhanced by later developments that boosted the capacity of the operator to manage uncertainty. Mei et al. (2023) made a step towards this direction by creating one valued neutrosophic Hamacher weighted average operator. This model enhances the neutrosophic modelling but retained the strengths in the treatment of indeterminate and inconsistent data. In the same way, Paul et al. (2023) introduced Pythagorean fuzzy Hamacher AOs that could increase the representational

power of the Hamacher operator by using the Pythagorean structures. Luqman & Shahzadi (2023) also expanded the application of Hamacher aggregation by applying it to an interval-valued fermatean fuzzy information that enhance decision stability by reflecting the uncertainty.

Geometric and structural extensions were further developed and gave a new theoretical insight to the Hamacher operator. Ali & Yang (2024) introduced the circular Pythagorean fuzzy operators using Hamacher aggregation by extending the geometrically symmetry representation of relationship among decision criteria. In the same year, Thilagavathy & Mohanaselvi (2024) combined T-spherical fuzzy set with Hamacher and HM operator to improve group DM under uncertainty and ambiguous information. Asif et al. (2025) proposed a Pythagorean fuzzy Hamacher model of MADM, which focuses on the computational efficiency, flexibility, and accuracy of AOs in the multidimensional decision criteria.

These studies demonstrate a distinct current trend on the evolution of the Hamacher operator potential in modelling of uncertainty, interdependence, ambiguity, and non-linear trade-offs in MCDM models. The extensions to fuzzy, neutrosophic, and geometric models have made the Hamacher operator as an effective and versatile computational mechanism, which brings a powerful methodological base of contemporary data-driven DM systems to which the framework suggest in this study.

2.6 Heronian Mean Operator in Aggregation Framework

Among the various mean-based AOs, the HM operator attracted growing attention due to its distinctive ability to represent the interaction between several input values by combining the concepts of the arithmetic, geometric, and harmonic means. Historically, Beliakov et al. (2007) presented the HM operator as an AOs in complex DM cases where the relationships between criteria are nonlinear or partially dependent. The HM operator is unlike the traditional AOs that assume linearity and independence based on the assumption that might influence input values. The operator also changes to the effect of each value depending on the positional relationship with other inputs and depending on how it interacts with other inputs. This feature is especially useful in real-world situations of MCDM, because the criteria for DM are not always completely independent. Table 2.5 presented the comparison of some mean operator utilized in MCDM area.

Table 2.5
Author's Contribution to Comparison of Mean Operator in MCDM

Operator	Capture interactions among criteria	Flexibility
Arithmetic mean		Low
Geometric mean		Moderate
Harmonic mean		Moderate
HM	✓	Moderate
GHM	✓	High
GWHM	✓	Very high

Based on Table 2.5, the arithmetic mean, geometric mean and the harmonic mean are simple and easily calculated averaging operators and are commonly used (Chakrabarty, 2025). But these operators combine information independently and don't reflect any underlying relations between criteria. The arithmetic mean treats all inputs linearly and may be sensitive to extreme inputs that can give rise to biased means. The geometric mean is appropriate for multiplicative relationships but is difficult to use when there are zero values and it is not necessarily appropriate for conflicting criteria. Likewise, harmonic mean is mainly used to average rates, or ratios, and is strongly affected by the smaller values. Most important, all three operators ignore the interactions between the criteria, which may lose information in complex decision environments (Chakrabarty, 2024).

In order to overcome these disadvantages, the HM operator is able to include interaction terms among aggregated values, so that the inter-criteria relationships are maintained in the aggregation process. The HM operator is a more balanced and more robust aggregation mechanism that combines the beneficial features of arithmetic, geometric and harmonic means and can cope with a broader spectrum of DM scenarios. However, there are still some drawbacks to the classical HM operator. It is of equal importance relative to the other criteria and has no parameters that can be adjusted to fit different decision situations, which diminishes its applicability in the face of unstructured or very complex information.

The restriction of HM operator stimulated the creation of the GHM operator that was proposed by Sykora (2009) that offered parameters that could be adjusted to make decision criteria with complex interactions more flexible. GHM operator enables differentiation of special cases in the decision problems, which leads to enhancement

of adaptability in MCDM scenarios. Even with this improvement, GHM operator continues to use constant parameters in aggregating individual inputs, which constrained its sensitivity to dynamic or context specific relationships. To solve the problems, GWHM operator was proposed incorporating weight to indicate the relative importance of each criterion. It is a significant extension since it enables the decision-makers to emphasize attributes of importance, reflect a hierarchical emphasis and change the behaviour of aggregation as required according to the nature of the problem. This allows GWHM operator to increase the applicability of heronian-based operators in complex MCDM problems to capture the intercriteria correlation, nonlinearity, and compensatory effects in a more accurate way.

The ability of FS and NS theories to combine with other mathematical tools is yet another important indication of the increased potential of this operator in dealing with uncertainty, ambiguity, interdependence in the DM contexts. This flexibility allows the operator to be adapted to a wide range of DM problems with varying levels of complexity and uncertainty. HM based operators would be able to pool certain data whilst maintaining internal consistency on dependent criteria and therefore would be great importance in multi-dimensional evaluation scenarios. Table 2.6 presents an overview of existing studies that utilized HM operators' integration and their emerging applications in various fuzzy and neutrosophic models.

Table 2.6
Author's Contribution to HM Operator in MCDM

Author(s)	Year	Description	Set theory
Bilal et al.	2025	Developed quasi-rung orthopair fuzzy-based GHM	Quasi-rung orthopair fuzzy set
Tripura et al.	2025	Introduced geometric HM under picture fuzzy set	Picture fuzzy set
Yaacob et al.	2024	Introduced Dombi-based HM	BNS
Ali et al.	2024	Developed Archimedean AOs for complex IFS	IFS
Zhang et al.	2024	Proposed IVIFS Aczel Alsina-based HM	IVIFS
Ali & Alsager	2024	Proposed novel m-polar fuzzy AOs based-generalized geometric HM	m-polar fuzzy set
Wang & Feng	2024	Introduced generalized intuitionistic fuzzy Yager weighted HM	IFS

According to Table 2.6, the recent contributions to the integration of GM operators in MCDM are published within 2024 and 2025. The input to this problem provided by Wang & Feng (2024), who proposed a generalized intuitionistic fuzzy Yager weighted HM operator that the compensatory power of the Yager operator and the relationship-handling power of the HM operator. Expanding this line, Ali & Alsager (2024) proposed a new m-polar fuzzy aggregation model which used an extended geometric HM operator. This methodology applied the HM concept to m-polar fuzzy settings where the framework allows simultaneous consideration of multifaced poles of information like positive, neutral, and negative in MCDM situations.

Moreover, Zhang et al. (2024) introduced a HM operator with interval-valued fuzzy Aczel Alsina that overcome the deficiency of the HM operator in the interval-valued settings and increased the expressive power of the HM operator. This expansion enhanced the capacity of the operators to manage parameterized uncertainty and variability in membership degrees. Through the incorporation of the Aczel Alsina functionality, the study attained a more flexible and formidable aggregation system that further enhancing the quality of interpretative power of MCDM assessments. Ali et al. (2024) extended this direction to create the Archimedean AOs on the complex IFS that include the HM operator as a major structural component. The integration represents more closely complex valued information and enables the operator to measure both amplitude and phase relations in MCDM data.

Besides, more innovation was introduced by Yaacob et al. (2024), who introduced a Dombi based BNS using HM operators. This structure was a combination of the flexibility of the Dombi operator and the expressive power of the bipolar neutrosophic environment that can successfully operate positive and negative information when uncertainty arise. The HM operator balanced the aggregation where the correlation between bipolar components is taken into consideration. Continuing into 2025, Tripura et al. (2025) proposed a geometric HM operator within picture fuzzy set and extended the scope of HM-based aggregation into fuzzy settings with an even greater uncertainty and hesitation. Membership, non-membership, and the neutrality degrees were proportional to be represented by the geometric formulation.

The latest integration is by Bilal et al. (2025), who came up with quasi-rung orthopair fuzzy based generalized HM operator. The innovation is the newest development in HM operator integration, whereby it promotes the ability of the operators to handle the complex and layer uncertainty by using the advanced FS

structures. This integration was more accurate and stable compared to existing AOs in large scale and high dimensional MCDM problems by accommodating more flexible membership and non-membership relationships.

These contributions together show the growing versatility and flexibility of HM operator in terms of fuzzy and neutrosophic extensions. The HM operator group has become an asset in the current modern theory of aggregation by capturing the nonlinear interactions and confounding intercriteria dependencies. It increases the interpretability, strength, and accuracy of MCDM models especially when uncertainty and incomplete information are present. The HM operator becomes one of the main elements that allow representing intercriteria correlations and enhance the aggregation procedure.

2.7 Application of MCDM and AOs in Learning English Vocabulary

English has become the global second language since its simplicity, versatility, and acceptance across nations. The skill of studying and learning English as a second language has been critical field in facilitating the interaction between diverse cultures and professions as a communication tool. Vocabulary acquisition is one of the aspects of language learning that are central to the acquisition of learning, speaking, reading, and writing proficiency (Chiew & Ismail, 2021). A strong vocabulary foundation not only enhances language comprehension but also enables learners to communicate more accurately and confidently in different contexts. Lack of vocabulary knowledge usually makes the learners unable to understand texts, convey ideas correctly, and blend in the communication process.

The reason why gaining vocabulary does not only involve the application of memorization strategies but rather a complex mental process which requires the selection and implementation of the relevant learning strategies. Cognitive, metacognitive, memory-based and social strategies vary in effectiveness according to learners' proficiency, motivation and learning environments. Understanding the strengths and limitations of these strategies can help educators design more effective vocabulary instruction tailored to different learners' needs. It is crucial to discover and comprehend the ways these strategies can collaborate with the individual characteristics of learners and ensure greater efficiency and sustainability of vocabulary learning. Table 2.7 summarises the existing English language evaluation studies by employing FS and NS theories with MCDM models.

Table 2.7

Author's Contribution to English Evaluation Studies using MCDM and AOs

Author(s)	Year	Description	Set theory	Method
He	2025	Evaluation of English language and literature quality	HSS and SVNS	CRITIC
Huang	2025	Developed and optimized teaching technique to enhance English vocabulary teaching	Fuzzy decision system	N/A
Wen & Pan	2025	Evaluate the effectiveness of fuzzy logic in improving English learning outcomes	Fuzzy Bayesian	N/A
Fu & Pan	2024	Proposed a decision model for mobile English education	N/A	Network DEA
Zhao	2024	Introduced IT solution for English language	Fermatean fuzzy	CRITIC and AROMAN
Wu & Luo	2023	Design an improved fuzzy model to evaluate English teaching quality	Fuzzy logic	AHP
Wang	2021	Evaluation of English teaching quality	IVIFS	CODAS and CRITIC

According to the Table 2.7, He (2025) developed a Hyper Soft Set (HSS)-based MCDM framework integrated with NSs and the CRITIC weighting method to evaluate English language and literature teaching quality under uncertainty. The study found that pedagogical competence of English language educators was the most important evaluation criterion, demonstrating that the proposed approach provides an objective and reliable framework for teaching quality assessment in higher education. To enhance the efficiency of teaching English, Huang (2025) developed Progression Teaching System (PTS) with the help of Fuzzy Decision (FD) approach. The findings showed that the proposed model enhanced lexical precision, reduced lexical placement errors, provided personalized learning recommendations, and improved teaching efficiency while reducing computational time. The system also generated personalized recommendations for learners with lower precision levels.

Wen & Pan (2025) compare the effectiveness of the Fuzzy Bayesian Intelligent Tutoring System (FB-ITS) to enhance the outcomes of English learning activities in comparison with the traditional practices. These findings demonstrated that, in general, students on FB-ITS spent less time to fulfil the post-test compared to students on the traditional e-learning system. In the English language teaching field, the modern state of the art AI systems like the FB-ITS demonstrate a significant improvement in the

learning effectiveness. To enhance both the learning process and effectiveness of teaching English in higher education, Wu & Luo (2023) developed a better Fuzzy AHP (IF-AHP) model that relies on the game-based approach to improve the quality of teaching English. The findings confirm that the proposed IF-AHP approach increases student engagement ratio of 98.5 percent, teacher knowledge level ratio of 97.9 percent, teacher performance ratio of 98.9 percent, accuracy ratio of 96.7 percent and teacher involvement ratio of 95.5 percent over the other existing methods.

Furthermore, Fu & Pan (2024) design a network DEA model to optimise decision-making in mobile English language teaching within an information technology environment. The proposed framework systematically evaluated the efficiency of multiple stages in the teaching decision process, including educational stimuli, motivation determination, scheme evaluation, decision-making, and subsequent learning behaviour, providing a comprehensive approach for improving teaching effectiveness. Zhao (2024) carries out a systematic and thorough assessment of the numerous IT-based alternatives that can be adopted in classes of English as Second Language. Each of these alternatives is thoroughly considered based on eight criteria. These requirements are as follows, cost-effectiveness, flexibility, customization, security, integration, and scaling. One of the key elements of the DM process is the CRITIC-AROMAN method which operates within a Fermatean Fuzzy (FF) setting. This study is based on six major criteria and 23 sub-criteria that can be considered and choose the criterion that is of utmost importance in DM problem. The criteria are assessed on three experts. The findings reveal that Pedagogical Competence of Educators of English Language has the best rate.

Whereas some studies have used intelligent and fuzzy model like FD, FB-ITS, IF-AHP, and network DEA to improve English teaching and learning quality, there is still a conspicuous gap in research that uses MCDM techniques or AOs to appraise and rank performance in studying or teaching English vocabulary. The vocabulary acquisition learned through MCDM or AO frameworks is not the topic of systematic and criteria assessment of English education, but most of the existing research is concerned with more general aspects, like the quality of teaching, the efficiency of a system, or the learning process. Limited attention has been given to developing comprehensive DM frameworks that can evaluate and prioritise English vocabulary learning strategies. Figure 2.2 illustrates the selected criteria in evaluating English vocabulary learning strategies.

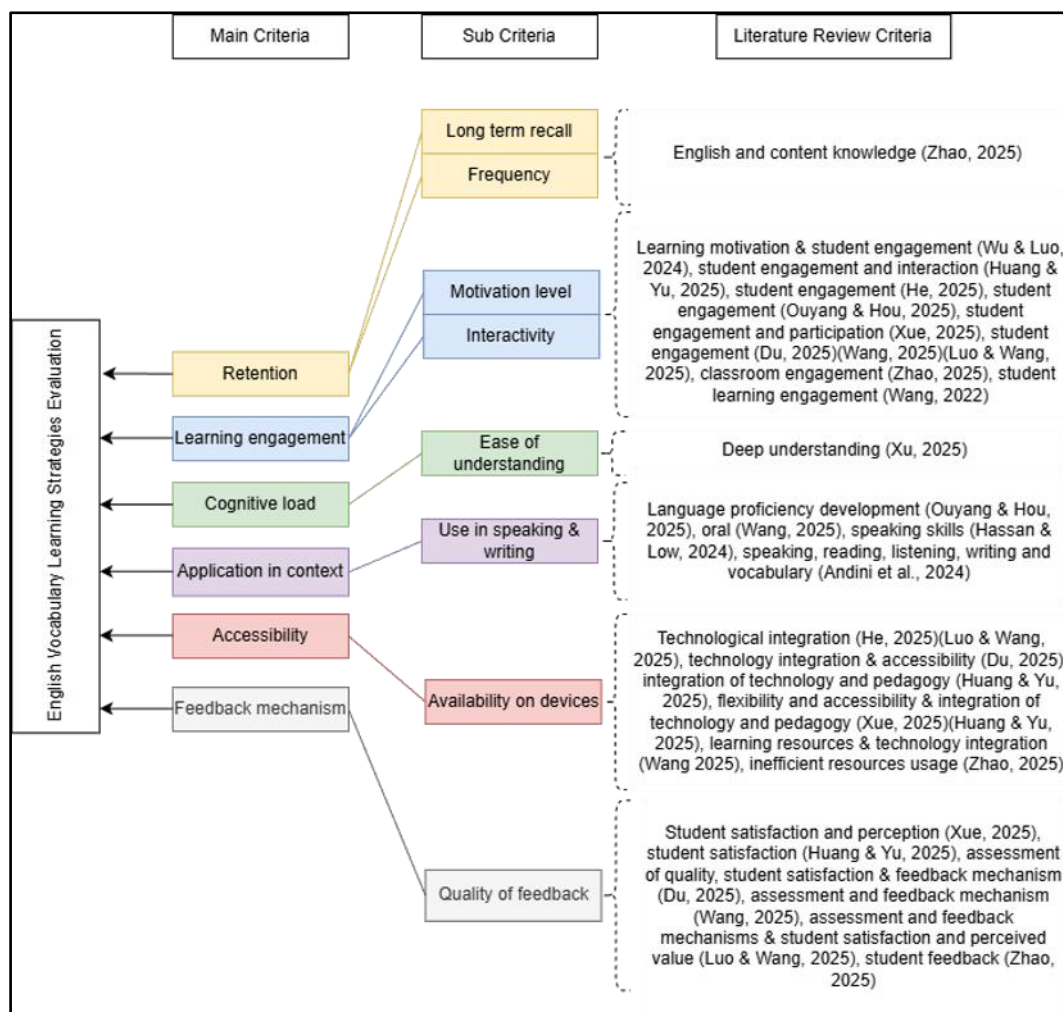


Figure 2.2 Author's Contribution to Application of Selected Criteria for English Vocabulary Learning Strategies

Based on Figure 2.2, the selected criteria used in this study are summarized and visualized. Six main criteria such as retention, learning engagement, cognitive load, application in context, accessibility and feedback mechanisms are now considered as the necessary components in assessing English vocabulary learning strategies. These criteria were established through a synthesis of recent literature and collectively reflect the cognitive, behavioural, and technological factors that influence vocabulary acquisition.

The studies of vocabulary and content knowledge, Zhao (2025) point to the process of vocabulary development boosting the long-term recall and leading to a more profound academic comprehension. This suggests that effective vocabulary learning strategies should promote durable retention rather than short-term memorisation. Empirical research on the topic of interaction and engagement, (Du, 2025; He, 2025;

Huang & Yu, 2025; Luo & Wang, 2025; Ouyang & Hou, 2025; Wang, 2022; Wang, 2025; Wu & Luo, 2023; Xue, 2025; Zhao, 2025) repeatedly proves that interactive and motivating learning environments contribute to effective vocabulary retrieval and long-term involvement of learners. These interaction related aspects ensure that vocabulary knowledge is more significant and social thus supporting long term memory.

Mastery of vocabulary is another process dependent on cognition. It demonstrates that processes associated with inferencing, semantic elaboration, and conceptual clarity decrease cognitive load and enhance profound understanding (Xu, 2025). Consequently, vocabulary learning strategies should facilitate meaningful understanding without overwhelming learners' cognitive capacity. Research on speaking abilities (Hassan & Low, 2024), vocabulary-specific learning (Luo & Wang, 2025), oral proficiency (Wang, 2025), and general vocabulary development (Andini et al., 2024) consistently shows that contextual usage bridges the gap between passive word recognition and active language production.

Other than that, vocabulary learning strategies are also more effective when there are accessibility and feedback mechanisms. Research on mobile learning, digital platforms, and the integration of the pedagogical system and technology, (Du, 2025; He, 2025; Huang & Yu, 2025; Luo & Wang, 2025; Wang, 2025; Xue, 2025; Zhao, 2025) suggests that available tools offer flexible and individualized practice opportunities and exposure to a larger amount of vocabulary. This accessibility enables learners to engage in continuous vocabulary practice regardless of time and location, thereby promoting greater learning flexibility and consistency.

Furthermore, studies of feedback and assessment, (Du, 2025; Huang & Yu, 2025; Luo & Wang, 2025; Wang, 2025; Xue, 2025; Zhao, 2025), indicate that prompt, concise, and positive feedback can help to control self-regulation and achieve better vocabulary results. Effective feedback also enables learners to identify their vocabulary weaknesses, monitor their learning progress, and make appropriate adjustments to improve language performance over time. When combined, these studies provide a comprehensive framework where retention, engagement, cognition, contextual application, accessibility and feedback interact to influence effective vocabulary acquisition. As such, the definition of English vocabulary learning strategies applied in this study are demonstrated in Table 2.8.

Table 2.8
 Author's Contribution to the Definitions of Key Criteria for Assessing English Vocabulary Learning Strategies

Main criteria	Sub-criteria	Description	Academic sources
Retention	Long term recall	How well students can retrieve key knowledge weeks or months later	Ilxomovich (2024)
	Frequency	How often students revisit or practice the content	Gantari (2024)
Learning engagement	Motivation level	How strong and sustained learners' interest is to use the tool and complete activities over	Huseinović (2023)
	Interactivity	Two-way interaction with content, peers, or system	Eli (2021); Zaynitdinovna, (2024)
Cognitive load	Ease of understanding	Clarity and simplicity that reduce mental effort to grasp concepts quickly	Asma & Dallel (2020)
Application in context	Use in speaking & writing	How well the tool enables applying knowledge in real tasks	Albogami & Algethami (2022); Criollo-C et al. (2022); Crompton et al. (2024); Rosyada-AS & Apoko (2023)
Accessibility	Availability on devices	Reliable access across phone/table/desktop, including bandwidth or offline support	Xodabande & Hashemi (2023)
Feedback mechanism	Quality of feedback	Timely, specific and actionable guidance on what to improve and how	AlGhamdi (2024); Mun (2024)

Table 2.8 synthesizes key dimensions and sub-criteria derived from recent literature on English vocabulary learning strategies between 2020 and 2024. The criteria that were identified based on retention and engagement to accessibility and feedback depict the complexity of vocabulary acquisition. One of the most essential areas of learning a vocabulary is retention. It demonstrates the capacity of the learners to remember and recall words correctly after some time. According to Ilxomovich (2024) long-term recall is one of the factors that establish the ability of the learners to memorise the lexical information weeks or months after the exposure which is the sign of the profound and stable learning. To add to this point, Gantari (2024) emphasizes the frequency factor, stating that the more frequently the information is reviewed, the

stronger the memory traces become and the more automatic they are recalled. The ability to transfer words between short term and long-term memory is enhanced by frequent use of vocabulary in writing activities, reading or using the digital learning tools. Collectively, these results underscore the fact that long-term exposure and repetitive recall are critical in long-term vocabulary memory retention. Moreover, the involvement in learning also contributes to the improvement of the vocabulary learning results.

Furthermore, Huseinović (2023) says that the high level of motivation keeps the learners interested and motivated, and the persistence can result in the increased regularity of involvement in the vocabulary activity. Motivation makes learners engage more with materials, and to press further through obstacles with learning. Moreover, Eli (2021) and Zaynitdinovna (2024) underline the importance of interactivity as a way of encouraging active participation. Two-way communication with content, peers or learning systems stimulates the learners to use new words in a meaningful context. Interactive resources like games or shared digital environments, can be useful to turn the passive process of vocabulary learning into the active one, making it easier to remember and engage.

Cognitive load is another important factor as it has to do with the mental effort needed to process and understand new information. Asma & Dallel (2020) indicate that simplicity of comprehension might greatly influence the capacity of learners in the efficient acquisition of vocabulary. Learners can have reduced cognitive strain when learning materials are well-organized and the concepts are explained in simplified forms. This enables them to concentrate on learning and internalizing the new words and not being taken up by the process of learning itself. By minimizing cognitive load, which is unnecessary, focus, understanding, and formation of long-term memory is increased.

The contextual use of vocabulary is also significant in changing the passive knowledge to the active language use. The articles are based on the studies, Albogami & Algethami (2022); Criollo-C et al. (2022); Crompton et al. (2024); Rosyada-AS & Apoko (2023) that emphasize the importance of vocabulary when learning speaking and writing activities. Using acquired words in real contexts like essays, discussions or a presentation will aid in cementing the knowledge and the interrelation between the form, meaning, and application. Contextual practice is not solely conducive towards

memorization, but also functional fluency, so that learners can access and use words in a natural way when communicating.

The criteria of accessibility also affect the effectiveness of the learners when engaging with the vocabulary materials language. According to Xodabande & Hashemi (2023), devices availability allows learners to access vocabulary materials anywhere and at any time even when accessing to low-bandwidth or offline conditions. This flexibility and self-paced nature of learning facilitating such access promotes continuous interaction with learning content. By means of the ability to easily reread the materials using numerous gadgets, students can be expected to sustain their studying patterns and ensure a higher level of development in their vocabulary.

Finally, the feedback mechanism is one of the key factors in aiding successful vocabulary acquisition. AlGhamdi (2024) and Mun (2024) characterize that quality of feedback that is determined by its timeliness, specificity and clarity is a decisive factor in helping the learners to improve. Positive feedback helps learners to note the mistakes, track and use corrective measures in vocabulary. Immediate and actionable feedback encourages a metacognitive awareness that helps learners to contemplate their learning strategies and make the appropriate changes. This continuous feedback loop enhances both learner autonomy and vocabulary proficiency over time.

All these factors define a holistic assistive model that includes psychological and technological factors of successful vocabulary acquisition. There have been several different instructional methods that are suggested to improve the learning of the English Language, and these different methods have their own definite benefits in language proficiency, engagement, and retention. Traditional teaching methods offer structured teaching and basic language skills, and contextual lessons focus on the use of language in real-life contexts. Gamification and the use of multimedia have attracted the attention and interest of researchers because they are seen to enhance learner motivation and engagement by using interactive and technology-based activities. In recent years, AI-powered learning has become a personalized learning strategy that offers real-time feedback and adaptive assistance. Mnemonic techniques are aimed at better word retention and memory while peer-assisted learning involves collaborative and communicative learning. Given the growing diversity of instructional approaches, it is important to examine evidence from previous studies regarding their effectiveness in English language learning. A summary of relevant past studies investigating these alternatives is presented in Table 2.9.

Table 2.9
Author's Contribution to Evaluating English Vocabulary Learning Strategies

Alternative	Supporting literature	Findings	Relevance to current study
Traditional method	Owen, (2025)	The traditional model of classroom instructional interventions remains successful in improving the accuracy of writing and the proficiency of speaking, as teacher-guided interactions and peer interactions are more effective in supporting productive language skills development	Provides a reference for assessing innovative vocabulary learning strategies and identifies strengths and weaknesses that should be considered in the assessment framework
Contextual learning	Haque et al. (2025)	Contextual language learning techniques affect in enhancing students' confidence in learning and self-study, which reduces the students' reliance on using dictionary and other external aids	Explains the value of motivating contexts in learning vocabulary and defines indicators for evaluating vocabulary learning strategies
Gamification method	Mythili et al. (2026)	Gamification helps to build up vocabulary knowledge and communicative competence as well as creates a learner-centric and low stress learning environment	Proposes the inclusion of the three key skills of engagement, motivation and communication among the important assessment criteria
Artificial intelligence	Jomaa et al. (2024)	The quantitative data showed that the English instructors have a positive attitude toward the advantages of the use of using AI tools in teaching English	Validates the use of technology approach and adaptive approach in assessing the vocabulary learning strategies
Mnemonic	Gonzaga et al. (2026)	There was a highly significant difference increase in the students' retention of English vocabulary and concepts before and retention after the implementation of the application of mnemonic strategy	Supplies proof of vocabulary retention as measure of strategic efficacy
Multimedia	Raghavendra & Deepakumari, (2025)	The findings strongly support the integration use of innovative, learner-centric approaches centred instructional strategies, such as using multimedia instruction	Acknowledges the importance of considering the various needs of learners and learning support
Peer	Jalalzai et al. (2023)	English second language teachers use a variety of positive peer feedback techniques that enable students to improve their language skills and gain in vocabulary	Supports the use of collaborative learning and language reinforcement factors in a vocabulary learning strategies assessment

The results of the English vocabulary learning strategies previously studied were summarized in Table 2.9. The literature examined has shown that there are different pedagogical methods that can be used to facilitate vocabulary learning, such as traditional method, contextual method, gamification, AI-supported learning, mnemonic method, multimedia learning, and peer-assisted method. While these methods vary in their instructional design, they all aim to improve learners' vocabulary learning and motivation. According to Goyibova et al. (2025), there is not a single best instructional method that can be applied to all learning environments because all learners have varying cognitive, motivational and technological needs.

On closer inspection of the literature, each strategy highlights other aspects of the learning process. While traditional methods are mostly geared towards structured language and vocabulary acquisition, contextual learning emphasizes using vocabulary in meaningful contexts. AI-assisted learning emphasizes personalization and adaptive feedback, and gamification is often linked to higher learner motivation and engagement, and multimedia learning is linked with enhanced student motivation and learning engagement. Mnemonic methods are often associated with vocabulary acquisition skills, while peer learning involves group participation and social interaction among pupils. The effectiveness of vocabulary learning is not only determined by what is being taught in the classroom but also by how well a strategy allows the students to be engaged and participate in learning.

Although there is a growing body of research, most studies have a specific learning context and test these strategies individually. Comparisons across strategies are rather limited and the evaluation is frequently limited to just one outcome measure, such as vocabulary retention, motivation or language proficiency. As noted by Oc & Hassen (2025), single factor assessments influence the strengths and weaknesses of the different learning strategies. However, existing evidence may be useful for understanding the effectiveness of individual strategies and is less useful in determining which strategies are most suited where multiple educational factors are present at the same time.

Moreover, past research studies usually do not combine pedagogical, technological, cognitive and social dimensions in the same framework. This is in line with the studies of Simonnet et al. (2025) who claimed that vocabulary learning outcomes are affected by several dimensions that are not affected by only one element of instruction. Likewise, Saito & Uchihara (2025) noted that studies on vocabulary learning usually tend to examine the development of vocabulary knowledge in relation

to a single learning goal or criterion, instead of multiple evaluation criteria. While Owen (2025) discussed some issues associated with English vocabulary learning, few comprehensive comparative studies have been conducted. Learners have been faced with difficulties in choosing educational strategies when evidence is scattered across the various dimensions of evaluations as noted by Wendy & Rosli (2023).

Therefore, there is no systematic MCDM framework that can evaluate and rank vocabulary learning strategies in an uncertain and multidimensional condition. Multiple scholars have thus called for more comprehensive evaluation methods that would be able to capture a number of criteria and uncertainty in the context of educational decisions (Maral & Özdemir, 2025b; Srivastava et al., 2024). Based on the research gap identified, a systematic research framework was developed to assess and rank English vocabulary learning strategies using the proposed MCDM method and advanced AOs under complex and uncertain environment.

2.8 Research Methodology

This section presents the detailed of three phases illustrated in Figure 2.3 where each phase related to the research objectives respectively.

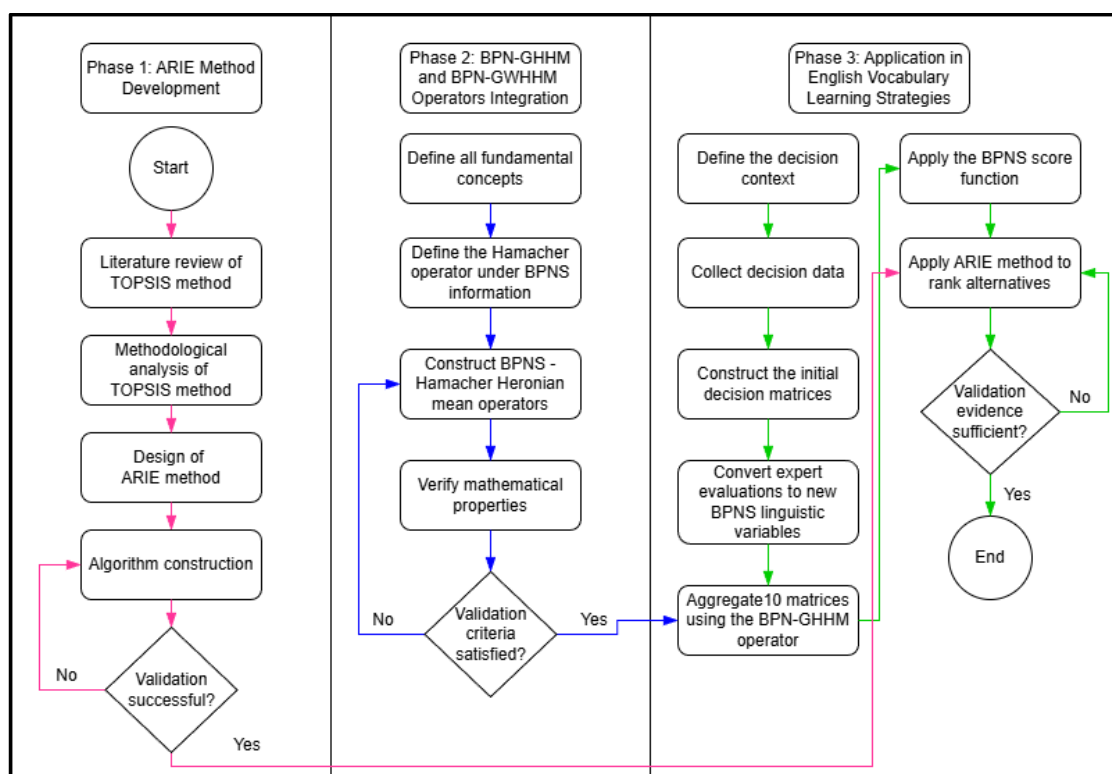


Figure 2.3 Research Methodology Phases

Based on Figure 2.3, Phase 1 focuses on the development of modified MCDM method known as the ARIE method, which is conceptually inspired by the widely recognized classical TOPSIS method. This phase serves as the theoretical foundation of the entire study, where the structural components, mathematical formulations, and operational logic of the ARIE method are systematically designed. The development process involves examining the limitations of TOPSIS and its extensions by identifying the essential computational requirements for handling uncertainty and formulating a DM approach capable of adapting to various data characteristics. Through this process, the ARIE method is constructed to enhance DM performance by improving stability, reducing sensitivity to extreme values, and offering greater interpretability, while still preserving the core strengths and intuitive appeal of existing MCDM frameworks. The resulting model provides a refined alternative that is both theoretically grounded and practically applicable across diverse decision environments.

Phase 2 involves the integration of the AOs with the BPNS model, aiming to improve the treatment of uncertainty and vagueness in DM contexts. This integration ensures the model can accommodate more nuanced and flexible input information, thereby increasing its applicability and robustness. Although this study introduces both the BPN-GHHM and BPN-GWHHM operators, only the BPN-GHHM operator is applied in the application of Phase 3. This choice is made to maintain methodological clarity and to avoid redundancy, as the BPN-GHHM operator is sufficient for demonstrating the applicability of the ARIE based aggregation in the selected English vocabulary learning dataset. The BPN-GWHHM operator is conceptually introduced as an extended form, but it is not implemented in the empirical analysis due to the targeted scope of Objective 3.

In Phase 3, the application of the proposed models demonstrated through a comprehensive application in the education field. In this phase, ARIE method, in combination with the AOs operator and BPNS linguistic variable, is applied to real-world DM scenarios, illustrating how the proposed methodology can provide meaningful and actionable insights. Each phase of the study has been rigorously verified and validated through multiple experimental analyses, including comparative analysis against established methods, sensitivity analysis to assess stability under varying conditions, and statistical validation using Spearman correlation coefficients to evaluate consistency and reliability.

CHAPTER 3

ADAPTIVE RANKING WITH IDEAL EVALUATION (ARIE) METHOD

3.1 Introduction

This chapter focused on the introduction and development of a modified MCDM method that is ARIE method conceptually based on the traditional TOPSIS method. Section 3.2 presents the general development process of ARIE. Section 3.3 presents the detailed formulation and computational steps of the proposed ARIE method in an illustrative example. Section 3.4 discuss the sensitivity analysis, while Section 3.5 presents the comparative analysis of the proposed model with other MCDM methods as well as validation through Kendall's τ coefficients with permutation test. This chapter concluded in Section 3.6.

To make explicit the logical evolution from TOPSIS to the proposed method, ARIE retains the ideal-solution philosophy of TOPSIS but restructures its core computational steps. Classical TOPSIS proceeds through vector normalization, construction of the weighted normalized matrix, computation of Euclidean distances to the PIS, NIS, and ranking by RC. ARIE preserves the normalization and weighting stages but replaces the distance-to-ideal and RC steps with a target-based normalization and a similarity measure and further introduces dual adjustable sensitivity parameters to control ranking stability. This restructuring directly addresses the inconsistency of the RC measure and the compensatory behaviour identified in the problem statement. In this sense, the ARIE method is the modified TOPSIS approach named in the title of this thesis, and the acronym ARIE is used throughout the remaining chapters for brevity.

3.2 Proposed Adaptive Ranking with Ideal Evaluation (ARIE) Method

Inspired by the advantages and weaknesses of the classical TOPSIS technique, this study proposes an extended version of TOPSIS termed as the ARIE method. Although TOPSIS method offers an intuitive and widely used approach for ranking alternatives using the concept of ideal and anti-ideal solution, its use of linear distance measures and the assumption of complete compensability among criteria can restrict its

effectiveness in more complex and realistic decision problems. In particular, these assumptions may oversimplify interactions among criteria and reduce the method's sensitivity in distinguishing closely competing alternatives. The proposed ARIE approach maintains the conceptual basis of ideal-based evaluation, while introducing a new decision logic to address these limitations. It aims to improve flexibility, possess more discrimination power in modelling MCDM scenarios. By refining the evaluation mechanism, the ARIE method seeks to overcome certain structural constraints of TOPSIS and better capture the nuances of real-world decision problems. To achieve an easier comprehension of the offered framework, the general methodological process of the ARIE method illustrated in Figure 3.1.

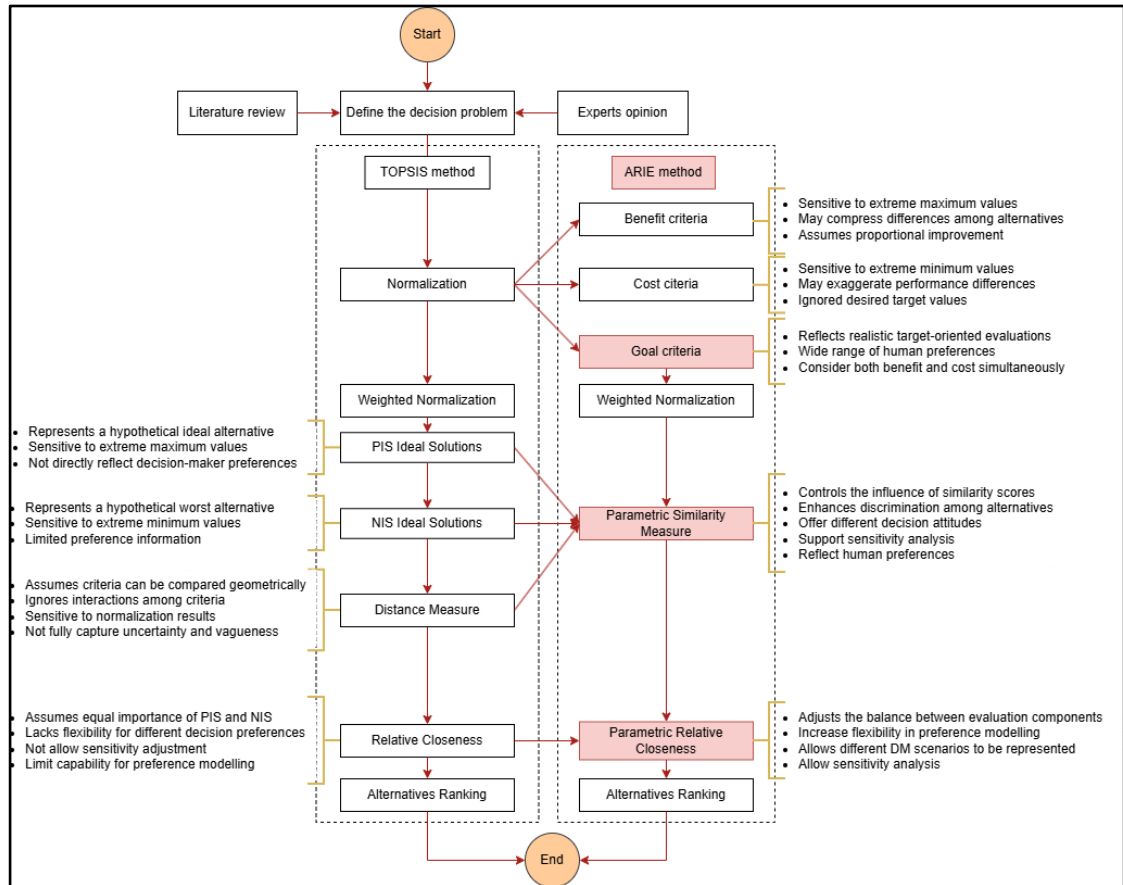


Figure 3.1 Flowchart of the TOPSIS and Proposed ARIE Method

Overall framework of the proposed ARIE method is shown in Figure 3.1. The red-coloured boxes highlight the methodological enhancements introduced by incorporating concepts from the TOPSIS method into the original ARIE method, representing the key contributions of this study. The initial stage of the proposed

development process involves conducting a thorough literature review and consulting with domain experts to establish the decision problem being addressed and the requirements for the proposed DM model. After a review of the literature, several limitations of the conventional TOPSIS method have been identified which is sensitivity in extreme values of the normalized data, the inability to explicitly include target criteria and the restriction of the conventional RC measure in terms of flexibility. The problems may affect the ability of TOPSIS to model practical DM problems with different evaluation preferences. TOPSIS method is applied as the mathematical basis, and the corresponding mathematical calculation steps are sequentially analysed and then redefined to build the proposed ARIE method. In this section, the mathematical formulations of TOPSIS are therefore presented and the corresponding ARIE formulations are presented to strengthen the identified limitations in TOPSIS. The step-by-step presentation clearly illustrates the logical progression of the mathematical transformation from the conventional TOPSIS to the proposed ARIE framework.

The proposed ARIE method involves the following steps, as illustrated in Figure 3.1, such as decision matrix formulation, normalization, weighted normalization, parametric similarity measurement, parametric RC computation, and alternative ranking. In contrast to the traditional TOPSIS approach, ARIE explicitly models benefit, cost, and goal criteria by using aspiration-based normalizations, and adds parametric similarity and RC formulations to offer the flexibility in the modelling of the decision-maker's preferences and enhance the discrimination between the alternatives. The theoretical motivation, advantages and disadvantages of these reformulated components have been critically discussed in the previous chapter. The next subsection concentrates on the mathematical development of each ARIE formulation from its respective TOPSIS formulation.

3.2.1 Decision Matrix

The construction of the decision matrix in the proposed ARIE method follows the same procedure as the conventional TOPSIS method. Since this step simply organizes the performance values, x_{ij} , of alternatives, $i = (1, 2, \dots, m)$, with respect to the evaluation criteria, $j = (1, 2, \dots, n)$, and does not influence the subsequent methodological improvements, no modification is required.

$$X = [x_{ij}]_{m \times n} \quad (3.1)$$

3.2.2 Normalization

TOPSIS consists of dimensionless values of the original decision matrix, x_{ij} , obtained by vector normalization, n_{ij} , that related to the Euclidean norm of the vector.

$$n_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^m x_{ij}^2}}$$

Vector normalization allows comparison of criteria measured in different units while maintaining the proportionality between alternatives. However, there are several limitations of vector normalization. The normalization process in TOPSIS only differentiates between benefit and cost criteria in a unified model,

$$n_{ij} \in \{\text{benefit, cost}\}$$

Which assumes that all the criteria are to be maximized or minimized. This is limiting for target or desired value by decision-makers in the real-world applications where the desired performance based on the set value rather than the maximum or minimum value. For the denominator of vector normalization,

$$\sqrt{\sum x_{ij}^2}$$

Which is dependent on the size of all observations. As a result, criteria with very high observations can distort the normalized values of the other alternatives, making it difficult to distinguish between some moderate performances. In addition, the normalized values generated by the vector normalization have meaning only with respect to relative size, not decision objectives. Hence, this normalization approach lacks in measuring the level of proximity of an alternative to a desired target explicitly. The normalization stage plays a critical role in MADM because the selected normalization procedure directly influences the entropy weights, ranking stability, and final decision outcomes (Roszkowska, 2026). Therefore, to overcome these limitations, this study proposes a unified normalization model that contains three criterion types simultaneously,

$$n_{ij} \in \{\text{benefit, cost, target}\}$$

Compared with the conventional TOPSIS method, the proposed normalization method clearly considers target-type criteria for which optimal values are predetermined as desired values rather than maximum or minimum values. Many practical decision problems have criteria for which the optimum value is not a maximum or minimum, but a target value. The inclusion of the explicit target-type criteria therefore offers more modelling flexibility in the context of normalization. The normalized value, n_{ij} , for each decision matrix, x_{ij} , in ARIE is defined as follows,

i. Max-type (benefit) criterion:

$$n_{ij} = \frac{x_{ij}}{x_j^{\max}} \quad (3.2)$$

ii. Min-type (cost) criterion:

$$n_{ij} = \frac{x_j^{\min}}{x_{ij}} \quad (3.3)$$

Although linear normalization is widely used in MADM approaches, it is found that there is a lot of criticism regarding the dependence on extreme values and transformation of scales of original data. However, the normalized approach is proposed as it offers a straightforward and efficient way of transforming heterogeneous criteria into a common dimensionless scale. Using maximum and minimum values provides consistent reference limits for the evaluation of alternative performances in terms of benefit, cost and goal.

Specifically, when using goal criteria, an assessment is based on how close an alternative is to the desired goal value, and not so much on the extremes themselves. This new addition enables a more realistic and wider representation of human judgment during DM processes. There are real life scenarios where certain criteria cannot be purely divided as either beneficial, in which case higher values are always desirable or cost in which case lower values are always desirable.

As an example, the purchase of a car. Normally, criteria that are evaluated in buying a care include engine performance, fuel efficiency, price, and cost and maintenance. Though the performance of the engines and fuel consumption can be considered as benefit criteria and cost criteria respectively, while the price criterion can be viewed as a dilemma. The high price can be associated with high quality and reliability of the engine and, consequently, it can lead to an increased financial load and maintenance costs. On the contrary, low car price would lower the short-term costs yet may compromise quality and a greater cost of maintenance in the long run. Thus, such cases when there is a desired optimal target value that is not too high or too low are applied with the goal criterion. The new criterion in ARIE defined as,

iii. Target-type (goal) criterion:

$$n_{ij} = 1 - \frac{|x_{ij} - x_j^T|}{\max\left(|x_j^{\max} - x_j^T|, |x_j^{\min} - x_j^T|\right)} \quad (3.4)$$

The following properties describe the mathematical behaviour of the proposed normalization formula and demonstrate its suitability for the ARIE method.

3.2.2.2 Properties of the Proposed Target Normalization

Property 3.1 (Identity) Let $n_{ij} = 1 - \frac{|x_{ij} - x_j^T|}{\max\left(|x_j^{\max} - x_j^T|, |x_j^{\min} - x_j^T|\right)}$, where

$$D_j = \max\left\{|x_j^{\max} - x_j^T|, |x_j^{\min} - x_j^T|\right\} > 0.$$

Proof Assume $x_{ij} = x_j^T$. By the definition of the absolute value,

$$|x_{ij} - x_j^T| = |x_j^T - x_j^T| = 0. \text{ Thus,}$$

$$\begin{aligned} n_{ij} &= 1 - \frac{|x_{ij} - x_j^T|}{D_j} \\ &= 1 - \frac{0}{D_j} \\ &= 1. \end{aligned}$$

Hence, the proposed target normalization satisfies the identity property.

Property 3.2 (Boundedness) Let $n_{ij} = 1 - \frac{|x_{ij} - x_j^T|}{\max(|x_j^{\max} - x_j^T|, |x_j^{\min} - x_j^T|)}$, where

$D_j > 0$. Let $0 \leq n_{ij} \leq 1$.

Proof If $x_{ij} \in [x_j^{\min}, x_j^{\max}]$, then

$$\begin{aligned} 0 &\leq |x_{ij} - x_j^T| \leq D_j \\ 0 &\leq \frac{|x_{ij} - x_j^T|}{D_j} \leq 1 \\ 0 &\leq 1 - \frac{|x_{ij} - x_j^T|}{D_j} \leq 1 \end{aligned}$$

Thus, $0 \leq n_{ij} \leq 1$.

Hence, every normalized value is contained within the unit interval, and therefore the proposed normalization satisfies the boundedness property.

Property 3.3 (Symmetry) Let x_a and x_b be any two criterion values satisfying

$$r(x_a) = 1 - \frac{|x_a - x_j^T|}{D_j} \text{ and } r(x_b) = 1 - \frac{|x_b - x_j^T|}{D_j}.$$

Proof Let $|x_a - x_j^T| = |x_b - x_j^T|$, then,

$$\begin{aligned} r(x_a) &= 1 - \frac{|x_a - x_j^T|}{D_j} \\ &= 1 - \frac{|x_b - x_j^T|}{D_j} \\ &= r(x_b) \end{aligned}$$

Hence, the proposed target normalization satisfies the symmetry property. ARIE method is more capable of accommodating ambiguous or wavering judgment, improving the model flexibility and strength in responding to realistic preferences of decision-makers.

3.2.3 Weighted Normalized

The weighted normalized decision matrix in ARIE, v_{ij} , remain similar of TOPSIS, is obtained by multiplying normalized values according to corresponding weights, w_j , is as follows,

$$v_{ij} = w_j \cdot n_{ij}, \quad (3.5)$$

Here, $\sum_{j=1}^n w_j = 1$. The weights represent the relative significance of each criterion which may be established by the opinion of the decision-makers, objective methods such as entropy or MMethod of Removal Effects of Criteria (MEREC) or subjective methods such as AHP or BWM. This step of adding weights is what will provide the basis of insightful similarity computations and incorporate the preferences of the stakeholders or prioritized data. Moreover, the preservation of the environment helps to keep comparisons between options clear and understandable.

3.2.4 Similarity Measure

In TOPSIS, the ideal (PIS) and anti-ideal (NIS) solutions where J_1 denotes for benefit criteria and J_2 for cost criteria are as follows,

$$PIS = \{v_1^+, \dots, v_j^+, \dots, v_n^+\} = \left\{ \max_{\forall i} v_{ij} \mid j \in J_1, \min_{\forall i} v_{ij} \mid j \in J_2 \right\}$$

$$NIS = \{v_1^-, \dots, v_j^-, \dots, v_n^-\} = \left\{ \min_{\forall i} v_{ij} \mid j \in J_1, \max_{\forall i} v_{ij} \mid j \in J_2 \right\}$$

The separation of PIS (D_i^+) and NIS (D_i^-) are based on the Euclidean distance,

$$D_i^+ = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^+)^2}$$

$$D_i^- = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^-)^2}$$

Here, the first weakness in TOPSIS is the distance measures absolute deviation instead of proportional similarity. Euclidean distance computes $|v_{ij} - v_j^+|$. Suppose $v_j^+ = 100$ with two alternatives, $v_{1j} = 99$, and $v_{2j} = 49$. Distances will be,

$$|v_{ij} - v_j^+| = |99 - 100| = 1, \text{ and } |v_{ij} - v_j^+| = |49 - 100| = 51$$

Now suppose another criterion has $v_j^+ = 0.05$ for two alternatives, $v_{3j} = 0.04$, and $v_{4j} = 0.03$. Distances are

$$|v_{ij} - v_j^+| = |0.04 - 0.05| = 0.01, \text{ and } |v_{ij} - v_j^+| = |0.03 - 0.05| = 0.02$$

Although $\frac{99}{100} = 0.99$ and $\frac{0.04}{0.05} = 0.80$. TOPSIS compares 1 with 0.01, which are measures on completely different absolute values. ARIE replaces subtraction in Euclidean distance by ratio. If $v_{ij} = 99$, $v_j^{\max} = 100$, then,

$$\left(\frac{99}{100}\right)^\gamma = (0.99)^\gamma$$

If $v_{ij} = 0.04$, $v_j^{\max} = 0.05$, then

$$\left(\frac{0.04}{0.05}\right)^\gamma = (0.80)^\gamma$$

Both alternatives are now evaluated proportionally. Hence, similarity measures relative agreement rather than absolute deviation. Next, the squaring in Euclidean distance exaggerates large deviations. TOPSIS uses $(v_{ij} - v_j^+)^2$, suppose criterion 1 with $v_{ij} = 9$, and $v_j^+ = 10$. Then,

$$\begin{aligned} (v_{ij} - v_j^+)^2 &= (9 - 10)^2 \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

If criterion 2 with $v_{ij} = 5$, and $v_j^+ = 10$. Then,

$$\begin{aligned} (v_{ij} - v_j^+)^2 &= (5 - 10)^2 \\ &= (-5)^2 \\ &= 25 \end{aligned}$$

Even though criterion 2 differs only five times more, its influence becomes bigger with 25 times. Generally, $(ax)^2 = a^2 x^2$. Hence, Euclidean distance magnifies large deviations quadratically. Therefore, one criterion with a large deviation can dominate

the overall distance. In ARIE, the computation of $\left(\frac{v_{ij}}{v_j^{\max}}\right)^\gamma$ for criterion 1 will be

$\left(\frac{9}{10}\right)^\gamma = (0.9)^\gamma$ and for criterion 2, $\left(\frac{5}{10}\right)^\gamma = (0.5)^\gamma$. The parameter γ controls discrimination,

however the values remain bounded $0 \leq \left(\frac{v_{ij}}{v_j^{\max}}\right)^\gamma \leq 1$. Thus, no criterion can produce arbitrarily large contributions.

For ARIE, similarity measure is used to compare the distance between each alternative and the best (ideal) and worst (anti-ideal) alter natives using a similarity-based approach. Let:

$$Sim_i^{best} = \sum_{j=1}^n \left(\frac{v_{ij}}{v_j^{\max}} \right)^\gamma \quad (3.6)$$

$$Sim_i^{worst} = \sum_{j=1}^n \left(\frac{v_j^{\min}}{v_{ij}} \right)^\gamma \quad (3.7)$$

Where $v_j^{\max} = \max_i v_{ij}$ and $v_j^{\min} = \min_i v_{ij}$. ARIE method proposes parametric similarity measurement to determine how close each alternative is to the adaptive ideal solution and how far away it is to anti-ideal solution. A higher similarity value indicates greater correspondence with the ideal solution, and vice versa. The parametric similarity measure provides greater flexibility in representing decision-makers' preferences and improves evaluation stability. The mathematical properties of the proposed similarity measure are presented as follows.

3.2.4.1 Properties of Similarity Measures

Property 3.4 (Boundedness) For every criterion, let $0 \leq Sim_i \leq 1$.

Proof For benefit criterion, $v_j^{\max} \geq v_{ij}$, then $0 \leq v_{ij} \leq v_j^{\max}$. Hence, $0 \leq \frac{v_{ij}}{v_j^{\max}} \leq 1$. Since

$\gamma > 0$, raising both sides to the positive power preserves the inequality, $0 \leq \left(\frac{v_{ij}}{v_j^{\max}}\right)^\gamma \leq 1$.

Similarly, for the cost criteria, $v_j^{\min} \leq v_{ij}$, thus, $0 \leq \frac{v_j^{\min}}{v_{ij}} \leq 1$, and $0 \leq \left(\frac{v_j^{\min}}{v_{ij}} \right)^\gamma \leq 1$. Hence,

$$0 \leq Sim_i \leq 1.$$

Property 3.5 (Identity) Assume $Sim_i = 1$ if and only if $v_{ij} = v_j^{\max}$.

Proof Suppose $v_{ij} = v_j^{\max}$, then

$$\begin{aligned} Sim_i &= \left(\frac{v_{ij}}{v_j^{\max}} \right)^\gamma \\ &= \left(\frac{v_j^{\max}}{v_j^{\max}} \right)^\gamma \\ &= 1 \end{aligned}$$

Hence, $Sim_i = 1$.

Property 3.6 (Monotonicity) If $v_a > v_b$, then $Sim(v_a) > Sim(v_b)$.

Proof Consider $f(x) = x^\gamma$, where $x \in [0, 1]$. By differentiate, $f'(x) = \gamma x^{\gamma-1}$, since $\gamma > 0$ and $x^{\gamma-1} > 0$, we obtain $f'(x) > 0$. Thus, $v_a > v_b \Rightarrow Sim(v_a) > Sim(v_b)$.

Step 5. Computation of the RC coefficient to rank alternatives

In TOPSIS, the ranking index with $0 \leq RC_i^{(0)} \leq 1$ as follows,

$$RC_i^{(0)} = \frac{D_i^-}{D_i^- + D_i^+}.$$

TOPSIS follows descending order where the highest RC value represents the best alternatives. The classical coefficient assumes $D^- = D^+ = 0.5$, thus, $\frac{\partial RC_i}{\partial D_i} = 0$. Hence, TOPSIS fail to represent different DM attitudes. ARIE introduces $\kappa \in [0, 1]$ and computes,

$$RC_i = \frac{\kappa \cdot Sim_i^{best}}{\kappa \cdot Sim_i^{best} + (1 - \kappa) Sim_i^{worst}} \quad (3.8)$$

This enhancement refines the classical RC formulation employed in the TOPSIS method. When the sensitivity parameter $\kappa = 0.5$, the model operates in a neutral state, assigning equal importance to both the similarity to the ideal solution and the separation from the negative ideal. Conversely, when $\kappa > 0.5$, greater emphasis is placed on proximity to the ideal alternative, reflecting an optimistic or goal-oriented decision perspective. In contrast, a value of $\kappa < 0.5$, prioritizes distance from the worst alternative, aligning with a conservative or risk-averse decision attitude. The resulting RC_i scores are normalized within the range of $(0,1)$, where alternatives are ranked in descending order of RC_i to determine their relative performance. The next section applies the ARIE method to the selected DM problem and provides a detailed calculation process, followed by an analysis of the resulting findings.

3.3 Illustrative Example

ARIE method has been applied in this section based on the Mijalkovski et al. (2024) application. Calculation process and figures are all conducted using Microsoft Excel for Microsoft 365 (Version 2605). The company in the application oversees controlling irrigation and drainage systems across nation is the focus of the study. In addition to maintaining riverbeds and wastewater, its operations guarantee the supply of water for industrial processes, energy production, drinking water utilities, and agricultural activities. The company is run by 14 regional branches that are dispersed throughout various geographic areas. Within these, the most notable and strategically significant branches are, Tikves (A_1), Bregalnica (A_2), Sveti Nikole (A_3), and Skopsko (A_4), stand out as the most influential in managing irrigation and drain system. Several evaluation criteria are utilized to determine the significance of the evaluation including:

- a) Average size to cover and serve municipalities (K_1)
- b) Average annual revenues of the subsidiary company (K_2)
- c) Average distance to all branches (K_3)
- d) Strategic importance of the subsidiary at the state level (K_4)
- e) Average cost of warehouse service (K_5)

- f) Need for new employment (K_6)
- g) Average delivery of spare parts and consumables (K_7)
- h) Need for new facilities (K_8)

Since the coverage area, K_1 , should not be too small or too large, it is considered as a goal criterion. Large coverage areas increase costs and inefficiency, whereas small areas restrict service capacity. K_2 , K_4 and K_7 are classified as benefit criteria, while whereas the remaining factors are viewed as cost criteria, K_3 , K_6 and K_8 . The establishment of a new central warehouse for consumables and spare parts is being explored for these four branches. The calculation process and generated figure were performed using Microsoft Excel for Microsoft 365 (Version 2605). The following provides the step of ARIE method's calculation process,

Step 1. Decision matrix construction

Table 3.1 displays the alternatives' performance scores for each criterion. This matrix serves as the initial input based on (3.1) in the ARIE method.

Table 3.1
Raw Decision Matrix

Alternative	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
A_1	5	79	103	9	1	1	9	1
A_2	5	38	124	7	1	3	7	3
A_3	4	24	101	9	3	1	7	1
A_4	8	23	112	7	3	3	5	3

The value of maximum, minimum, and target were determined on the data presented in the Table 3.1. The calculations for alternative A_1 with respect to the criteria K_1, K_2 , and K_3 are demonstrated below to obtain the normalized values presented in Table 3.2. The remaining values in the table were calculated following the same procedure.

For $n_{11} : x_1^{\max} = 8, x_1^{\min} = 4, x_1^T = 6$

$$\begin{aligned} n_{11} &= 1 - \frac{|5-6|}{\max(|8-6|, |4-6|)} \\ &= 1 - \frac{|-1|}{\max(|2|, |-2|)} \\ &= 1 - \frac{1}{2} \\ &= 0.5000 \end{aligned}$$

For $n_{12} : x_2^{\max} = 79, x_2^{\min} = 23$

$$\begin{aligned} n_{12} &= \frac{79}{79} \\ &= 1.0000 \end{aligned}$$

For $n_{13} : x_3^{\max} = 124, x_3^{\min} = 101$

$$\begin{aligned} n_{13} &= \frac{101}{103} \\ &= 0.9806 \end{aligned}$$

Next, the decision matrix is normalized using the proposed normalization method. The normalization formulas for benefit, cost, and target criteria are given as follows:

Step 2. Normalization

Following the normalization process, the converted values are shown in Table 3.2 to ensure every criterion is presented on an equivalent scale from 0 to 1 by following (3.2)-(3.4).

Table 3.2
Normalized Decision Matrix

Alternative	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
A_1	0.5000	1.0000	0.9806	1.0000	1.0000	1.0000	1.0000	1.0000
A_2	0.5000	0.4810	0.8145	0.7778	1.0000	0.3333	0.7778	0.3333
A_3	0.0000	0.3038	1.0000	1.0000	0.3333	1.0000	0.7778	1.0000
A_4	0.0000	0.2911	0.9018	0.7778	0.3333	0.3333	0.5556	0.3333

Table 3.2 presents the normalized decision matrix. The normalized values are then used to construct the weighted normalized decision matrix in the next step.

Step 3. Weighted normalization

The weighted normalized values in (3.5) are computed by multiplying each normalized value by its corresponding criterion weight. Calculations for v_{11} , v_{12} , and v_{13} are shown below:

For v_{11} :

$$\begin{aligned} v_{11} &= 0.1 \times 0.5000 \\ &= 0.0500 \end{aligned}$$

For v_{12} :

$$\begin{aligned} v_{12} &= 0.15 \times 1.0000 \\ &= 0.1500 \end{aligned}$$

For v_{13} :

$$\begin{aligned} v_{13} &= 0.14 \times 0.9806 \\ &= 0.1373 \end{aligned}$$

Table 3.3 presented the weighted normalized decision matrix resulting from applying the criterion weights.

Table 3.3
Weighted Normalized Decision Matrix

Alternative	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
w_j	0.1	0.15	0.14	0.13	0.12	0.09	0.16	0.11
A_1	0.0500	0.1500	0.1373	0.1300	0.1200	0.0900	0.1600	0.1100
A_2	0.0500	0.0722	0.1140	0.1011	0.1200	0.0300	0.1244	0.0367
A_3	0.0000	0.0456	0.1400	0.1300	0.0400	0.0900	0.1244	0.1100
A_4	0.0000	0.0437	0.1263	0.1011	0.0400	0.0300	0.0889	0.0367

The weighted normalized decision matrix in Table 3.3 is then used to calculate the similarity measures of each alternative with respect to the ideal and anti-ideal solutions.

Step 4. Similarity measure

Here, by letting $\gamma = 0.6$ with the values of $\nu^{\max}$ and $\nu^{\min}$ are obtained from the Table 3.3, the similarity to ideal-solution and anti-ideal solution based on (3.6) and (3.7) are shown. Sensitivity parameter value for $\gamma \in (0, \infty)$ can be assigned at any value according to the decision-makers' preferences.

$$\begin{aligned}\nu^{\max} &= \{0.0500, 0.1500, 0.1400, 0.1300, 0.1200, 0.0900, 0.1600, 0.1100\} \\ \nu^{\min} &= \{0.0000, 0.0437, 0.1140, 0.1011, 0.0400, 0.0300, 0.0889, 0.0367\}\end{aligned}$$

The calculation of similarity to the ideal solution for A_1 is as follows:

$$\begin{aligned}Sim_1^{best} &= \left(\frac{0.0500}{0.0500}\right)^{0.6} + \left(\frac{0.1500}{0.1500}\right)^{0.6} + \left(\frac{0.1373}{0.1400}\right)^{0.6} + \left(\frac{0.1300}{0.1300}\right)^{0.6} + \left(\frac{0.1200}{0.1200}\right)^{0.6} + \left(\frac{0.0900}{0.0900}\right)^{0.6} + \\ &\quad \left(\frac{0.1600}{0.1600}\right)^{0.6} + \left(\frac{0.1100}{0.1100}\right)^{0.6} \\ &= 7.9883\end{aligned}$$

The calculation of similarity to the anti-ideal solution for A_1 is as follows:

$$\begin{aligned}Sim_1^{worst} &= \left(\frac{0.0000}{0.0500}\right)^{0.6} + \left(\frac{0.0437}{0.1500}\right)^{0.6} + \left(\frac{0.1140}{0.1373}\right)^{0.6} + \left(\frac{0.1011}{0.1300}\right)^{0.6} + \left(\frac{0.0400}{0.1200}\right)^{0.6} + \left(\frac{0.0300}{0.0900}\right)^{0.6} + \\ &\quad \left(\frac{0.0889}{0.1600}\right)^{0.6} + \left(\frac{0.0367}{0.1100}\right)^{0.6} \\ &= 4.4863\end{aligned}$$

Alternatives that are closer to the adaptive ideal solution are indicated by higher similarity values, while alternatives that are farther from the ideal and closer to the anti-ideal are indicated by lower values. The resulting similarity measures are summarized in Table 3.4.

Table 3.4
Similarities to Ideal Solutions and Anti-Ideal Solutions

Alternative	Sim^{best}	Sim^{worst}
A_1	7.9883	4.4863
A_2	6.2834	6.0744
A_3	5.8666	5.5707
A_4	4.5315	6.9408

After determining the similarity measures, the RC values are calculated to obtain the final ranking of the alternatives.

Step 5. RC and ranking

The κ parameter is flexible and can be adjusted to any value $\kappa \in (0,1)$ that aligns with the decision-makers' preferences. The calculation of RC values for A_1 by using (3.8) and let $\kappa = 0.8$ is computed as below:

$$RC_1 = \frac{0.8 \cdot 7.9883}{0.8 \cdot 7.9883 + (1 - 0.8) \cdot 4.4863} = 0.8769$$

The performance results are presented in Table 3.5 and visualized in Figure 3.2.

Table 3.5
Relative Closeness and Ranking

Alternative	RC_i	Ranking
A_1	0.8769	1
A_2	0.8054	3
A_3	0.8082	2
A_4	0.7231	4

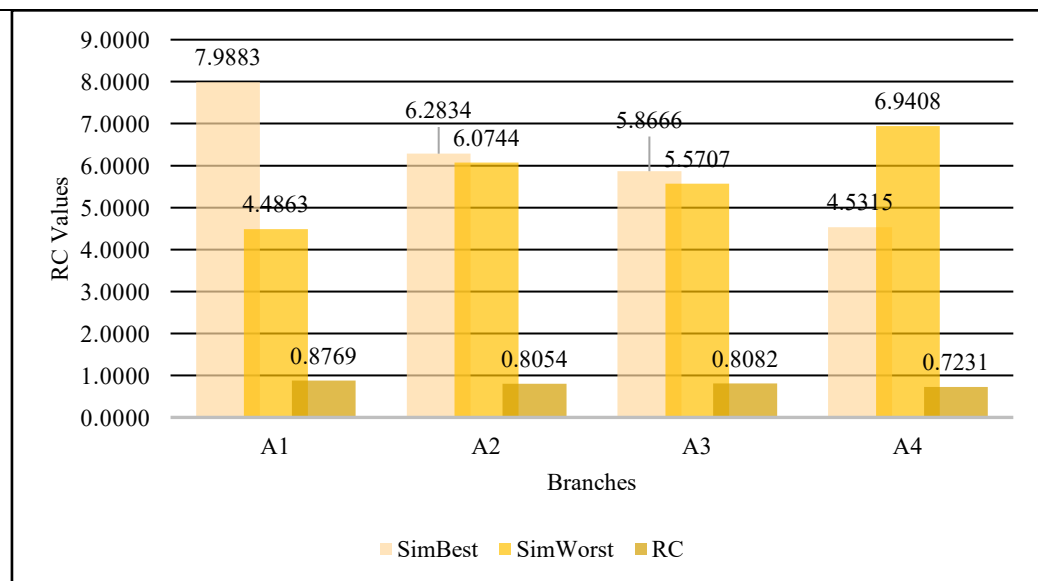


Figure 3.2 Alternative Ranking Results using the Proposed ARIE Method

From the Figure 3.2, the ranking of the company follows: Tikves (A_1) $\succ$ Sveti Nikole (A_3) $\succ$ Bregalnica (A_2) $\succ$ Skopsko Field (A_4). Tikves branch is the top-performing alternative with the highest value of RC and was chosen as the most acceptable location for the construction of the main warehouse of spare parts and consumables for the needs of companies from dispersed centres. Skopsko Field branch has the lowest RC value and being the unfavourable location for the construction of the main warehouse. Sveti Nikole branch ranked second, followed by Bregalnica branch in third ranked location. Selected MCDM method is applied and verified the stability of ARIE method in the subsequent section through sensitivity and comparative analysis.

3.4 Sensitivity Analysis

This section analyses the effect of the parameters γ and κ values on the ranking order by using the ARIE method to test the stability and flexibility of the method. It was performed using Microsoft Excel for Microsoft 365 (Version 2605). This sensitivity analysis evaluates whether changes in the parameter values influence the ranking results and assesses the robustness of the proposed decision-making method. The effect of parameter variations summarized in Table 3.6 and graphically presented in Figure 3.3 where κ value remain at 0.8.

Table 3.6
Effect of Parameter γ Values on Ranking Order

γ values	A_1	A_2	A_3	A_4	Ranking
0.2	0.8422	0.8153	0.8024	0.7750	$A_1 \succ A_2 \succ A_3 \succ A_4$
0.5	0.8692	0.8078	0.8066	0.7362	$A_1 \succ A_2 \succ A_3 \succ A_4$
1.0	0.9023	0.7961	0.8145	0.6714	$A_1 \succ A_3 \succ A_2 \succ A_4$
5.0	0.9769	0.7350	0.8665	0.3649	$A_1 \succ A_3 \succ A_2 \succ A_4$
10.0	0.9924	0.6940	0.8959	0.2165	$A_1 \succ A_3 \succ A_2 \succ A_4$
15.0	0.9973	0.4962	0.9094	0.1315	$A_1 \succ A_3 \succ A_2 \succ A_4$

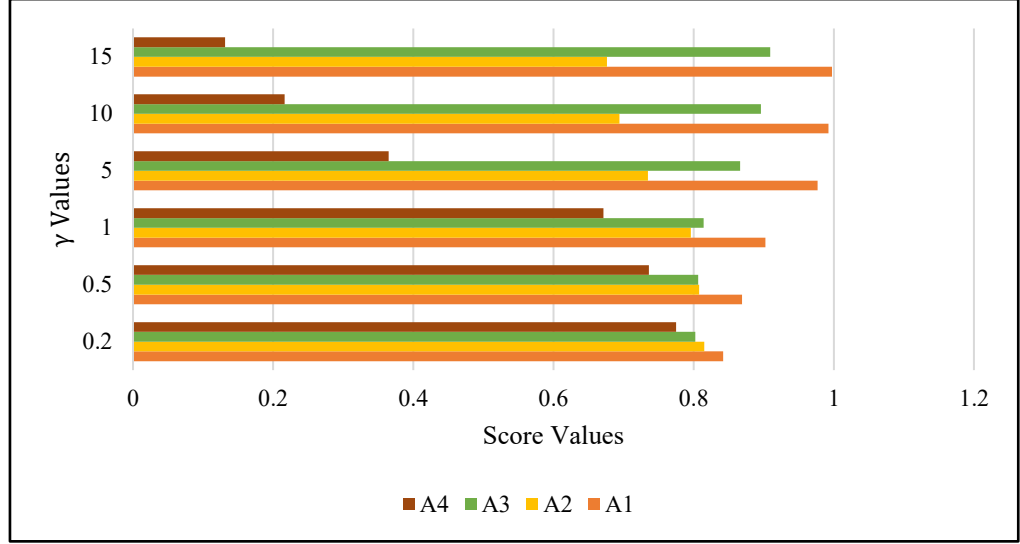


Figure 3.3 Graphical Representation of the Changes in Ranking Order with Respect to Different γ Parameter Values

Figure 3.3 illustrates the effect of different γ parameter values on the ranking order of the ARIE method. The results indicate that when $\gamma < 1$, a change in ranking is observed between alternatives A_2 and A_3 , suggesting that the method is more sensitive to parameter variations in this region. However, when $\gamma \geq 1$, the ranking order becomes stable and remains unchanged even when γ is increased up to 15 and further extended to 100. In terms of score behaviour, increasing γ results in an increase in the scores of A_1 and A_3 , while the scores of A_2 and A_4 decrease. This indicates that the parameter γ enhances the contrast between alternatives by amplifying the effect of the similarity measure. Higher values of γ strengthen the contribution of higher performing alternatives while suppressing lower-performing ones.

The observed behaviour suggests that γ functions as a sensitivity control parameter that regulates the discrimination intensity of the ARIE method. At lower values of γ , the differences between alternatives are less pronounced, leading to potential ranking instability. As γ increases, the separation between alternatives becomes more distinct, resulting in stable and consistent ranking outcomes. Therefore, the ARIE method demonstrates robustness for $\gamma \geq 1$, where the ranking results become stable and insensitive to further increases in the parameter value. The parameter κ values in the range of $[0,1]$ influence the ranking order with fixed γ value at 0.6 are shown in Table 3.7 and illustrated in Figure 3.4.

Table 3.7
Effect of Parameter κ Values on Ranking Order

κ values	A_1	A_2	A_3	A_4	Ranking
0.1	0.1652	0.1031	0.1048	0.0676	$A_1 \succ A_3 \succ A_2 \succ A_4$
0.3	0.4328	0.3072	0.3110	0.2186	$A_1 \succ A_3 \succ A_2 \succ A_4$
0.4	0.5428	0.4081	0.4125	0.3033	$A_1 \succ A_3 \succ A_2 \succ A_4$
0.5	0.6404	0.5085	0.5129	0.3950	$A_1 \succ A_3 \succ A_2 \succ A_4$
0.7	0.8060	0.7071	0.7108	0.6037	$A_1 \succ A_3 \succ A_2 \succ A_4$
0.9	0.9413	0.9030	0.9046	0.8546	$A_1 \succ A_3 \succ A_2 \succ A_4$

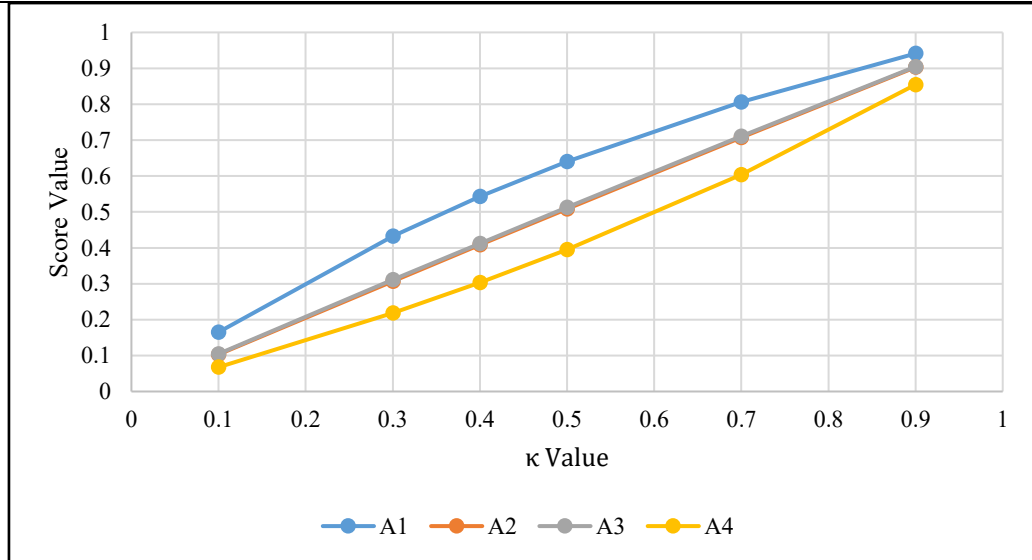


Figure 3.4 Graphical Representation of the Changes in Ranking Order with Respect to Different κ Parameter Values

Based on the Table 3.7 and Figure 3.4, the impact of varying κ parameter values on the ranking order of ARIE method remains stable. Like the previous figure, A_1 remains the top-ranked location, reinforcing its suitability as the main house site, while A_4 remains the least preferred option. Although the RC scores of all alternatives increase progressively as κ increases, the overall ranking order remains unchanged. This indicates that the parameter κ influences the magnitude of the scores but does not alter the relative preference among the alternatives. The stability of the ranking order across the examined range of κ values demonstrate the robustness of ARIE method

against parameter variations. Therefore, it can be concluded that ARIE method exhibits a high degree of stability and consistency in ranking the alternatives. To strengthen the ARIE method applicability, this study compares ARIE method and the existing MCDM approaches within the same numerical example.

3.5 Comparative Analysis

This section aimed at evaluating the stability, reliability, and consistency of the proposed ARIE method to several existing MCDM approaches, such as ARAS, CRADIS, MABAC, MOORA, SAW, TOPSIS, and VIKOR. All computational procedures, including the ranking calculations and the graphical visualizations were generated using MATLAB R2024a. Table 3.8 presented the comparative results, while the corresponding graphical representation is illustrated in Figure 3.5.

Table 3.8
Comparison of Company Ranking Results

Alternative	ARAS	ARIE	CRADIS	MABAC	MOORA	SAW	TOPSIS	VIKOR
A_1	1.0000	0.6979	1.0000	0.4483	0.2067	0.9128	0.8338	1.0000
A_2	0.6667	0.4940	0.2046	-0.1994	-0.0001	0.2652	0.3883	0.2289
A_3	0.7191	0.5234	0.4002	0.0881	0.0352	0.5527	0.3892	0.5899
A_4	0.5544	0.3381	0.1475	-0.2915	-0.0663	0.1730	0.2146	0.0000

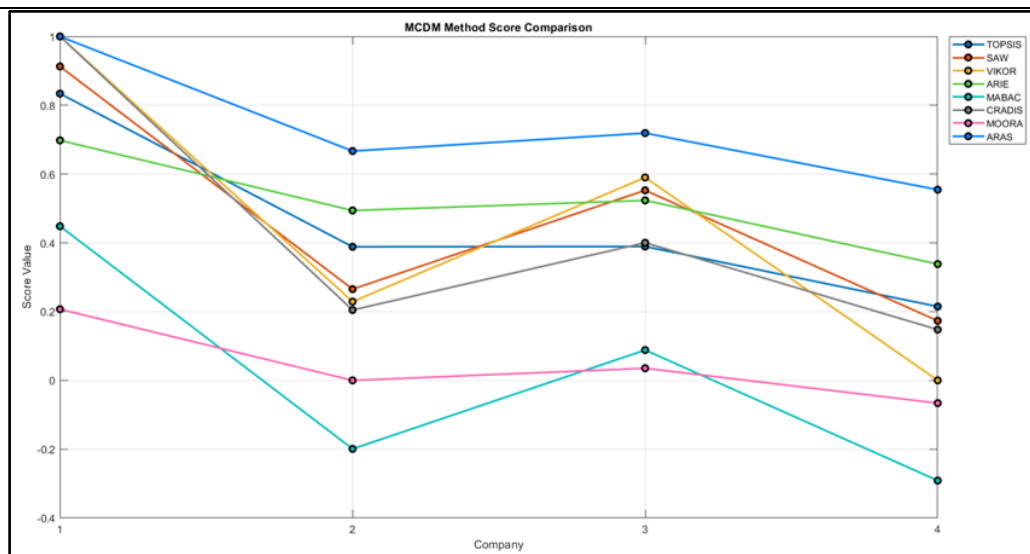


Figure 3.5 Comparison of Company Rankings of the ARIE and Other Methods

Based on Table 3.8 and Figure 3.5, the comparison of scores of four companies that were appraised through multiple MCDM techniques. In most of the approaches, Tikves branch, A_1 , has always had the highest score followed by Sveti Nikole, A_3 , Bregalnica, A_2 , and Skopsko Filed branch, A_4 , had the lowest score. Although the absolute score values vary between one method and another, because of various normalization and AO processes, the relative ranking pattern is mostly preserved between the best and the worst alternatives. There are slight differences in the Bregalnica and Sveti Nikole concerning some methods, but they do not significantly change the preference structure.

Consequently, the ranking order can be regarded as largely consistent across the MCDM techniques applied. Specifically, the ARIE method exhibits smooth transitions between scores and preserves the same relative ordering of alternatives as most other methods, without extreme fluctuations. This indicates that the ARIE method can be considered stable, as it does not produce abrupt changes in scores. Furthermore, the ARIE method appears robust, as it is consistent with other established MCDM methods, and flexible, as it generates discriminative yet balanced evaluations of all companies. The consistency of the ranking structures was further examined through Kendall's τ correlation analysis, with statistical significance evaluated via permutation testing.

3.5.1 Kendall's τ Correlation Analysis with Permutation Test

Kendall's τ rank correlation coefficient was used to analyse the consistency of the rankings obtained by ARIE and MCDM techniques. The Kendall's τ is a non-parametric value that measures the level of ordinal association between two variables or rankings (Xu et al., 2023). The observed τ values were determined for statistical significance using a permutation test, in which the rankings were randomly permuted to generate a null distribution representing the absence of a genuine ranking effect. The p values were approximated with the use of 10,000 random permutations. Table 3.9 presents Kendall's τ coefficients along with the corresponding permutation based on p values quantifying the ordinal agreement between the ARIE method and the selected MCDM techniques. Figure 3.6 presents permutation distributions of Kendall's τ measuring the agreement between benchmark MCDM methods and the ARIE ranking.

Table 3.9

Kendall's τ Rank Correlation and Permutation Test Results Between ARIE and Selected MCDM Methods

Method	ARAS	CRADIS	MABAC	MOORA	SAW	TOPSIS	VIKOR
Kendall τ	1	1	1	1	1	1	1
Permutation p	0.0880	0.0790	0.0840	0.0780	0.0850	0.0820	0.0770

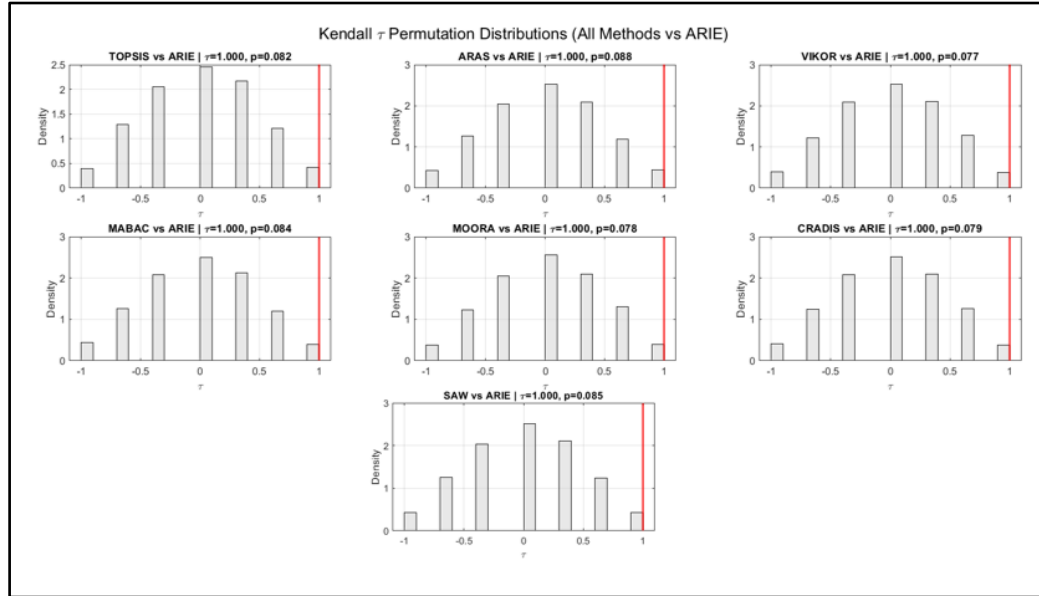


Figure 3.6 Permutation-Based Kendall's τ Distributions for Assessing the Agreement between ARIE and MCDM Methods

The figure shows the permutation distribution of the τ of Kendall, for each of the MCDM methods compared to the ARIE method. Every histogram represents the range of values of τ that were achieved by randomizing the ranks 10,000 times. All subplots have a red vertical line indicating the observed tau and $\tau = 1.000$ indicating the perfect agreement in the real rankings in all methods. Despite this perfect alignment, the histograms show that τ can take on many possible values between -1 and 1 if we consider random permutations. The number of alternatives (4) is small, which means that there are not many possible rank permutations, and even a τ value of 1 does not result in a very low p value. That is why the observed τ is not very large compared to most of the permuted τ values, but not too large to achieve a traditional level of significance, and thus this is why all of the p values are in the range of approximately 0.077 to 0.088.

In short, the figure shows that the MCDM methods are perfectly concordant with the ARIE ranking at $\tau = 1$. As the permutation histograms compute the same

ranking patterns and a similar statistical uncertainty, the findings show that rankings obtained using the ARIE method are completely consistent with rankings obtained using traditional MCDM methods as seen by the perfect Kendall's τ consistency of rankings across all comparisons. The perfect ordinal concordance, stable behaviour under permutation testing, and consistent performance in sensitivity and comparative analyses collectively demonstrate the robustness, reliability, and methodological soundness of the ARIE method in addressing the logistics application. These findings further validate the effectiveness of the proposed approach in producing consistent and dependable decision outcomes under varying evaluation conditions.

3.6 Conclusion

Based on the above findings, it can be concluded that the ARIE method was successfully developed to overcome the limitations of the TOPSIS method to become the alternative distance-based MCDM method. The analyses presented in this chapter prove that the proposed method can produce stable and consistent ranking results. The robustness in this sensitivity analysis, along with good agreement with the existing MCDM methods and the statistical validation of this ranking method via Kendall's τ and permutation testing, support the validity and reliability of the proposed ranking method. Hence, the ARIE method is a credible and effective method that can be used for the support of DM processes that involve several evaluation criteria.

The limitations of ARIE method, especially its limited scope to represent uncertainty, indeterminacy, and incomplete information comprehensively in complex DM environments, motivate the development of an enhanced DM framework. Thus, the next chapter has been devoted to the proposed BPN-GHHM and BPN-GWHHM operators. These AOs, which are specifically developed to reinforce the DM model with bipolar Pythagorean neutrosophic information and generalized mean operations, can provide a more effective representation and aggregation of uncertain, inconsistent, and incomplete information that may not be well represented by the ARIE method. It is therefore expected that the proposed operators will contribute to the strengthening of the robustness and reliability of the DM process in complex and uncertain conditions.

CHAPTER 4

INTEGRATED BIPOLAR PYTHAGOREAN NEUTROSOPHIC – GENERALIZED HAMACHER HERONIAN MEAN OPERATOR

4.1 Introduction

This section introduces new operators that integrate the BPNS with the Hamacher and HM operators. Section 4.2 presents the basic concepts and operational rules of set theory and related concept used in this study, while Section 4.3 defines the BPN-GHHM and BPN-GWHHM operators. The proposed operators are then evaluated through three key properties, idempotency, monotonicity, and boundedness, along with a numerical example, comparative and sensitivity analyses in Sections 4.4, 4.5 and 4.6 respectively. Finally, section 4.7 summarises this chapter.

4.2 Preliminaries

This section systematically presents the fundamental concepts of NS and their extension theories, the Hamacher and HM operators, and a newly defined Hamacher operator under BPNS information.

Definition 4.1 (Smarandache, 1998) Let Z be a universe of discourse, then NS is defined as,

$$D = \{ \langle z, \mu_D(z), \eta_D(z), \nu_D(z) \rangle : z \in Z \} \quad (4.1)$$

which is characterized by a truth-membership degree $\mu_D : Z \rightarrow]0^-, 1^+]$, an indeterminacy-membership degree $\eta_D : Z \rightarrow]0^-, 1^+]$, and a falsity-membership degree $\nu_D : Z \rightarrow]0^-, 1^+]$.

Definition 4.2 (Wang et al., 2010) Let Z be a space of points (objects), with a generic element in Z denoted by z . A SVN, D in Z is characterized by truth-membership degree $\mu_D(z)$, indeterminacy-membership degree, $\eta_D(z)$, and falsity-membership degree $\nu_D(z)$ defined as,

$$D = \left\{ \langle z, \mu_D(z), \eta_D(z), \nu_D(z) \rangle : z \in Z \right\} \quad (4.2)$$

where $\mu_D(z), \eta_D(z), \nu_D(z) \in [0, 1]$ and $0 \leq \mu_D(z) + \eta_D(z) + \nu_D(z) \leq 3$.

Definition 4.3 (Deli et al., 2015) Let D be a BNS in Z . It is defined as an object of the following form:

$$D = \left\{ \langle z, \mu_D^+(z), \eta_D^+(z), \nu_D^+(z), \mu_D^-(z), \eta_D^-(z), \nu_D^-(z) \rangle : z \in Z \right\} \quad (4.3)$$

where $\mu^+, \eta^+, \nu^+ : Z \rightarrow [0, 1]$ and $\mu^-, \eta^-, \nu^- : Z \rightarrow [-1, 0]$.

Definition 4.4 (Radha & Stanis, 2021) A PNS with μ and ν as dependent neutrosophic components D in Z as an object of the form:

$$D = \left\{ \langle z, \mu_D(z), \eta_D(z), \nu_D(z) \rangle : z \in Z \right\} \quad (4.4)$$

where $\mu_D(z), \eta_D(z), \nu_D(z)$ are the truth, indeterminacy and false membership respectively such that $\mu, \eta, \nu \in [0, 1]$.

Definition 4.5 (Ahmad et al., 2024) Let Z be a non-empty set. A BPNS with components D in Z is defined as:

$$D = \left\{ \langle z, \mu_D^+(z), \eta_D^+(z), \nu_D^+(z), \mu_D^-(z), \eta_D^-(z), \nu_D^-(z) \rangle : z \in Z \right\} \quad (4.5)$$

where $\mu_D^+(z), \eta_D^+(z), \nu_D^+(z) \rightarrow [0, 1]$ and $\mu_D^-(z), \eta_D^-(z), \nu_D^-(z) : Z \rightarrow [-1, 0]$. $\mu_D^+(z), \eta_D^+(z), \nu_D^+(z)$ represents positive membership degree, while $\mu_D^-(z), \eta_D^-(z), \nu_D^-(z)$ represents negative membership degree of the element $z \in Z$ to the set D with the conditions,

$$0 \leq \mu_D^+(x)^2 + \nu_D^+(x)^2 \leq 1 \quad (4.6)$$

$$0 \leq \mu_D^-(x)^2 + \nu_D^-(x)^2 \leq 1 \quad (4.7)$$

$$0 \leq \mu_D^+(x)^2 + \eta_D^+(x)^2 + \nu_D^+(x)^2 \leq 2 \quad (4.8)$$

$$0 \leq \mu_D^-(x)^2 + \eta_D^-(x)^2 + \nu_D^-(x)^2 \leq 2 \quad (4.9)$$

Definition 4.6 (Ahmad et al., 2024) Let

$$D_1 = \{z, \langle \mu_1^+(z), \eta_1^+(z), \nu_1^+(z), \mu_1^-(z), \eta_1^-(z), \nu_1^-(z) \rangle\} \quad \text{and}$$

$$D_2 = \{z, \langle \mu_2^+(z), \eta_2^+(z), \nu_2^+(z), \mu_2^-(z), \eta_2^-(z), \nu_2^-(z) \rangle\} \text{ be two BPNSs. The inequality property}$$

defined as $D_1 \subseteq D_2$ if and only if $\mu_1^+(z) \leq \mu_2^+(z)$, $\eta_1^+(z) \leq \eta_2^+(z)$, $\nu_1^+(z) \leq \nu_2^+(z)$ and $\mu_1^-(z) \geq \mu_2^-(z)$, $\eta_1^-(z) \geq \eta_2^-(z)$, $\nu_1^-(z) \geq \nu_2^-(z)$ for all z in Z .

Definition 4.7 (Ahmad et al., 2024) Let

$$D_1 = \{z, \langle \mu_1^+(z), \eta_1^+(z), \nu_1^+(z), \mu_1^-(z), \eta_1^-(z), \nu_1^-(z) \rangle\} \quad \text{and}$$

$$D_2 = \{z, \langle \mu_2^+(z), \eta_2^+(z), \nu_2^+(z), \mu_2^-(z), \eta_2^-(z), \nu_2^-(z) \rangle\} \text{ be two BPNSs. The equality property}$$

defined as $D_1 = D_2$ if and only if $\mu_1^+(z) = \mu_2^+(z)$, $\eta_1^+(z) = \eta_2^+(z)$, $\nu_1^+(z) = \nu_2^+(z)$ and $\mu_1^-(z) = \mu_2^-(z)$, $\eta_1^-(z) = \eta_2^-(z)$, $\nu_1^-(z) = \nu_2^-(z)$ for all z in Z .

Definition 4.8 (Ahmad et al., 2024) Let

$$D_1 = \{z, \langle \mu_1^+(z), \eta_1^+(z), \nu_1^+(z), \mu_1^-(z), \eta_1^-(z), \nu_1^-(z) \rangle\} \quad \text{and}$$

$$D_2 = \{z, \langle \mu_2^+(z), \eta_2^+(z), \nu_2^+(z), \mu_2^-(z), \eta_2^-(z), \nu_2^-(z) \rangle\} \text{ be two BPNSs. The union is defined as,}$$

$$D_1 \cup D_2 = \left\{ \begin{array}{l} \max(\mu_1^+(z), \mu_2^+(z)), \min(\eta_1^+(z), \eta_2^+(z)), \min(\nu_1^+(z), \nu_2^+(z)), \\ \min(\mu_1^-(z), \mu_2^-(z)), \min(\eta_1^-(z), \eta_2^-(z)), \max(\nu_1^-(z), \nu_2^-(z)) \end{array} \right\}$$

for all z in Z .

Definition 4.9 (Ahmad et al., 2024) Let

$$D_1 = \{z, \langle \mu_1^+(z), \eta_1^+(z), \nu_1^+(z), \mu_1^-(z), \eta_1^-(z), \nu_1^-(z) \rangle\} \quad \text{and}$$

$$D_2 = \{z, \langle \mu_2^+(z), \eta_2^+(z), \nu_2^+(z), \mu_2^-(z), \eta_2^-(z), \nu_2^-(z) \rangle\} \text{ be two BPNSs. The intersection is}$$

defined as,

$$D_1 \cap D_2 = \left\{ \begin{array}{l} \min(\mu_1^+(z), \mu_2^+(z)), \max(\eta_1^+(z), \eta_2^+(z)), \max(\nu_1^+(z), \nu_2^+(z)), \\ \max(\mu_1^-(z), \mu_2^-(z)), \max(\eta_1^-(z), \eta_2^-(z)), \min(\nu_1^-(z), \nu_2^-(z)) \end{array} \right\}$$

for all z in Z .

Definition 4.10 (Ahmad et al., 2024) Let D be a BPNS in Z . Thus, the complement of Z is defined by $D^C = \{z, \langle v_D^+(z), \eta_D^+(z), \mu_D^+(z), v_D^-(z), \eta_D^-(z), \mu_D^-(z) \rangle\}$ for all z in Z .

Definition 4.11 (Ahmad et al., 2024) Let $s = \{\mu_s^+, \eta_s^+, \nu_s^+, \mu_s^-, \eta_s^-, \nu_s^-\}$ and $h = \{\mu_h^+, \eta_h^+, \nu_h^+, \mu_h^-, \eta_h^-, \nu_h^-\}$ be two BPNSs and t is a scalar. The following is the definition of the operational rules, which include addition, multiplication, scalar multiplication and power operations.

$$s \oplus h = \left\{ \begin{array}{l} \sqrt{(\mu_s^+)^2 + (\mu_h^+)^2 - (\mu_s^+)^2 (\mu_h^+)^2}, \quad \eta_s^+ \eta_h^+, \quad \nu_s^+ \nu_h^+, \\ -(\mu_s^- \mu_h^-), \quad -(\eta_s^- \eta_h^- - \eta_s^- \eta_h^-), \quad -\sqrt{(\nu_s^-)^2 + (\nu_h^-)^2 - (\nu_s^-)^2 (\nu_h^-)^2} \end{array} \right\} \quad (4.10)$$

$$s \otimes h = \left\{ \begin{array}{l} \mu_s^+ \mu_h^+, \quad \eta_s^+ + \eta_h^+ - \eta_s^+ \eta_h^+, \quad \sqrt{(\nu_s^+)^2 + (\nu_h^+)^2 - (\nu_s^+)^2 (\nu_h^+)^2}, \\ -\sqrt{(\mu_s^-)^2 + (\mu_h^-)^2 - (\mu_s^-)^2 (\mu_h^-)^2}, \quad -(\eta_s^- \eta_h^-), \quad -(\nu_s^- \nu_h^-) \end{array} \right\} \quad (4.11)$$

$$t(s) = \left\{ \sqrt{1 - \left(1 - (\mu_s^+)^2\right)^t}, (\eta_s^+)^t, (\nu_s^+)^t, -(\mu_s^-)^t, -(\eta_s^-)^t, \sqrt{1 - \left(1 - (\nu_s^-)^2\right)^t} \right\} \quad (4.12)$$

$$(s)^t = \left\{ (\mu_s^+)^t, 1 - (1 - \eta_s^+)^t, \sqrt{1 - \left(1 - (\nu_s^+)^2\right)^t}, \sqrt{1 - \left(1 - (\mu_s^-)^2\right)^t}, -(\eta_s^-)^t, -(\nu_s^-)^t \right\} \quad (4.13)$$

This study utilized the score, accuracy and certainty functions presented for BNSs. This adoption is justified as the BPNSs include the same six membership degrees as BNSs, positive truth, positive indeterminacy, positive falsity, negative truth, negative indeterminacy and negative falsity. The key distinction is the Pythagorean constraints that is imposed in the BPNS, which extends the representation of uncertainty.

Definition 4.12 (Deli et al., 2015) Assume $A_1 = \{\mu_1^+, \eta_1^+, \nu_1^+, \mu_1^-, \eta_1^-, \nu_1^-\}$ is a BNS, the score, accuracy, and certain function of A_1 are represented as:

$$s(A_1) = \frac{\mu_1^+ + 1 - \eta_1^+ + 1 - \nu_1^+ + 1 - \mu_1^- + \eta_1^- + \nu_1^-}{6} \quad (4.14)$$

$$a(A_1) = \mu_1^+ - \nu_1^+ + \nu_1^- - \mu_1^- \quad (4.15)$$

$$c(A_1) = \mu_1^+ - \nu_1^+ \quad (4.16)$$

Definition 4.13 (Hamacher, 1978) Hamacher operators extend the t-norm and t-conorm within the framework of Archimedean t-norms and t-conorms. The Hamacher product, $\otimes$, is a t-norm and the Hamacher sum, $\oplus$, is a t-conorm, are two real numbers a_i and a_j are defined as follows:

$$T(a_i, a_j) = a_i \otimes a_j = \frac{a_i a_j}{\zeta + (1 - \zeta)(a_i + a_j - a_i a_j)}, \zeta > 0 \quad (4.17)$$

$$T^*(a_i, a_j) = a_i \oplus a_j = \frac{a_i + a_j - a_i a_j - (1 - \zeta)(a_i a_j)}{1 - (1 - \zeta)(a_i a_j)}, \zeta > 0 \quad (4.18)$$

If $\zeta = 1$, the Hamacher t-norm and t-conorm reduce to the algebraic t-norm and t-conorm, respectively:

$$T(a_i, a_j) = a_i \otimes a_j = a_i a_j$$

$$T^*(a_i, a_j) = a_i \oplus a_j = a_i + a_j - a_i a_j$$

For $\zeta = 2$, the Hamacher t-norm and t-conorm become equivalent to the Einstein t-norm and t-conorm, respectively:

$$T(a_i, a_j) = a_i \otimes a_j = \frac{a_i a_j}{1 + (1 - a_i)(1 - a_j)}$$

$$T^*(a_i, a_j) = a_i \oplus a_j = \frac{a_i + a_j}{1 + a_i a_j}$$

Definition 4.14 (Beliakov et al., 2007) For a set $x_i (i = 1, 2, \dots, n)$ of non-negative real numbers, the HM is defined by,

$$HM(x_1, x_2, \dots, x_n) = \frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n \sqrt{x_i x_j} \quad (4.19)$$

Definition 4.15 (Sykora, 2009) Let $x_i (i = 1, 2, \dots, n)$ denotes a set of non-negative real numbers, and $p, q \geq 0$. Then, the GHM is defined as,

$$GHM^{p,q}(x_1, x_2, \dots, x_n) = \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n x_i^p x_j^q \right]^{\frac{1}{p+q}} \quad (4.20)$$

Definition 4.16 (Liu & Pei, 2012) Consider a weight vector $w_i = (w_1, w_2, \dots, w_n)^T$ of non-negative numbers $x_i (i = 1, 2, \dots, n)$ that satisfies $p, q \geq 0, w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Then, GWHM is defined as follows,

$$GWHM^{p,q}(x_1, x_2, \dots, x_n) = \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (w_i x_i)^p \otimes (w_j x_j)^q \right]^{\frac{1}{p+q}} \quad (4.21)$$

Definition 4.17 is formulated in reference to the Hamacher operations and HM operator, which were combined to create new AOs for the T-Spherical Fuzzy by Thilagavathy & Mohanaselvi (2024). Motivated by the methodology, the proposed Hamacher operations are built in the framework of BPNS.

Definition 4.17 Hamacher operations of BPNSs. For any two BPNSs $\Upsilon_1 = \{\mu_1^+, \eta_1^+, \nu_1^+, \mu_1^-, \eta_1^-, \nu_1^-\}$ and $\Upsilon_2 = \{\mu_2^+, \eta_2^+, \nu_2^+, \mu_2^-, \eta_2^-, \nu_2^-\}$, any real number $\zeta > 0$, the Hamacher operators under BPNSs are defined by,

$$\Upsilon_1 \oplus \Upsilon_2 = \left(\begin{aligned} &\sqrt{\frac{(\mu_1^+)^2 + (\mu_2^+)^2 - (1-\zeta)(\mu_1^+)^2 (\mu_2^+)^2}{1-(1-\zeta)(\mu_1^+)^2 (\mu_2^+)^2}}, \frac{\eta_1^+ \eta_2^+}{\zeta + (1-\zeta)(\eta_1^+ + \eta_2^+ - \eta_1^+ \eta_2^+)}, \\ &\frac{\nu_1^+ \nu_2^+}{\zeta + (1-\zeta)(\nu_1^+ + \nu_2^+ - \nu_1^+ \nu_2^+)}, \frac{-(\mu_1^- \mu_2^-)}{\zeta + (1-\zeta)(\mu_1^- + \mu_2^- - \mu_1^- \mu_2^-)}, \\ &\frac{-(-\eta_1^- - \eta_2^- - \eta_1^- \eta_2^- - (1-\zeta)(\eta_1^- \eta_2^-))}{1-(1-\zeta)(\eta_1^- \eta_2^-)}, -\sqrt{\frac{(\nu_1^-)^2 + (\nu_2^-)^2 - (\nu_1^-)^2 (\nu_2^-)^2 - (1-\zeta)(\nu_1^-)^2 (\nu_2^-)^2}{1-(1-\zeta)(\nu_1^-)^2 (\nu_2^-)^2}} \end{aligned} \right) \quad (4.22)$$

$$\Upsilon_1 \otimes \Upsilon_2 = \left(\begin{aligned} &\frac{\mu_1^+ \mu_2^+}{\zeta + (1-\zeta)(\mu_1^+ + \mu_2^+ - \mu_1^+ \mu_2^+)}, \frac{\eta_1^+ + \eta_2^+ - \eta_1^+ \eta_2^+ - (1-\zeta)(\eta_1^+ \eta_2^+)}{1-(1-\zeta)(\eta_1^+ \eta_2^+)}, \\ &\sqrt{\frac{(\nu_1^+)^2 + (\nu_2^+)^2 - (\nu_1^+)^2 (\nu_2^+)^2 - (1-\zeta)(\nu_1^+)^2 (\nu_2^+)^2}{1-(1-\zeta)(\nu_1^+)^2 (\nu_2^+)^2}}, \\ &-\sqrt{\frac{(\mu_1^-)^2 + (\mu_2^-)^2 - (\mu_1^-)^2 (\mu_2^-)^2 - (1-\zeta)(\mu_1^-)^2 (\mu_2^-)^2}{1-(1-\zeta)(\mu_1^-)^2 (\mu_2^-)^2}}, \\ &\frac{\eta_1^- \eta_2^-}{\zeta - (1-\zeta)(\eta_1^+ + \eta_2^+ - \eta_1^+ \eta_2^+)}, \frac{-(\nu_1^- \nu_2^-)}{\zeta - (1-\zeta)(\nu_1^+ + \nu_2^+ - \nu_1^+ \nu_2^+)} \end{aligned} \right) \quad (4.23)$$

$$\delta(Y_1) = \left(\frac{\sqrt{\frac{\left(1+(\zeta-1)(\mu_1^+)^2\right)^\delta - \left(1-(\mu_1^+)^2\right)^\delta}{\left(1+(\zeta-1)(\mu_1^+)^2\right)^\delta + (\zeta-1)\left(1-(\mu_1^+)^2\right)^\delta}} \cdot \frac{\zeta(\eta_1^+)^\delta}{\left(1+(\zeta-1)(1-\eta_1^+)\right)^\delta + (\zeta-1)(\eta_1^+)^\delta}, \right. \\ \left. \frac{\zeta(v_1^+)^\delta}{\left(1+(\zeta-1)(1-v_1^+)\right)^\delta + (\zeta-1)(v_1^+)^\delta}, \frac{\zeta|\mu_1^-|^\delta}{\left(1+(\zeta-1)(1+\mu_1^-)\right)^\delta + (\zeta-1)|\mu_1^-|^\delta}, \right. \\ \left. \frac{\zeta|\eta_1^-|^\delta}{\left(1+(\zeta-1)(1+\eta_1^-)\right)^\delta + (\zeta-1)|\eta_1^-|^\delta}, -\frac{\sqrt{\frac{\left(1+(\zeta-1)(v_1^-)^2\right)^\delta - \left(1+(v_1^-)^2\right)^\delta}{\left(1+(\zeta-1)(v_1^-)^2\right)^\delta - (\zeta-1)\left(1+(v_1^-)^2\right)^\delta}}}{\left(1+(\zeta-1)(v_1^-)^2\right)^\delta - (\zeta-1)\left(1+(v_1^-)^2\right)^\delta} \right) \quad (4.24)$$

$$(Y_1)^\delta = \left(\frac{(\mu_1^+)^\delta}{\left(1+(\zeta-1)(1+\mu_1^+)\right)^\delta + (\zeta-1)(\mu_1^+)^\delta}, \frac{\left(1+(\zeta-1)\eta_1^+\right)^\delta - (\zeta-1)(\eta_1^+)^\delta}{\left(1+(\zeta-1)\eta_1^+\right)^\delta + (\zeta-1)(1-\eta_1^+)^\delta}, \right. \\ \left. \frac{\sqrt{\frac{\left(1+(\zeta-1)(-v_1^+)^2\right)^\delta - \left(1-(-v_1^+)^2\right)^\delta}{\left(1+(\zeta-1)(-v_1^+)^2\right)^\delta + (\zeta-1)\left(1-(-v_1^+)^2\right)^\delta}}}{\left(1+(\zeta-1)(-v_1^+)^2\right)^\delta + (\zeta-1)\left(1-(-v_1^+)^2\right)^\delta}, \right. \\ \left. -\frac{\sqrt{\frac{\left(1+(\zeta-1)(\mu_1^-)^2\right)^\delta - \left(1-(\mu_1^-)^2\right)^\delta}{\left(1+(\zeta-1)(\mu_1^-)^2\right)^\delta + (\zeta-1)\left(1-(\mu_1^-)^2\right)^\delta}}}{\left(1+(\zeta-1)(\mu_1^-)^2\right)^\delta + (\zeta-1)\left(1-(\mu_1^-)^2\right)^\delta}, \right. \\ \left. \frac{-\zeta|\eta_1^-|^\delta}{\left(1+(\zeta-1)(1+\eta_1^-)\right)^\delta + (\zeta-1)|\eta_1^-|^\delta}, \frac{-\zeta|v_1^-|^\delta}{\left(1+(\zeta-1)(1+v_1^-)\right)^\delta + (\zeta-1)|v_1^-|^\delta} \right) \quad (4.25)$$

Each BPNS operations use Hamacher t-norm and t-conorm structures to model conjunction and aggregation behaviours. Opposite structures preserve bipolarity and logical consistency.

4.3 Proposed Generalized Hamacher Heronian Mean Operator for BPNSs

This section presents the proposed operator, its formulation in detail, the fundamental properties with proofs. Figure 4.1 illustrates the systematic procedure of the BPN–GHHM and BPN–GWHHM operators.

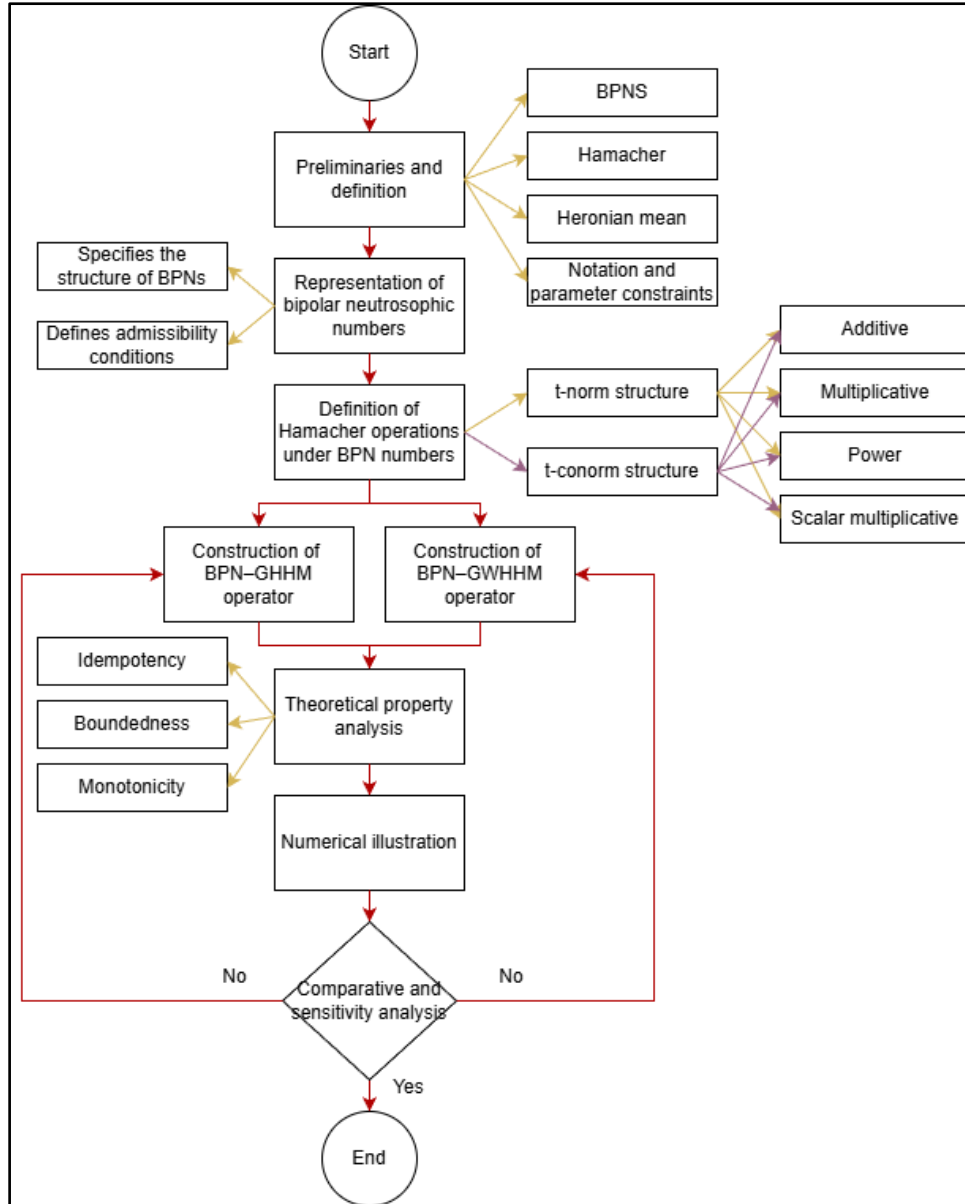


Figure 4.1 Flowchart of the Proposed BPN-GHHM and BPN-GWHHM Operators

According to Figure 4.1, the integration process of the new AOs is started by establishing fundamental concepts related to each of the components, which are BPNS, Hamacher, HM operator and its constraints. The next step is to determine the addition, multiplication, scalar multiplication, power operations. These operations are

responsible for changes of the six membership components of BPNS, positive truth, negative truth, indeterminacy degree and falsity degree. The current operational rules of Bipolar Pythagorean Neutrosophic Number (BPNN), however, do not include a sensitivity parameter nor a generalized operational structure. In order to make BPNN more flexible, the Hamacher operator is added, based on which the BPNN is extended with parameterized t-norm and t-conorm mechanisms. By this integration, new operational laws of BPNS under the Hamacher context as in Definition 4.12 are built. The development process was carefully conducted by referring to relevant previous studies and through repeated refinement of mathematical calculations to ensure that the proposed operations preserve the inherent behaviour and constraints of BPNS.

The operations are designed to preserve the validity of the characteristics of BPNNs, as the positive membership degree is kept within $[0,1]$ and the negative membership degree within $[-1,0]$. The t-conorm or conjunction is used for the positive truth degree, with the t-norm or disjunction used for the positive indeterminacy and falsity degrees of the addition and scalar multiplication. The same operating structure is then followed for the other bipolar membership elements. The multiplication and power operations start with the t-norm structure for the positive truth degree, and then the t-conorm mechanisms for the remaining membership degrees. The methodologies of the proposed integration are referred to the methodologies presented by Jamil et al. (2019); Jamil et al. (2022); Thilagavathy & Mohanaselvi (2024); Yaacob et al. (2024) in order to ensure the consistency of the proposed integration.

Previous works have not specifically discussed the application of BPNS with the advanced AOs, but the same AOs were also used in the works concerning BNSs, which can be regarded as appropriate references owing to the close relationship between BNS and BPNS, in the latter the membership degrees obtained Pythagorean constraints. The proposed AOs are tested for theoretical validity by their properties such as idempotency, monotonicity and boundedness with examples. The proposed AOs are assessed for their effectiveness by comparative and sensitivity analysis, which shows that the proposed AOs are suitable for dealing with uncertainty and incomplete information in MCDM problems.

4.3.1 Bipolar Pythagorean Neutrosophic – Generalized Hamacher Heronian Mean (BPN-GHHM) Operator

Definition 4.18 Let $p, q \geq 0$ and $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i = 1, 2, \dots, n)$ be a collection of BPNSs, BPN-GHHM operator defined as follows:

$$BPN-GHHM^{p,q}(x_1, x_2, \dots, x_n) = \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q \right]^{\frac{1}{p+q}} \quad (4.26)$$

Theorem 4.1 Let $p, q \geq 0$ and $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i = 1, 2, \dots, n)$ be a collection of BPNSs, the BPN-GHHM aggregated value is computed using the Hamacher operations in Definition 4.17 as defined in Definition 4.18.

$$\begin{aligned}
BPN - GHM^{p,q}(x_1, x_2, \dots, x_n) &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q \right]^{p+q} = \\
&= \left[\frac{\left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1-\mu_i^+))^p (1+(\zeta-1)(1-\mu_j^+))^q \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \right. \\
&\quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)\eta_i^+)^p (1+(\zeta-1)\eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n (1-\eta_i^+)^p (1-\eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)\eta_i^+)^p (1+(\zeta-1)\eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n (1-\eta_i^+)^p (1-\eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
&\quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(-v_i^+)^2)^p (1+(\zeta-1)(-v_j^+)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n (1-(-v_i^+)^2)^p (1-(-v_j^+)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(-v_i^+)^2)^p (1+(\zeta-1)(-v_j^+)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n (1-(-v_i^+)^2)^p (1-(-v_j^+)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
&\quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(\mu_i^-)^2)^p (1+(\zeta-1)(\mu_j^-)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n (1-(\mu_i^-)^2)^p (1-(\mu_j^-)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(\mu_i^-)^2)^p (1+(\zeta-1)(\mu_j^-)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n (1-(\mu_i^-)^2)^p (1-(\mu_j^-)^2)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
&\quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1+\eta_i^-))^p (1+(\zeta-1)(1+\eta_j^-))^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1+\eta_i^-))^p (1+(\zeta-1)(1+\eta_j^-))^q \right]^{\frac{2(p+q)}{n(n+1)}} - \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
&\quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1+v_i^-))^p (1+(\zeta-1)(1+v_j^-))^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n |v_i^-|^p |v_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1+v_i^-))^p (1+(\zeta-1)(1+v_j^-))^q \right]^{\frac{2(p+q)}{n(n+1)}} - \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |v_i^-|^p |v_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}} \Bigg] \quad (4.27)
\end{aligned}$$

Proof By substituting the scalar with A_i into (4.25), the results can be derived as follows,

$$(A_i)^P = \left[\begin{array}{c} \frac{(\mu_i^+)^P}{\left(1+(\zeta-1)(1+\mu_i^+)\right)^P - (\zeta-1)(\mu_i^+)^P}, \frac{\left(1+(\zeta-1)\eta_i^+\right)^P - (\zeta-1)(\eta_i^+)^P}{\left(1+(\zeta-1)\eta_i^+\right)^P + (\zeta-1)(1-\eta_i^+)^P}, \\ \sqrt{\frac{\left(1+(\zeta-1)(-v_i^+)^2\right)^P - \left(1-(-v_i^+)^2\right)^P}{\left(1+(\zeta-1)(-v_i^+)^2\right)^P + (\zeta-1)\left(1-(-v_i^+)^2\right)^P}}, -\sqrt{\frac{\left(1+(\zeta-1)(\mu_i^-)^2\right)^P - \left(1-(\mu_i^-)^2\right)^P}{\left(1+(\zeta-1)(\mu_i^-)^2\right)^P + (\zeta-1)\left(1-(\mu_i^-)^2\right)^P}}, \\ \frac{-\zeta|\eta_i^-|^P}{\left(1+(\zeta-1)(1+\eta_i^-)\right)^P + (\zeta-1)|\eta_i^-|^P}, \frac{-\zeta|v_i^-|^P}{\left(1+(\zeta-1)(1+v_i^-)\right)^P + (\zeta-1)|v_i^-|^P} \end{array} \right]$$

$$(A_j)^q = \left[\begin{array}{c} \frac{(\mu_j^+)^q}{\left(1+(\zeta-1)(1+\mu_j^+)\right)^q - (\zeta-1)(\mu_j^+)^q}, \frac{\left(1+(\zeta-1)\eta_j^+\right)^q - (\zeta-1)(\eta_j^+)^q}{\left(1+(\zeta-1)\eta_j^+\right)^q + (\zeta-1)(1-\eta_j^+)^q}, \\ \sqrt{\frac{\left(1+(\zeta-1)(-v_j^+)^2\right)^q - \left(1-(-v_j^+)^2\right)^q}{\left(1+(\zeta-1)(-v_j^+)^2\right)^q + (\zeta-1)\left(1-(-v_j^+)^2\right)^q}}, -\sqrt{\frac{\left(1+(\zeta-1)(\mu_j^-)^2\right)^q - \left(1-(\mu_j^-)^2\right)^q}{\left(1+(\zeta-1)(\mu_j^-)^2\right)^q + (\zeta-1)\left(1-(\mu_j^-)^2\right)^q}}, \\ \frac{-\zeta|\eta_j^-|^q}{\left(1+(\zeta-1)(1+\eta_j^-)\right)^q + (\zeta-1)|\eta_j^-|^q}, \frac{-\zeta|v_j^-|^q}{\left(1+(\zeta-1)(1+v_j^-)\right)^q + (\zeta-1)|v_j^-|^q} \end{array} \right]$$

Referring to (4.23), the next step is a multiplicative aggregation. The multiplication shouldn't multiply inner components directly. In the Hamacher setting, the constituent elements are carried forward unchanged, what changes is the AO, the 'outer shell', which now becomes multiplication. So don't substitute the entire earlier formula wholesale into the new one. Instead, keep the evaluated elements and apply the Hamacher-style multiplicative composition at this stage.

$$\begin{aligned}
(A_i)^p \otimes (A_j)^q = & \left[\begin{aligned} & \frac{(\mu_i^+)^p (\mu_j^+)^q}{\left(1+(\zeta-1)(1-\mu_i^+)\right)^p \left(1+(\zeta-1)(1-\mu_j^+)\right)^q - (\zeta-1)(\mu_i^+)^p (\mu_j^+)^q}, \\ & \frac{(1+(\zeta-1)\eta_i^+)^p (1+(\zeta-1)\eta_j^+)^q - (1-\eta_i^+)^p (1-\eta_j^+)^q}{\left(1+(\zeta-1)\eta_i^+\right)^p \left(1+(\zeta-1)\eta_j^+\right)^q + (\zeta-1)(1-\eta_i^+)^p (1-\eta_j^+)^q}, \\ & \sqrt{\frac{\left(1+(\zeta-1)(-v_i^+)^2\right)^p \left(1+(\zeta-1)(-v_j^+)^2\right)^q - \left(1-(-v_i^+)^2\right)^p \left(1-(-v_j^+)^2\right)^q}{\left(1+(\zeta-1)(-v_i^+)^2\right)^p \left(1+(\zeta-1)(-v_j^+)^2\right)^q + (\zeta-1)\left(1-(-v_i^+)^2\right)^p \left(1-(-v_j^+)^2\right)^q}}, \\ & - \sqrt{\frac{\left(1+(\zeta-1)(\mu_i^-)^2\right)^p \left(1+(\zeta-1)(\mu_j^-)^2\right)^q - \left(1-(\mu_i^-)^2\right)^p \left(1-(\mu_j^-)^2\right)^q}{\left(1+(\zeta-1)(\mu_i^-)^2\right)^p \left(1+(\zeta-1)(\mu_j^-)^2\right)^q + (\zeta-1)\left(1-(\mu_i^-)^2\right)^p \left(1-(\mu_j^-)^2\right)^q}}, \\ & \frac{-\zeta |\eta_i^-|^p |\eta_j^-|^q}{\left(1+(\zeta-1)(1+\eta_i^+)\right)^p \left(1+(\zeta-1)(1+\eta_j^+)\right)^q + (\zeta-1) |\eta_i^-|^p |\eta_j^-|^q}, \\ & \frac{-\zeta |v_i^-|^p |v_j^-|^q}{\left(1+(\zeta-1)(1+v_i^+)\right)^p \left(1+(\zeta-1)(1+v_j^+)\right)^q + (\zeta-1) |v_i^-|^p |v_j^-|^q} \end{aligned} \right]
\end{aligned}$$

According to (4.22), we now introduce the double summation as the aggregation step. However, this isn't a naïve 'just add everything' move. In the Hamacher framework, the inner components remain as they are; what changes is the aggregation shell, which here is addition implemented in the specific way prescribed by (4.22). In other words, this study doesn't substitute the entire previous expression and add it term-by-term. This study carries forward the evaluated elements and then applies the Hamacher-consistent additive composition at this stage.

$$\begin{aligned}
& \sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q = \\
& \left[\sqrt{\frac{\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 - \mu_i^+)\right)^p \left(1 + (\zeta - 1)(1 - \mu_j^+)\right)^q - (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q}}, \right. \\
& \quad \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)\eta_i^+\right)^p \left(1 + (\zeta - 1)\eta_j^+\right)^q - \sum_{i=1}^n \sum_{j=1}^n (1 - \eta_i^+)^p (1 - \eta_j^+)^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)\eta_i^+\right)^p \left(1 + (\zeta - 1)\eta_j^+\right)^q + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n (1 - \eta_i^+)^p (1 - \eta_j^+)^q}, \\
& \quad \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-v_i^+)\right)^p \left(1 + (\zeta - 1)(-v_j^+)\right)^q - \sum_{i=1}^n \sum_{j=1}^n \left(1 - (-v_i^+)\right)^p \left(1 - (-v_j^+)\right)^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-v_i^+)\right)^p \left(1 + (\zeta - 1)(-v_j^+)\right)^q + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n \left(1 - (-v_i^+)\right)^p \left(1 - (-v_j^+)\right)^q}, \\
& \quad \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(\mu_i^-)^2\right)^p \left(1 + (\zeta - 1)(\mu_j^-)^2\right)^q - \sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2\right)^p \left(1 - (\mu_j^-)^2\right)^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(\mu_i^-)^2\right)^p \left(1 + (\zeta - 1)(\mu_j^-)^2\right)^q + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2\right)^p \left(1 - (\mu_j^-)^2\right)^q}, \\
& \quad \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-v_i^-)^2\right)^p \left(1 + (\zeta - 1)(-v_j^-)^2\right)^q - \sum_{i=1}^n \sum_{j=1}^n \left(1 - (-v_i^-)^2\right)^p \left(1 - (-v_j^-)^2\right)^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-v_i^-)^2\right)^p \left(1 + (\zeta - 1)(-v_j^-)^2\right)^q + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n \left(1 - (-v_i^-)^2\right)^p \left(1 - (-v_j^-)^2\right)^q}, \\
& \quad \frac{-\zeta \sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 + \eta_i^-)\right)^p \left(1 + (\zeta - 1)(1 + \eta_j^-)\right)^q + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q}, \\
& \quad \left. - \sqrt{\frac{-\zeta \sum_{i=1}^n \sum_{j=1}^n |v_i^-|^p |v_j^-|^q}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 + v_i^-)\right)^p \left(1 + (\zeta - 1)(1 + v_j^-)\right)^q + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n |v_i^-|^p |v_j^-|^q}} \right]
\end{aligned}$$

Based on (4.24), this study now applies scalar multiplication. The outer shell is identical to the additive (4.22): the square-root and the negative sign remain in the same positions. Only the scalar factor rescales the preserved inner expression; nothing inside is re-expanded or re-substituted.

$$\begin{aligned}
& \frac{2}{n(n+1)} \left[\sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q \right] = \\
& \left[\begin{aligned}
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(\mu_i^+ \right)^p \left(\mu_j^+ \right)^q \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 - \mu_i^+ \right) \right)^p \left(1 + (\zeta - 1) \left(1 - \mu_j^+ \right) \right)^q \right]^{\frac{2}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(\mu_i^+ \right)^p \left(\mu_j^+ \right)^q \right]^{\frac{2}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \eta_i^+ \right)^p \left(1 + (\zeta - 1) \eta_j^+ \right)^q \right]^{\frac{2}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \eta_i^+ \right)^p \left(1 - \eta_j^+ \right)^q \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \eta_i^+ \right)^p \left(1 + (\zeta - 1) \eta_j^+ \right)^q \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \eta_i^+ \right)^p \left(1 - \eta_j^+ \right)^q \right]^{\frac{2}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(-v_i^+ \right)^2 \right)^p \left(1 + (\zeta - 1) \left(-v_j^+ \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(-v_i^+ \right)^2 \right)^p \left(1 - \left(-v_j^+ \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(-v_i^+ \right)^2 \right)^p \left(1 + (\zeta - 1) \left(-v_j^+ \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(-v_i^+ \right)^2 \right)^p \left(1 - \left(-v_j^+ \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^- \right)^2 \right)^p \left(1 + (\zeta - 1) \left(\mu_j^- \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^- \right)^2 \right)^p \left(1 - \left(\mu_j^- \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(-\mu_i^- \right)^2 \right)^p \left(1 + (\zeta - 1) \left(-\mu_j^- \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^- \right)^2 \right)^p \left(1 - \left(\mu_j^- \right)^2 \right)^q \right]^{\frac{2}{n(n+1)}}} \\
& \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^- \right|^p \left| \eta_j^- \right|^q \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 + \eta_i^- \right) \right)^p \left(1 + (\zeta - 1) \left(1 + \eta_j^- \right) \right)^q \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^- \right|^p \left| \eta_j^- \right|^q \right]^{\frac{2}{n(n+1)}}} \\
& \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| v_i^- \right|^p \left| v_j^- \right|^q \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 + v_i^- \right) \right)^p \left(1 + (\zeta - 1) \left(1 + v_j^- \right) \right)^q \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| v_i^- \right|^p \left| v_j^- \right|^q \right]^{\frac{2}{n(n+1)}}}
\end{aligned} \right]
\end{aligned}$$

Referring to (4.25), the final stage of the proposed operator is a power transformation. The inner components are kept intact; the exponent is applied to the

already-aggregated term within the same outer structure. This step modulates nonlinearity and emphasis without re-substituting or altering the inner expressions.

$$\begin{aligned}
& \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q \right]^{\frac{1}{p+q}} = \\
& \left[\frac{\left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 - \mu_i^+) \right)^p \left(1 + (\zeta - 1)(1 - \mu_j^+) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \right. \\
& \quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)\eta_i^+ \right)^p \left(1 + (\zeta - 1)\eta_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n (1 - \eta_i^+)^p (1 - \eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)\eta_i^+ \right)^p \left(1 + (\zeta - 1)\eta_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (1 - \eta_i^+)^p (1 - \eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
& \quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-v_i^+)^2 \right)^p \left(1 + (\zeta - 1)(-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (-v_i^+)^2 \right)^p \left(1 - (-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-v_i^+)^2 \right)^p \left(1 + (\zeta - 1)(-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (-v_i^+)^2 \right)^p \left(1 - (-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
& \quad \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(\mu_i^-)^2 \right)^p \left(1 + (\zeta - 1)(\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2 \right)^p \left(1 - (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(\mu_i^-)^2 \right)^p \left(1 + (\zeta - 1)(\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2 \right)^p \left(1 - (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
& \quad \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 + \eta_i^-) \right)^p \left(1 + (\zeta - 1)(1 + \eta_j^-) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}, \\
& \quad \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |v_i^-|^p |v_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 + v_i^-) \right)^p \left(1 + (\zeta - 1)(1 + v_j^-) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |v_i^-|^p |v_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \left. \right]
\end{aligned}$$

Consequently, (4.27) is proved. To assess the proposed operator, this study now possesses three fundamental properties.

4.3.1.1 Properties of Bipolar Pythagorean Neutrosophic Generalized Hamacher Heronian Mean Operator

Property 4.1 (Idempotency property) $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i = 1, 2, \dots, n)$ be a set of BPNSs. If $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\}$ for all i , then, $BPN-GHHM^{p,q}(A_1, A_2, \dots, A_n) = A$.

Proof Since $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i = 1, 2, \dots, n)$, then

$$\begin{aligned} BPN-GHHM^{p,q}(x_1, x_2, \dots, x_n) &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q \right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n A^p \otimes A^q \right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n A^{p+q} \right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)} \left(\frac{n(n+1)}{2} \right) (A^{p+q}) \right]^{\frac{1}{p+q}} \\ &= A \end{aligned}$$

$$\Rightarrow BPN-GHHM^{p,q}(A_1, A_2, \dots, A_n) = A.$$

Property 4.2 (Monotonicity property) For any two set of BPNSs, $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\}$ and $A_i^* = \{\mu_i^{+*}, \eta_i^{+*}, \nu_i^{+*}, \mu_i^{-*}, \eta_i^{-*}, \nu_i^{-*}\}$, such that $A_i \leq A_i^*$ for all $i = 1, 2, \dots, n$. If $\mu^+ \leq \mu^{+*}, \eta^+ \geq \eta^{+*}, \nu^+ \geq \nu^{+*}$ and $\mu^- \geq \mu^{-*}, \eta^- \leq \eta^{-*}, \nu^- \leq \nu^{-*}$, then

$$BPN-GHHM^{p,q}(A_1, A_2, \dots, A_n) \leq BPN-GHHM^{p,q}(A_1^*, A_2^*, \dots, A_n^*).$$

Proof If $\mu^+ \leq \mu^{+*}, \eta^+ \geq \eta^{+*}, \nu^+ \geq \nu^{+*}$ and $\mu^- \geq \mu^{-*}, \eta^- \leq \eta^{-*}, \nu^- \leq \nu^{-*}$, then

$$\begin{aligned}
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(\mu_i^+ \right)^p \left(\mu_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)(1-\mu_i^+) \right)^p \left(1+(\zeta-1)(1-\mu_j^+) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(\mu_i^+ \right)^p \left(\mu_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(\mu_i^{+*} \right)^p \left(\mu_j^{+*} \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)(1-\mu_i^{+*}) \right)^p \left(1+(\zeta-1)(1-\mu_j^{+*}) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(\mu_i^{+*} \right)^p \left(\mu_j^{+*} \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)\eta_i^+ \right)^p \left(1+(\zeta-1)\eta_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-\eta_i^+ \right)^p \left(1-\eta_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)\eta_i^+ \right)^p \left(1+(\zeta-1)\eta_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-\eta_i^+ \right)^p \left(1-\eta_j^+ \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)\eta_i^{+*} \right)^p \left(1+(\zeta-1)\eta_j^{+*} \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-\eta_i^{+*} \right)^p \left(1-\eta_j^{+*} \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)\eta_i^{+*} \right)^p \left(1+(\zeta-1)\eta_j^{+*} \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-\eta_i^{+*} \right)^p \left(1-\eta_j^{+*} \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)(-v_i^+)^2 \right)^p \left(1+(\zeta-1)(-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-(-v_i^+)^2 \right)^p \left(1-(-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)(-v_i^+)^2 \right)^p \left(1+(\zeta-1)(-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-(-v_i^+)^2 \right)^p \left(1-(-v_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)(-v_i^{+*})^2 \right)^p \left(1+(\zeta-1)(-v_j^{+*})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-(-v_i^{+*})^2 \right)^p \left(1-(-v_j^{+*})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1+(\zeta-1)(-v_i^{+*})^2 \right)^p \left(1+(\zeta-1)(-v_j^{+*})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1-(-v_i^{+*})^2 \right)^p \left(1-(-v_j^{+*})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^-)^2 \right)^p \left(1 + (\zeta - 1) (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2 \right)^p \left(1 - (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^-)^2 \right)^p \left(1 + (\zeta - 1) (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2 \right)^p \left(1 - (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}} \geq \\
& - \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^{*-})^2 \right)^p \left(1 + (\zeta - 1) (\mu_j^{*-})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^{*-})^2 \right)^p \left(1 - (\mu_j^{*-})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^{*-})^2 \right)^p \left(1 + (\zeta - 1) (\mu_j^{*-})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^{*-})^2 \right)^p \left(1 - (\mu_j^{*-})^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& - \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^-) \right)^p \left(1 + (\zeta - 1) (1 + \eta_j^-) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{- \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^{*-}|^p |\eta_j^{*-}|^q \right]^{\frac{2(p+q)}{n(n+1)}}} \leq \\
& \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^{*-}) \right)^p \left(1 + (\zeta - 1) (1 + \eta_j^{*-}) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^{*-}|^p |\eta_j^{*-}|^q \right]^{\frac{2(p+q)}{n(n+1)}}
\end{aligned}$$

and

$$\begin{aligned}
& - \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\nu_i^-|^p |\nu_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}} \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \nu_i^-) \right)^p \left(1 + (\zeta - 1) (1 + \nu_j^-) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\nu_i^-|^p |\nu_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{- \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\nu_i^{*-}|^p |\nu_j^{*-}|^q \right]^{\frac{2(p+q)}{n(n+1)}}} \leq \\
& \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \nu_i^{*-}) \right)^p \left(1 + (\zeta - 1) (1 + \nu_j^{*-}) \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\nu_i^{*-}|^p |\nu_j^{*-}|^q \right]^{\frac{2(p+q)}{n(n+1)}}
\end{aligned}$$

The validity of property 4.2 has been proven.

Property 4.3 (*Boundedness property*) For a collection of BPNSs, let $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i=1, 2, \dots, n)$ where $\zeta > 0$ and if

$$A^- = \left\langle \min(\mu^+), \max(\eta^+), \max(\nu^+), \max(\mu^-), \min(\eta^-), \min(\nu^-) \right\rangle$$

$$A^+ = \left\langle \max(\mu^{+*}), \min(\eta^{+*}), \min(\nu^{+*}), \min(\mu^{-*}), \max(\eta^{-*}), \max(\nu^{-*}) \right\rangle$$

Proof From the results presented in properties 4.1 and 4.2, the following can be inferred from:

$$A^+ = BPN - GHM^{p,q} (A^+, A^+, \dots, A^+),$$

$$A^- = BPN - GHM^{p,q} (A^-, A^-, \dots, A^-).$$

Thus, $BPN - GHM^{p,q} (A^-, A^-, \dots, A^-) \leq BPN - GHM^{p,q} (A_1, A_2, \dots, A_n)$

4.3.2 Bipolar Pythagorean Neutrosophic Generalized Weighted Hamacher Heronian Mean (BPN-GWHHM) Operator

Definition 4.19 Let $p, q \geq 0$ and $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i=1, 2, \dots, n)$ be a collection of BPNSs. The BPN-GWHHM operator defined as follows:

$$BPN - GWHHM^{p,q} (x_1, x_2, \dots, x_n) = \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (w_i A_i)^p \otimes (w_j A_j)^q \right]^{\frac{1}{p+q}} \quad (4.28)$$

Theorem 4.2 Let $p, q \geq 0$ and $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i=1, 2, \dots, n)$ be a collection of BPNSs. Let the BPNS information be aggregated using the BPN-GWHHM operator, as defined in Definition 4.19, based on the Hamacher operations outlined in Definition 4.17. The aggregated value is presented as follows:

$$\begin{aligned}
BPN - GWHHM^{p,q}(x_1, x_2, \dots, x_n) &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (w_i A_i)^p \otimes (w_j A_j)^q \right]^{\frac{1}{p+q}} = \\
&= \left\{ \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^+ \right)^2 \right)^{p w_i} \left(1 + (\zeta - 1) \left(\mu_j^+ \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^+ \right)^2 \right)^{p w_i} \left(1 - \left(\mu_j^+ \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^+ \right)^2 \right)^{p w_i} \left(1 + (\zeta - 1) \left(\mu_j^+ \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^+ \right)^2 \right)^{p w_i} \left(1 - \left(\mu_j^+ \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \right. \\
&\quad \left. \zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left(\eta_i^+ \right)^{p w_i} \left(\eta_j^+ \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \right. \\
&\quad \left. \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - \eta_i^+) \right)^{p w_i} \left(1 + (\zeta - 1) (1 - \eta_j^+) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(\eta_i^+ \right)^{p w_i} \left(\eta_j^+ \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left(v_i^+ \right)^{p w_i} \left(v_j^+ \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \right. \\
&\quad \left. \sqrt{\frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - v_i^+) \right)^{p w_i} \left(1 + (\zeta - 1) (1 - v_j^+) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(v_i^+ \right)^{p w_i} \left(v_j^+ \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \mu_i^- \right|^{p w_i} \left| \mu_j^- \right|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}} \right. \\
&\quad \left. \sqrt{\frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \mu_i^-) \right)^{p w_i} \left(1 + (\zeta - 1) (1 + \mu_j^-) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \mu_i^- \right|^{p w_i} \left| \mu_j^- \right|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^- \right|^{p w_i} \left| \eta_j^- \right|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}} \right. \\
&\quad \left. \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^-) \right)^{p w_i} \left(1 + (\zeta - 1) (1 + \eta_j^-) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^- \right|^{p w_i} \left| \eta_j^- \right|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(v_i^- \right)^2 \right)^{p w_i} \left(1 + (\zeta - 1) \left(v_j^- \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + \left(v_i^- \right)^2 \right)^{p w_i} \left(1 + \left(v_j^- \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \right. \\
&\quad \left. \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(v_i^- \right)^2 \right)^{p w_i} \left(1 + (\zeta - 1) \left(v_j^- \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + \left(v_i^- \right)^2 \right)^{p w_i} \left(1 + \left(v_j^- \right)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\right.} \left. \right\} \quad (4.29)
\end{aligned}$$

Proof Invoking (4.24), the result follows by replacing the function's coefficient with the desired scalar and performing the scalar-multiplication operation.

$$\begin{aligned}
w_i A_i &= \left\{ \begin{aligned} &\sqrt{\frac{\left(1+(\zeta-1)(\mu_i^+)^2\right)^{w_i} - \left(1-(\mu_i^+)^2\right)^{w_i}}{\left(1+(\zeta-1)(\mu_i^+)^2\right)^{w_i} + (\zeta-1)\left(1-(\mu_i^+)^2\right)^{w_i}}}, \frac{\zeta(\eta_i^+)^{w_i}}{\left(1+(\zeta-1)(1-\eta_i^+)\right)^{w_i} + (\zeta-1)(\eta_i^+)^{w_i}}, \\ &\frac{\zeta(v_i^+)^{w_i}}{\left(1+(\zeta-1)(1-v_i^+)\right)^{w_i} + (\zeta-1)(v_i^+)^{w_i}}, \frac{\zeta|\mu_i^-|^{w_i}}{\left(1+(\zeta-1)(1+\mu_i^-)\right)^{w_i} + (\zeta-1)|\mu_i^-|^{w_i}}, \\ &\frac{\zeta|\eta_i^-|^{w_i}}{\left(1+(\zeta-1)(1+\eta_i^-)\right)^{w_i} + (\zeta-1)|\eta_i^-|^{w_i}}, -\sqrt{\frac{\left(1+(\zeta-1)(v_i^-)^2\right)^{w_i} - \left(1+(v_i^-)^2\right)^{w_i}}{\left(1+(\zeta-1)(v_i^-)^2\right)^{w_i} - (\zeta-1)\left(1+(v_i^-)^2\right)^{w_i}}} \end{aligned} \right\} \\
w_j A_j &= \left\{ \begin{aligned} &\sqrt{\frac{\left(1+(\zeta-1)(\mu_j^+)^2\right)^{w_j} - \left(1-(\mu_j^+)^2\right)^{w_j}}{\left(1+(\zeta-1)(\mu_j^+)^2\right)^{w_j} + (\zeta-1)\left(1-(\mu_j^+)^2\right)^{w_j}}}, \frac{\zeta(\eta_j^+)^{w_j}}{\left(1+(\zeta-1)(1-\eta_j^+)\right)^{w_j} + (\zeta-1)(\eta_j^+)^{w_j}}, \\ &\frac{\zeta(v_j^+)^{w_j}}{\left(1+(\zeta-1)(1-v_j^+)\right)^{w_j} + (\zeta-1)(v_j^+)^{w_j}}, \frac{\zeta|\mu_j^-|^{w_j}}{\left(1+(\zeta-1)(1+\mu_j^-)\right)^{w_j} + (\zeta-1)|\mu_j^-|^{w_j}}, \\ &\frac{\zeta|\eta_j^-|^{w_j}}{\left(1+(\zeta-1)(1+\eta_j^-)\right)^{w_j} + (\zeta-1)|\eta_j^-|^{w_j}}, -\sqrt{\frac{\left(1+(\zeta-1)(v_j^-)^2\right)^{w_j} - \left(1+(v_j^-)^2\right)^{w_j}}{\left(1+(\zeta-1)(v_j^-)^2\right)^{w_j} - (\zeta-1)\left(1+(v_j^-)^2\right)^{w_j}}} \end{aligned} \right\}
\end{aligned}$$

Proceed by reusing the prior formula as the base; then apply the power operator, (4.25), to this aggregated term.

$$(w_i A_i)^P = \left\{ \frac{\left(\left(1 + (\zeta - 1) (\mu_i^+)^2 \right)^{w_i} \right)^P - \left(\left(1 - (\mu_i^+)^2 \right)^{w_i} \right)^P}{\left(\left(1 + (\zeta - 1) (\mu_i^+)^2 \right)^{w_i} \right)^P + (\zeta - 1) \left(\left(1 - (\mu_i^+)^2 \right)^{w_i} \right)^P}, \frac{\zeta \left((\eta_i^+)^{w_i} \right)^P}{\left(\left(1 + (\zeta - 1) (1 - \eta_i^+) \right)^{w_i} \right)^P + (\zeta - 1) \left((\eta_i^+)^{w_i} \right)^P}, \right. \\ \left. \frac{\zeta \left((\nu_i^+)^{w_i} \right)^P}{\left(\left(1 + (\zeta - 1) (1 - \nu_i^+) \right)^{w_i} \right)^P + (\zeta - 1) \left((\nu_i^+)^{w_i} \right)^P}, \frac{\zeta \left(|\mu_i^-|^{w_i} \right)^P}{\left(\left(1 + (\zeta - 1) (1 + \mu_i^-) \right)^{w_i} \right)^P + (\zeta - 1) \left(|\mu_i^-|^{w_i} \right)^P}, \right. \\ \left. \frac{\zeta \left(|\eta_i^-|^{w_i} \right)^P}{\left(\left(1 + (\zeta - 1) (1 + \eta_i^-) \right)^{w_i} \right)^P + (\zeta - 1) \left(|\eta_i^-|^{w_i} \right)^P}, - \frac{\left(\left(1 + (\zeta - 1) (\nu_i^-)^2 \right)^{w_i} \right)^P - \left(\left(1 + (\nu_i^-)^2 \right)^{w_i} \right)^P}{\left(\left(1 + (\zeta - 1) (\nu_i^-)^2 \right)^{w_i} \right)^P - (\zeta - 1) \left(\left(1 + (\nu_i^-)^2 \right)^{w_i} \right)^P} \right\}$$

$$(w_j A_j)^q = \left\{ \frac{\left(\left(1 + (\zeta - 1) (\mu_j^+)^2 \right)^{w_j} \right)^q - \left(\left(1 - (\mu_j^+)^2 \right)^{w_j} \right)^q}{\left(\left(1 + (\zeta - 1) (\mu_j^+)^2 \right)^{w_j} \right)^q + (\zeta - 1) \left(\left(1 - (\mu_j^+)^2 \right)^{w_j} \right)^q}, \frac{\zeta \left((\eta_j^+)^{w_j} \right)^q}{\left(\left(1 + (\zeta - 1) (1 - \eta_j^+) \right)^{w_j} \right)^q + (\zeta - 1) \left((\eta_j^+)^{w_j} \right)^q}, \right. \\ \left. \frac{\zeta \left((\nu_j^+)^{w_j} \right)^q}{\left(\left(1 + (\zeta - 1) (1 - \nu_j^+) \right)^{w_j} \right)^q + (\zeta - 1) \left((\nu_j^+)^{w_j} \right)^q}, \frac{\zeta \left(|\mu_j^-|^{w_j} \right)^q}{\left(\left(1 + (\zeta - 1) (1 + \mu_j^-) \right)^{w_j} \right)^q + (\zeta - 1) \left(|\mu_j^-|^{w_j} \right)^q}, \right. \\ \left. \frac{\zeta \left(|\eta_j^-|^{w_j} \right)^q}{\left(\left(1 + (\zeta - 1) (1 + \eta_j^-) \right)^{w_j} \right)^q + (\zeta - 1) \left(|\eta_j^-|^{w_j} \right)^q}, - \frac{\left(\left(1 + (\zeta - 1) (\nu_j^-)^2 \right)^{w_j} \right)^q - \left(\left(1 + (\nu_j^-)^2 \right)^{w_j} \right)^q}{\left(\left(1 + (\zeta - 1) (\nu_j^-)^2 \right)^{w_j} \right)^q - (\zeta - 1) \left(\left(1 + (\nu_j^-)^2 \right)^{w_j} \right)^q} \right\}$$

Referring to the multiplication on (4.23), apply the multiplicative aggregation to the previously obtained terms.

$$\begin{aligned}
& (w_i A_i)^p \otimes (w_j A_j)^q = \\
& \left[\begin{aligned}
& \frac{\left(1 + (\zeta - 1)(\mu_i^+)^2\right)^{pw_i} \left(1 + (\zeta - 1)(\mu_j^+)^2\right)^{qw_j} - \left(1 - (\mu_i^+)^2\right)^{pw_i} \left(1 - (\mu_j^+)^2\right)^{qw_j}}{\left(1 + (\zeta - 1)(\mu_i^+)^2\right)^{pw_i} \left(1 + (\zeta - 1)(\mu_j^+)^2\right)^{qw_j} + (\zeta - 1) \left(1 - (\mu_i^+)^2\right)^{pw_i} \left(1 - (\mu_j^+)^2\right)^{qw_j}}, \\
& \frac{\zeta (\eta_i^+)^{pw_i} (\eta_j^+)^{qw_j}}{\left(1 + (\zeta - 1)(1 - \eta_i^+)\right)^{pw_i} \left(1 + (\zeta - 1)(1 - \eta_j^+)\right)^{qw_j} + (\zeta - 1) (\eta_i^+)^{pw_i} (\eta_j^+)^{qw_j}}, \\
& \frac{\zeta (\nu_i^+)^{pw_i} (\nu_j^+)^{qw_j}}{\left(1 + (\zeta - 1)(1 - \nu_i^+)\right)^{pw_i} \left(1 + (\zeta - 1)(1 - \nu_j^+)\right)^{qw_j} + (\zeta - 1) (\nu_i^+)^{pw_i} (\nu_j^+)^{qw_j}}, \\
& \frac{\zeta |\mu_i^-|^{pw_i} |\mu_j^-|^{qw_j}}{\left(1 + (\zeta - 1)(1 + \mu_i^-)\right)^{pw_i} \left(1 + (\zeta - 1)(1 + \mu_j^-)\right)^{qw_j} + (\zeta - 1) |\mu_i^-|^{pw_i} |\mu_j^-|^{qw_j}}, \\
& \frac{\zeta |\eta_i^-|^{pw_i} |\eta_j^-|^{qw_j}}{\left(1 + (\zeta - 1)(1 + \eta_i^-)\right)^{pw_i} \left(1 + (\zeta - 1)(1 + \eta_j^-)\right)^{qw_j} + (\zeta - 1) |\eta_i^-|^{pw_i} |\eta_j^-|^{qw_j}}, \\
& - \frac{\left(1 + (\zeta - 1)(\nu_i^-)^2\right)^{pw_i} \left(1 + (\zeta - 1)(\nu_j^-)^2\right)^{qw_j} - \left(1 + (\nu_i^-)^2\right)^{pw_i} \left(1 + (\nu_j^-)^2\right)^{qw_j}}{\left(1 + (\zeta - 1)(\nu_i^-)^2\right)^{pw_i} \left(1 + (\zeta - 1)(\nu_j^-)^2\right)^{qw_j} - (\zeta - 1) \left(1 + (\nu_i^-)^2\right)^{pw_i} \left(1 + (\nu_j^-)^2\right)^{qw_j}}
\end{aligned} \right]
\end{aligned}$$

Based on the summation (4.22), aggregate the previously evaluated terms using the prescribed additive form.

$$\begin{aligned}
& \sum_{i=1}^n \sum_{j=1}^n (w_i A_i)^P \otimes (w_j A_j)^Q = \\
& \left[\begin{aligned}
& \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^+)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (\mu_j^+)^2\right)^{Qw_j} - \sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2\right)^{Pw_i} \left(1 - (\mu_j^+)^2\right)^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^+)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (\mu_j^+)^2\right)^{Qw_j} + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2\right)^{Pw_i} \left(1 - (\mu_j^+)^2\right)^{Qw_j}}, \\
& \frac{\zeta \sum_{i=1}^n \sum_{j=1}^n (\eta_i^+)^{Pw_i} (\eta_j^+)^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - \eta_i^+)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (1 - \eta_j^+)^2\right)^{Qw_j} + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n (\eta_i^+)^{Pw_i} (\eta_j^+)^{Qw_j}}, \\
& \frac{\zeta \sum_{i=1}^n \sum_{j=1}^n (\nu_i^+)^{Pw_i} (\nu_j^+)^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - \nu_i^+)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (1 - \nu_j^+)^2\right)^{Qw_j} + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n (\nu_i^+)^{Pw_i} (\nu_j^+)^{Qw_j}}, \\
& \frac{\zeta \sum_{i=1}^n \sum_{j=1}^n |\mu_i^-|^{Pw_i} |\mu_j^-|^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \mu_i^-)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (1 + \mu_j^-)^2\right)^{Qw_j} + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n |\mu_i^-|^{Pw_i} |\mu_j^-|^{Qw_j}}, \\
& \frac{\zeta \sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{Pw_i} |\eta_j^-|^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^-)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (1 + \eta_j^-)^2\right)^{Qw_j} + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{Pw_i} |\eta_j^-|^{Qw_j}}, \\
& \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^-)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (1 + \eta_j^-)^2\right)^{Qw_j} + (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{Pw_i} |\eta_j^-|^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\nu_i^-)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (\nu_j^-)^2\right)^{Qw_j} - \sum_{i=1}^n \sum_{j=1}^n \left(1 + (\nu_i^-)^2\right)^{Pw_i} \left(1 + (\nu_j^-)^2\right)^{Qw_j}}, \\
& \frac{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\nu_i^-)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (\nu_j^-)^2\right)^{Qw_j} - (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n \left(1 + (\nu_i^-)^2\right)^{Pw_i} \left(1 + (\nu_j^-)^2\right)^{Qw_j}}{\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\nu_i^-)^2\right)^{Pw_i} \left(1 + (\zeta - 1) (\nu_j^-)^2\right)^{Qw_j} - (\zeta - 1) \sum_{i=1}^n \sum_{j=1}^n \left(1 + (\nu_i^-)^2\right)^{Pw_i} \left(1 + (\nu_j^-)^2\right)^{Qw_j}}
\end{aligned} \right]
\end{aligned}$$

From the scalar multiplication (4.24), insert the prior expression as the argument and execute the scalar rescaling accordingly.

$$\begin{aligned}
& \frac{2}{n(n+1)} \left[\sum_{i=1}^n \sum_{j=1}^n (w_i A_i)^P \otimes (w_j A_j)^Q \right] = \\
& \left\{ \begin{aligned} & \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^+)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2 \right)^{p w_i} \left(1 - (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^+)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2 \right)^{p w_i} \left(1 - (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}}}, \\ & \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n (\eta_i^+)^{p w_i} (\eta_j^+)^{q w_j} \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - \eta_i^+) \right)^{p w_i} \left(1 + (\zeta - 1) (1 - \eta_j^+) \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (\eta_i^+)^{p w_i} (\eta_j^+)^{q w_j} \right]^{\frac{2}{n(n+1)}}}, \\ & \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n (v_i^+)^{p w_i} (v_j^+)^{q w_j} \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - v_i^+) \right)^{p w_i} \left(1 + (\zeta - 1) (1 - v_j^+) \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (v_i^+)^{p w_i} (v_j^+)^{q w_j} \right]^{\frac{2}{n(n+1)}}}, \\ & \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\mu_i^-|^{p w_i} |\mu_j^-|^{q w_j} \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \mu_i^-) \right)^{p w_i} \left(1 + (\zeta - 1) (1 + \mu_j^-) \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\mu_i^-|^{p w_i} |\mu_j^-|^{q w_j} \right]^{\frac{2}{n(n+1)}}}, \\ & \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{p w_i} |\eta_j^-|^{q w_j} \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^-) \right)^{p w_i} \left(1 + (\zeta - 1) (1 + \eta_j^-) \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{p w_i} |\eta_j^-|^{q w_j} \right]^{\frac{2}{n(n+1)}}}, \\ & \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (v_i^-)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (v_j^-)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (v_i^-)^2 \right)^{p w_i} \left(1 + (v_j^-)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (v_i^-)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (v_j^-)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (v_i^-)^2 \right)^{p w_i} \left(1 + (v_j^-)^2 \right)^{q w_j} \right]^{\frac{2}{n(n+1)}}} \end{aligned} \right\}
\end{aligned}$$

According to the power operation (4.25), the result follows by exponentiating the aggregated quantity with the designated power while preserving the existing structure.

$$\begin{aligned}
& \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (w_i A_i)^p \otimes (w_j A_j)^q \right]^{\frac{1}{p+q}} = \\
& \left\{ \begin{aligned} & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^+)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2 \right)^{p w_i} \left(1 - (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \right. \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^+)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2 \right)^{p w_i} \left(1 - (\mu_j^+)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \zeta \left[\sum_{i=1}^n \sum_{j=1}^n (\eta_i^+)^{p w_i} (\eta_j^+)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - \eta_i^+) \right)^{p w_i} \left(1 + (\zeta - 1) (1 - \eta_j^+) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (\eta_i^+)^{p w_i} (\eta_j^+)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \zeta \left[\sum_{i=1}^n \sum_{j=1}^n (\nu_i^+)^{p w_i} (\nu_j^+)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 - \nu_i^+) \right)^{p w_i} \left(1 + (\zeta - 1) (1 - \nu_j^+) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (\nu_i^+)^{p w_i} (\nu_j^+)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\mu_i^-|^{p w_i} |\mu_j^-|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \mu_i^-) \right)^{p w_i} \left(1 + (\zeta - 1) (1 + \mu_j^-) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\mu_i^-|^{p w_i} |\mu_j^-|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & -\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{p w_i} |\eta_j^-|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (1 + \eta_i^-) \right)^{p w_i} \left(1 + (\zeta - 1) (1 + \eta_j^-) \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^{p w_i} |\eta_j^-|^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\nu_i^-)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (\nu_j^-)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\nu_i^-)^2 \right)^{p w_i} \left(1 + (\nu_j^-)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \\ & \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\nu_i^-)^2 \right)^{p w_i} \left(1 + (\zeta - 1) (\nu_j^-)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\nu_i^-)^2 \right)^{p w_i} \left(1 + (\nu_j^-)^2 \right)^{q w_j} \right]^{\frac{2(p+q)}{n(n+1)}} \end{aligned} \right\}
\end{aligned}$$

With the BPN-GWHHM operator established, this study proceeds to present the operator's properties.

4.3.2.1 Properties of Bipolar Pythagorean Neutrosophic Generalized Weighted Hamacher Heronian Mean Operator

Property 4.4 (Idempotency property) Let $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i=1, 2, \dots, n)$ be a set of BPNSs. If $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\}$ and $w_i=1$ for all i , then $BPN-GWHHM^{p,q}(A_1, A_2, \dots, A_n) = A$.

Proof Since $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\} (i=1, 2, \dots, n)$, then

$$\begin{aligned} BPN-GWHHM^{p,q}(x_1, x_2, \dots, x_n) &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n (w_i A_i)^p \otimes (w_j A_j)^q \right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n A^p \otimes A^q \right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=1}^n A^{p+q} \right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)} \left(\frac{n(n+1)}{2} \right) (A^{p+q}) \right]^{\frac{1}{p+q}} \\ &= A. \end{aligned}$$

$$\Rightarrow BPN-GWHHM^{p,q}(A_1, A_2, \dots, A_n) = A.$$

Property 4.5 (Monotonicity property) For any two set of BPNS, $A_i = \{\mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^-\}$ and $A_i^* = \{\mu_i^{+*}, \eta_i^{+*}, \nu_i^{+*}, \mu_i^{-*}, \eta_i^{-*}, \nu_i^{-*}\}$, such that $A_i \leq A_i^*$ for all $i=1, 2, \dots, n$. If $\mu^+ \leq \mu^{+*}, \eta^+ \geq \eta^{+*}, \nu^+ \geq \nu^{+*}$ and $\mu^- \geq \mu^{-*}, \eta^- \leq \eta^{-*}, \nu^- \leq \nu^{-*}$, then

$$BPN-GWHHM^{p,q}(A_1, A_2, \dots, A_n) \leq BPN-GWHHM^{p,q}(A_1^*, A_2^*, \dots, A_n^*).$$

Proof If $\mu^+ \leq \mu^{+*}, \eta^+ \geq \eta^{+*}, \nu^+ \geq \nu^{+*}$ and $\mu^- \geq \mu^{-*}, \eta^- \leq \eta^{-*}, \nu^- \leq \nu^{-*}$, then

$$\begin{aligned}
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^+ \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(\mu_j^+ \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^+ \right)^2 \right)^{pw_i} \left(1 - \left(\mu_j^+ \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^+ \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(\mu_j^+ \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^+ \right)^2 \right)^{pw_i} \left(1 - \left(\mu_j^+ \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \leq \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^{+*} \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(\mu_j^{+*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^{+*} \right)^2 \right)^{pw_i} \left(1 - \left(\mu_j^{+*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(\mu_i^{+*} \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(\mu_j^{+*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - \left(\mu_i^{+*} \right)^2 \right)^{pw_i} \left(1 - \left(\mu_j^{+*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left(\eta_i^+ \right)^{pw_i} \left(\eta_j^+ \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 - \eta_i^+ \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 - \eta_j^+ \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(\eta_i^+ \right)^{pw_i} \left(\eta_j^+ \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \geq \\
& \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left(\eta_i^{+*} \right)^{pw_i} \left(\eta_j^{+*} \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 - \eta_i^{+*} \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 - \eta_j^{+*} \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(\eta_i^{+*} \right)^{pw_i} \left(\eta_j^{+*} \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \\
& \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left(v_i^+ \right)^{pw_i} \left(v_j^+ \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 - v_i^+ \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 - v_j^+ \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(v_i^+ \right)^{pw_i} \left(v_j^+ \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \geq \\
& \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left(v_i^{+*} \right)^{pw_i} \left(v_j^{+*} \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 - v_i^{+*} \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 - v_j^{+*} \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(v_i^{+*} \right)^{pw_i} \left(v_j^{+*} \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}
\end{aligned}$$

$$\begin{aligned}
& \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \mu_i^- \right|^{pw_i} \left| \mu_j^- \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\sqrt{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 + \mu_i^- \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 + \mu_j^- \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \mu_i^- \right|^{pw_i} \left| \mu_j^- \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}} \geq \\
& \frac{\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \mu_i^{-*} \right|^{pw_i} \left| \mu_j^{-*} \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\sqrt{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 + \mu_i^{-*} \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 + \mu_j^{-*} \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \mu_i^{-*} \right|^{pw_i} \left| \mu_j^{-*} \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}} \\
& \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^- \right|^{pw_i} \left| \eta_j^- \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 + \eta_i^- \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 + \eta_j^- \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^- \right|^{pw_i} \left| \eta_j^- \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \leq \\
& \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^{-*} \right|^{pw_i} \left| \eta_j^{-*} \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(1 + \eta_i^{-*} \right) \right)^{pw_i} \left(1 + (\zeta - 1) \left(1 + \eta_j^{-*} \right) \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left| \eta_i^{-*} \right|^{pw_i} \left| \eta_j^{-*} \right|^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(v_i^- \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(v_j^- \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + \left(v_i^- \right)^2 \right)^{pw_i} \left(1 + \left(v_j^- \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(v_i^- \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(v_j^- \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + \left(v_i^- \right)^2 \right)^{pw_i} \left(1 + \left(v_j^- \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \leq \\
& \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(v_i^{-*} \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(v_j^{-*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + \left(v_i^{-*} \right)^2 \right)^{pw_i} \left(1 + \left(v_j^{-*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) \left(v_i^{-*} \right)^2 \right)^{pw_i} \left(1 + (\zeta - 1) \left(v_j^{-*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + \left(v_i^{-*} \right)^2 \right)^{pw_i} \left(1 + \left(v_j^{-*} \right)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}
\end{aligned}$$

Property 4.6 (*Boundedness property*) For a collection of BPNSs, let

$A_i = \{ \mu_i^+, \eta_i^+, \nu_i^+, \mu_i^-, \eta_i^-, \nu_i^- \} (i = 1, 2, \dots, n)$ where $\zeta > 0$ and if

$$A^- = \left\{ \min(\mu^+), \max(\eta^+), \max(\nu^+), \max(\mu^-), \min(\eta^-), \min(\nu^-) \right\}$$

$$A^+ = \left\{ \max(\mu^+), \min(\eta^+), \min(\nu^+), \min(\mu^-), \max(\eta^-), \max(\nu^-) \right\}$$

Proof From the results presented in properties 4.4 and 4.5, the following can be inferred from:

$$A^+ = BPN - GWHHM^{p,q} \left(A^+, A^+, \dots, A^+ \right),$$

$$A^- = BPN - GWHHM^{p,q} \left(A^-, A^-, \dots, A^- \right).$$

Therefore, $BPN - GWHHM^{p,q} \left(A^-, A^-, \dots, A^- \right) \leq BPN - GWHHM^{p,q} \left(A_1, A_2, \dots, A_n \right) \leq BPN - GWHHM^{p,q} \left(A^+, A^+, \dots, A^+ \right)$. Hence, $A^- \leq BPN - GWHHM^{p,q} \left(A_1, A_2, \dots, A_n \right) \leq A^+$. The next section presents detail computation of the proposed operators in a problem that utilised existing numerical studies to evaluate its performance and effectiveness. The results obtained provide insights into the applicability and reliability of the proposed methods in addressing MCDM problems under uncertain conditions.

4.4 Illustrative Example

A numerical example is carried out following the example proposed by Yaacob et al. (2024), utilizing the proposed BPN-GHHM and BPN-GWHHM operators to select the most suitable alternative. The calculation process in this section and the following comparative analysis, were performed using Microsoft Excel for Microsoft 365 (Version 2605), where spreadsheet formulas were used to compute the results and generate the corresponding figures. Suppose an investment company seeks to allocate its funds to the most appropriate alternative. Four alternatives are considered: an automobile company (A_1), a green energy company (A_2), a computer company (A_3), and a biotech company (A_4). These alternatives are evaluated against four criteria, including market conditions (C_1), growth factor (C_2), return on investment (C_3), and environmental impact (C_4). The evaluations are expressed in the form of BPNNs. The proposed BPN-GHHM operator is applied to identify the optimal investment alternative. The obtained results are then compared with those of existing aggregation methods to demonstrate the effectiveness and applicability of the proposed approach.

Step 1. Construction of the BPNS decision matrix

Consider BPNN decision matrix involving m alternatives, A_i ($i = 1, 2, \dots, m$) and n criteria C_j ($j = 1, 2, \dots, n$). Let

$$X = [x_{ij}]_{m \times n}$$

Where $x_{ij} = (\mu_{ij}^+, \eta_{ij}^+, \nu_{ij}^+, \mu_{ij}^-, \eta_{ij}^-, \nu_{ij}^-)$. Thus, for the investment company problem, the initial BPNS decision matrix is constructed as,

$$X = [x_{ij}]_{4 \times 4}$$

Table 4.1 outlines the BPNSs decision matrix used for assessing the alternatives under the specified criteria.

Table 4.1
BPNSs Decision Matrix

Alternative	Criteria			
	C_1	C_2	C_3	C_4
A_1	(0.5,0.6,0.2, -0.7, -0.4, -0.6)	(0.5,0.4,0.4, -0.4, -0.8, -0.6)	(0.7,0.5,0.5, -0.8, -0.6, -0.7)	(0.7,0.4,0.4, -0.5, -0.3, -0.8)
A_2	(0.9,0.6,0.5, -0.7, -0.7, -0.1)	(0.8,0.6,0.7, -0.7, -0.4, -0.3)	(0.8,0.5,0.6, -0.7, -0.5, -0.2)	(0.7,0.5,0.4, -0.5, -0.9, -0.1)
A_3	(0.9,0.4,0.2, -0.6, -0.3, -0.1)	(0.5,0.3,0.2, -0.4, -0.7, -0.4)	(0.5,0.9,0.8, -0.6, -0.2, -0.5)	(0.3,0.7,0.5, -0.3, -0.2, -0.5)
A_4	(0.9,0.7,0.2, -0.8, -0.6, -0.1)	(0.5,0.7,0.3, -0.6, -0.8, -0.4)	(0.4,0.5,0.5, -0.2, -0.7, -0.1)	(0.4,0.8,0.3, -0.5, -0.6, -0.4)

Step 2. Rearrangement according to the alternatives

Since BPN-GHHM operator aggregates the evaluations of one alternative across all criteria, the matrix is reorganized into decision matrix by alternatives. For A_1 ,

$$X_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} = \begin{bmatrix} (0.5, 0.6, 0.2, -0.7, -0.4, -0.6) \\ (0.5, 0.4, 0.4, -0.4, -0.8, -0.6) \\ (0.7, 0.5, 0.5, -0.8, -0.6, -0.7) \\ (0.7, 0.4, 0.4, -0.5, -0.3, -0.8) \end{bmatrix}$$

Step 3. Apply the BPN-GHHM operator

The aggregated BPNS for A_1 is computed as,

$$\tilde{X}_1 = \text{BPN-GHHM}(X_1)$$

With the value of $p = 2$, $q = 2$, $\zeta = 0.9$, and $n = 4$, and the truth membership degree, μ_{11}^+ , is given by,

$$\mu_{11}^+ = \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1 + (\zeta - 1)(1 - \mu_i^+))^p (1 + (\zeta - 1)(1 - \mu_j^+))^q \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}$$

$$\mu_{11}^+ = \frac{\left[\sum_{i=1}^4 \sum_{j=1}^4 (\mu_i^+)^2 (\mu_j^+)^2 \right]^{\frac{2(2+2)}{4(4+1)}}}{\left[\sum_{i=1}^4 \sum_{j=1}^4 (1 + (0.9 - 1)(1 - \mu_i^+))^2 (1 + (0.9 - 1)(1 - \mu_j^+))^2 \right]^{\frac{2(2+2)}{4(4+1)}} - (0.9 - 1) \left[\sum_{i=1}^4 \sum_{j=1}^4 (\mu_i^+)^2 (\mu_j^+)^2 \right]^{\frac{2(2+2)}{4(4+1)}}}$$

$$\mu_{11}^+ = \frac{\left[(0.5)^2 (0.5)^2 + (0.5)^2 (0.5)^2 + (0.5)^2 (0.7)^2 + (0.5)^2 (0.7)^2 + \dots + (0.7)^2 (0.7)^2 \right]^{\frac{8}{20}}}{\left[(1 + (0.9 - 1)(1 - 0.5))^2 (1 + (0.9 - 1)(1 - 0.5))^2 + (1 + (0.9 - 1)(1 - 0.5))^2 (1 + (0.9 - 1)(1 - 0.5))^2 + (1 + (0.9 - 1)(1 - 0.5))^2 (1 + (0.9 - 1)(1 - 0.7))^2 + (1 + (0.9 - 1)(1 - 0.5))^2 (1 + (0.9 - 1)(1 - 0.7))^2 + \dots + (1 + (0.9 - 1)(1 - 0.7))^2 (1 + (0.9 - 1)(1 - 0.7))^2 \right]^{\frac{8}{20}} - (0.9 - 1) \left[(0.5)^2 (0.5)^2 + (0.5)^2 (0.5)^2 + (0.5)^2 (0.7)^2 + (0.5)^2 (0.7)^2 + \dots + (0.7)^2 (0.7)^2 \right]^{\frac{8}{20}}}$$

$$\mu_{11}^+ = \frac{[1.3978]^{\frac{2}{5}}}{[8.4960]^{\frac{2}{5}} - (0.9 - 1)[1.3978]^{\frac{2}{5}}}$$

$$\mu_{11}^+ = 0.4633$$

The aggregated values obtained using the proposed BPN-GHHM operator are presented in Table 4.2.

Table 4.2

Aggregated BPNN Values Using the Proposed BPN-GHHM Operator

Alternative	BPNN					
	μ^+	η^+	ν^+	μ^-	η^-	ν^-
A_1	0.4633	0.6290	0.4738	-0.6982	-0.4158	-0.5457
A_2	0.6795	0.7128	0.6563	-0.7567	-0.5269	-0.0797
A_3	0.4639	0.6582	0.5332	-0.5811	-0.2678	-0.2522
A_4	0.5923	0.7952	0.3912	-0.6989	-0.5457	-0.1036

As shown in Table 4.2, the proposed BPN-GHHM operator produces the aggregated values for each alternative. The final calculation using the BPN-GHHM operator for the remaining BPNNs are presented in the subsequent computation, where the score function in (4.14), as defined, is used to obtain a single representative value.

Step 4. Compute the score function

The score value of A_1 is computed as,

$$s(A_1) = \frac{\mu_1^+ + 1 - \eta_1^+ + 1 - \nu_1^+ + 1 - \mu_1^- + \eta_1^- + \nu_1^-}{6}$$

$$s(A_1) = \frac{0.4633 + 1 - 0.6290 + 1 - 0.4738 + 1 - (-0.6982) + (-0.4158) + (-0.5457)}{6}$$

$$s(A_1) = 0.3495$$

Step 5. Alternatives ranking

Finally,

$$S(A_i) > S(A_j) \Rightarrow A_i > A_j$$

The alternative having the largest score value is selected as the optimal alternative. The final calculation using the score function for the remaining companies is illustrated in Figure 4.2, which yields the ranking $A_2 \succ A_4 \succ A_3 \succ A_1$.

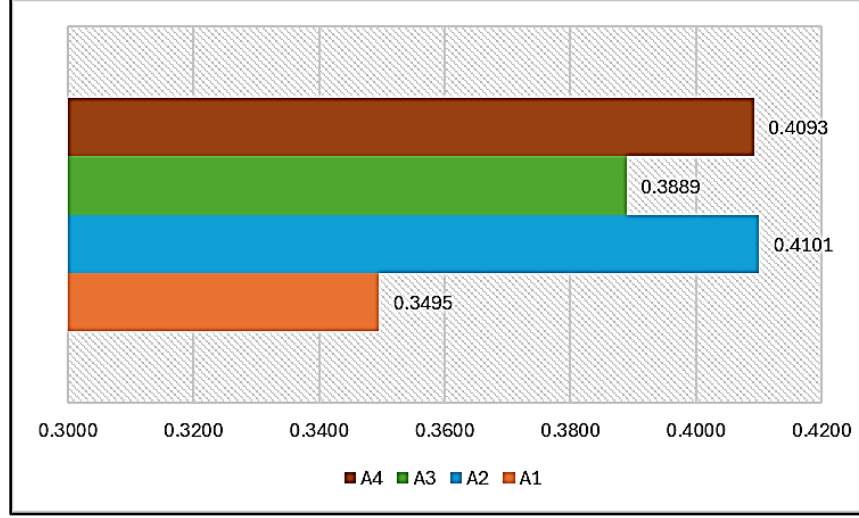


Figure 4.2 Company Ranking using BPN-GHHM Operator

Figure 4.2 demonstrate that the proposed BPN-GHHM operator can effectively differentiate the alternatives and identify the optimal choice under bipolar Pythagorean neutrosophic information. The small difference between A_2 and A_4 also indicates that both alternatives exhibit strong performance, although A_2 provides the most favourable aggregated evaluation. The implementation of the BPN-GWHHM operator follows the same computational framework as the BPN-GHHM operator. The only distinction is the inclusion of the criterion weight assignment step, which is incorporated prior to the aggregation process to account for the relative importance of each criterion.

Step 3. Apply the BPN-GWHHM operator

The aggregated BPNS for A_1 is computed as,

$$\tilde{X}_1 = \text{BPN-GWHHM}(X_1)$$

Assume the criteria weights are,

$$w = (w_1, w_2, w_3, w_4) = (0.3, 0.2, 0.3, 0.2)$$

Where $\sum_{j=1}^4 w_j = 1$. The calculation of the BPN-GWHHM operator for A_1 is shown below with $p = 2$, $q = 2$, $\zeta = 0.9$, $n = 4$. The corresponding truth membership degree calculation process is given by,

$$\begin{aligned} \mu_{11}^+ &= \frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(\mu_i^+)^2 \right)^{pw_i} \left(1 + (\zeta - 1)(\mu_j^+)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2 \right)^{pw_i} \left(1 - (\mu_j^+)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(\mu_i^+)^2 \right)^{pw_i} \left(1 + (\zeta - 1)(\mu_j^+)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^+)^2 \right)^{pw_i} \left(1 - (\mu_j^+)^2 \right)^{qw_j} \right]^{\frac{2(p+q)}{n(n+1)}}} \\ \mu_{11}^+ &= \frac{\left[\sum_{i=1}^4 \sum_{j=1}^4 \left(1 + (0.9 - 1)(\mu_i^+)^2 \right)^{2w_i} \left(1 + (0.9 - 1)(\mu_j^+)^2 \right)^{2w_j} \right]^{\frac{2(2+2)}{4(4+1)}} - \left[\sum_{i=1}^4 \sum_{j=1}^4 \left(1 - (\mu_i^+)^2 \right)^{2w_i} \left(1 - (\mu_j^+)^2 \right)^{2w_j} \right]^{\frac{2(2+2)}{4(4+1)}}}{\left[\sum_{i=1}^4 \sum_{j=1}^4 \left(1 + (0.9 - 1)(\mu_i^+)^2 \right)^{2w_i} \left(1 + (0.9 - 1)(\mu_j^+)^2 \right)^{2w_j} \right]^{\frac{2(2+2)}{4(4+1)}} + (0.9 - 1) \left[\sum_{i=1}^4 \sum_{j=1}^4 \left(1 - (\mu_i^+)^2 \right)^{2w_i} \left(1 - (\mu_j^+)^2 \right)^{2w_j} \right]^{\frac{2(2+2)}{4(4+1)}}} \\ \mu_{11}^+ &= \frac{\left[\begin{aligned} &\left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} \left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} + \\ &\left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} \left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.2)} + \\ &\left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} \left(1 + (0.9 - 1)(0.7)^2 \right)^{2(0.3)} + \\ &\dots + \\ &\left(1 + (0.9 - 1)(0.7)^2 \right)^{2(0.2)} \left(1 + (0.9 - 1)(0.7)^2 \right)^{2(0.2)} \end{aligned} \right]^{\frac{8}{20}} - \left[\begin{aligned} &\left(1 - (0.5)^2 \right)^{2(0.3)} \left(1 - (0.5)^2 \right)^{2(0.3)} + \\ &\left(1 - (0.5)^2 \right)^{2(0.3)} \left(1 - (0.5)^2 \right)^{2(0.2)} + \\ &\left(1 - (0.5)^2 \right)^{2(0.3)} \left(1 - (0.7)^2 \right)^{2(0.3)} + \\ &\dots + \\ &\left(1 - (0.7)^2 \right)^{2(0.2)} \left(1 - (0.7)^2 \right)^{2(0.2)} \end{aligned} \right]^{\frac{8}{20}}}{\left[\begin{aligned} &\left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} \left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} + \\ &\left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} \left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.2)} + \\ &\left(1 + (0.9 - 1)(0.5)^2 \right)^{2(0.3)} \left(1 + (0.9 - 1)(0.7)^2 \right)^{2(0.3)} + \\ &\dots + \\ &\left(1 + (0.9 - 1)(0.7)^2 \right)^{2(0.2)} \left(1 + (0.9 - 1)(0.7)^2 \right)^{2(0.2)} \end{aligned} \right]^{\frac{8}{20}} + (0.9 - 1) \left[\begin{aligned} &\left(1 - (0.5)^2 \right)^{2(0.3)} \left(1 - (0.5)^2 \right)^{2(0.3)} + \\ &\left(1 - (0.5)^2 \right)^{2(0.3)} \left(1 - (0.5)^2 \right)^{2(0.2)} + \\ &\left(1 - (0.5)^2 \right)^{2(0.3)} \left(1 - (0.7)^2 \right)^{2(0.3)} + \\ &\dots + \\ &\left(1 - (0.7)^2 \right)^{2(0.2)} \left(1 - (0.7)^2 \right)^{2(0.2)} \end{aligned} \right]^{\frac{8}{20}}} \\ \mu_{11}^+ &= \frac{[9.6299]^{\frac{2}{5}} - [6.2722]^{\frac{2}{5}}}{[9.6299]^{\frac{2}{5}} + (0.9 - 1)[6.2722]^{\frac{2}{5}}} \\ \mu_{11}^+ &= 0.1721 \end{aligned}$$

The aggregated BPNN values obtained using the proposed BPN-GHHM operator are presented in Table 4.3.

Table 4.3
Aggregated BPNN Values Using the Proposed BPN-GWHHM Operator

Alternative	BPNN					
	μ^+	η^+	ν^+	μ^-	η^-	ν^-
A_1	0.1721	0.7443	0.8172	-0.9058	-0.7624	-0.1640
A_2	0.3511	0.7856	0.8849	-0.9167	-0.8187	-0.0138
A_3	0.1886	0.7941	0.8335	-0.8628	-0.6321	-0.0609
A_4	0.2732	0.8359	0.7818	-0.8908	-0.8508	-0.0155

BPN-GWHHM operator produces the aggregated values for each alternative shown in Table 4.3. The final values are calculated using (4.14).

Step 4. Compute the score function

The score value of A_1 is computed as,

$$s(A_1) = \frac{\mu_1^+ + 1 - \eta_1^+ + 1 - \nu_1^+ + 1 - \mu_1^- + \eta_1^- + \nu_1^-}{6}$$

$$s(A_1) = \frac{0.1721 + 1 - 0.7443 + 1 - 0.8172 + 1 - (-0.9058) + (-0.7624) + (-0.1640)}{6}$$

$$s(A_1) = 0.2650$$

Step 5. Alternatives ranking

Figure 4.3 illustrated score function resulting in $A_2 \succ A_3 \succ A_4 \succ A_1$.

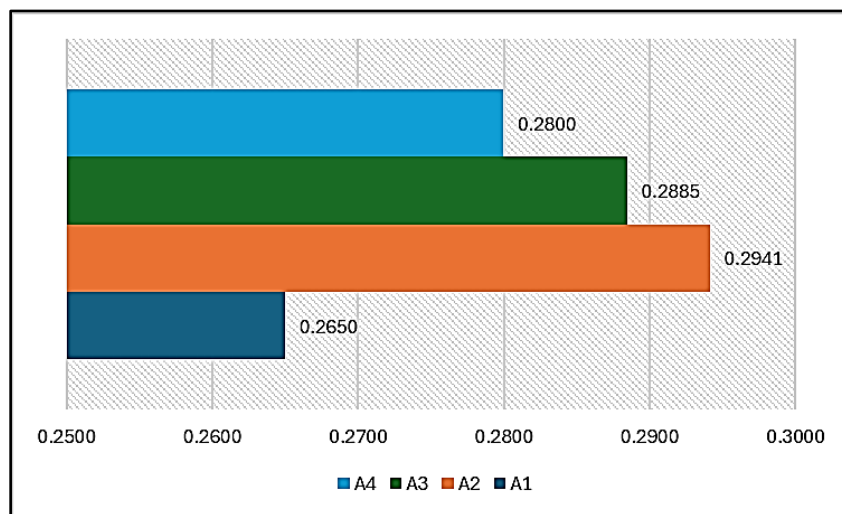


Figure 4.3 Company Ranking using BPN-GWHHM Operator

As shown in the figure, A_2 achieves the highest aggregated value of 0.2941, making it the most suitable alternative among the considered options. A_3 ranks second with a value of 0.2885, followed by A_4 with an aggregated value of 0.2800. Meanwhile, A_1 obtains the lowest value of 0.2650, indicating the least preferred alternative. The results indicate that the proposed BPN-GWHHM operator successfully distinguishes the performance of different alternatives and identifies A_2 as the optimal choice. Both proposed operators are compared in the next section.

4.5 Comparative Analysis

This section analyses and compares the two proposed operators presented in the previous section to evaluate their stability. The comparison between weighted and unweighted operators follows the approach of Garg et al. (2023). Table 4.4 shows the company ranking order using both proposed operators.

Table 4.4
Comparison of Results Between BPN-GHHM and BPN-GWHHM Operator

Proposed operator	Score value				Ranking
	A_1	A_2	A_3	A_4	
BPN-GHHM	0.3495	0.4101	0.3889	0.4093	$A_2 \succ A_4 \succ A_3 \succ A_1$
BPN-GWHHM	0.2650	0.2941	0.2885	0.2800	$A_2 \succ A_3 \succ A_4 \succ A_1$

The rankings are the same for both operators in Table 4.4, but the score values are different because the BPN-GWHHM operator accounts for criterion weights. This weighting system allows the aggregation process to take into consideration the different weights of the evaluation criteria and achieve more representative evaluation results. The fact that A_2 consistently ranks as the top choice under both BPN-GHHM and BPN-GWHHM operators indicates that the weighting scheme successfully highlights the criteria in which A_2 excels. The reliability of A_2 as the best option is strengthened by this, which suggests that the decision model is resilient to the chosen weight distribution.

The BPN-GHHM operator tends to balance extreme performance values across criteria, even though both operators produce comparable overall ranking patterns. The BPN-GWHHM operator, on the other hand, generalizes this procedure by adding a weight, offering more adaptable control over each criterion's influence. Therefore, rather than indicating a significant shift in the overall decision preference, the minor variations between the middle-ranked alternatives A_4 and A_3 show how sensitive the model is to these various aggregation behaviours. The proposed operators are consistent, as confirmed by the comparative analysis, but it does not fully describe the effect of the parameters on the results of the ranking. Thus, the sensitivity of the proposed model for various parameter values is explored in the following section.

4.6 Sensitivity Analysis

Sensitivity analysis was conducted using Microsoft Excel (Version 2605) and the figures were generated by MATLAB R2024a. The impact of parameters p , q , and ζ is discussed in this section using the proposed BPN-GHHM and BPN-GWHHM operators by varying their values in different configurations. The sensitivity analysis, summarized in Table 4.5, illustrates how changes in these parameters influence the decision results.

Table 4.5
Effect of Parameter Variation on Company Rankings

Parameters			Alternatives ranking of the operator	
p	q	ζ	BPN-GHHM	BPN-GWHHM
1	1	0.9	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$
1	1	0.4	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$
1	1	0.1	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$
1	1	3	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
1	1	10	$A_4 \succ A_2 \succ A_1 \succ A_3$	$A_2 \succ A_4 \succ A_3 \succ A_1$
0.5	1	0.5	$A_2 \succ A_4 \succ A_1 \succ A_3$	$A_3 \succ A_2 \succ A_4 \succ A_1$
2	1	0.5	$A_2 \succ A_4 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$

Parameters			Alternatives ranking of the operator	
7	1	0.5	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
10	1	0.5	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
15	1	0.5	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
1	0.5	0.5	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$
1	5	0.5	$A_4 \succ A_3 \succ A_2 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
1	10	0.5	$A_3 \succ A_4 \succ A_2 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
1	30	0.5	$A_3 \succ A_4 \succ A_1 \succ A_2$	$A_2 \succ A_4 \succ A_1 \succ A_3$
1	50	0.5	$A_3 \succ A_4 \succ A_1 \succ A_2$	$A_2 \succ A_4 \succ A_1 \succ A_3$

The decision analysis results presented in Table 4.5 are visualized in Figure 4.4- Figure 4.6. The x -axis represents the alternatives, which is companies, A_1 to A_4 , while the y -axis denotes the ranking order, where A_2 considers the highest rank and A_1 the lowest. The shortest bar corresponds to the top-performing company, and whereas the tallest bar represents the least favourable company.

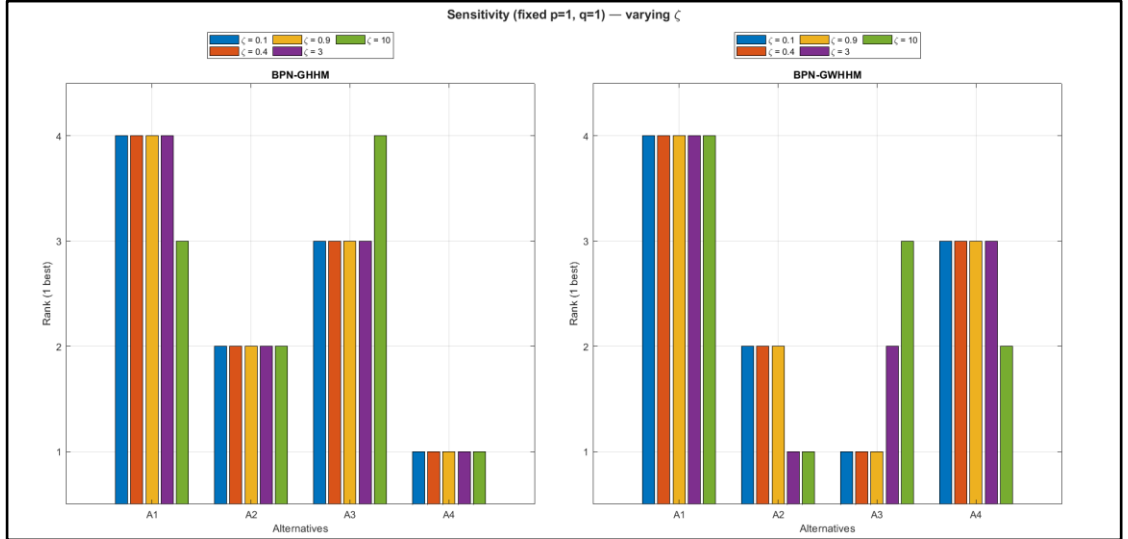


Figure 4.4 Effect of Varying ζ with Fixed p and q Values

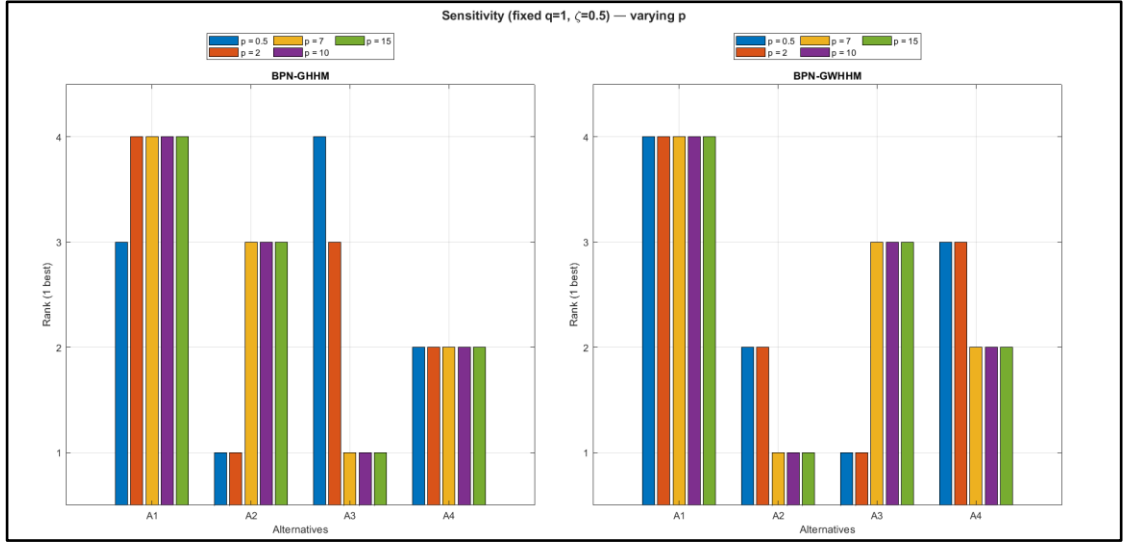


Figure 4.5 Effect of Varying p with Fixed q and ζ Values

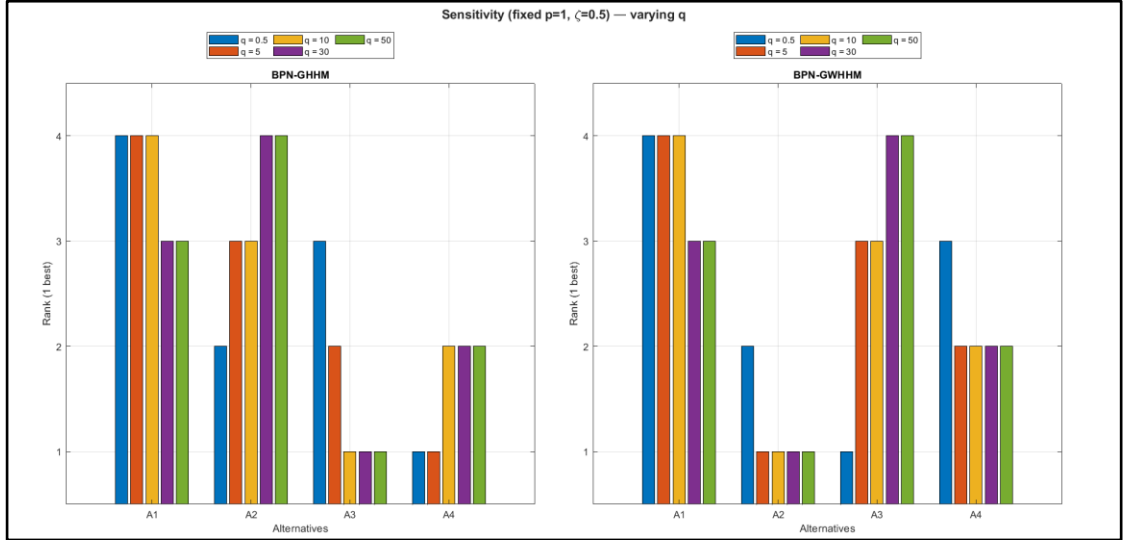


Figure 4.6 Effect of Varying q with Fixed p and ζ Values

Figure 4.4 displays the decision analysis for $p=1$, $q=1$ and $\zeta = 0.1, 0.4, 0.9, 1, 3, 10$. The findings show that the decision results remain consistent for BPN-GHHM operator within the range $\zeta = 0.1, 0.4, 0.9, 3$. However, slightly alter between $\zeta = 1$ and 10 , especially between companies A_2 and A_4 , where A_4 takes the top spot under smaller values but changes when $\zeta = 10$. For BPN-GWHHM operator, the ranking order remains stable for $\zeta = 0.1, 0.4, 0.9, 1$ and changes a bit when $\zeta = 3, 10$ for the first rank. Figure 4.5 illustrates the decision analysis for fixed $q=1$, $\zeta = 0.5$ and $p \in (0, 15]$. The decision order of the company changes little for BPN-GHHM and BPN-

GWHHM operators. Increasing p values compute variation of decision order between company A_2 and A_3 for the first rank, while the last rank remained unchanged across all p values. Figure 4.6 displays the decision analysis for $p = 1$, $\zeta = 0.5$ and $q \in (0, 50]$. The five values of q changes for BPN-GHHM operator, particularly between A_3 and A_4 for the top position, and between A_2 and A_3 for BPN-GWHHM operator. The variation also occurs between the order of the last rank company for both operators. The stable ranking results from the sensitivity analysis show that the proposed BPNS-based AOs can retain the key information from the input assessments.

4.7 Conclusion

In this chapter, two new bipolar Pythagorean neutrosophic AOs, namely BPN-GHHM and BPN-GWHHM operators, were introduced. The incorporation of BPNS theory in combination with Hamacher operational laws and HM operator allows the aggregation process to effectively reflect the interrelationships between attributes, while at the same time considering uncertainty, indeterminacy, bipolarity and incomplete information in DM environments. Several basic properties of the proposed operators, namely idempotency, boundedness and monotonicity were formally proved to establish the mathematical soundness of the operators. The proofs show that the operators proposed are suitable with regard to the required characteristics that are needed for the reliable information aggregation in the context of BPNS.

Moreover, a numerical example was carried out to show the practical application of the operators proposed in a real DM problem. The results of comparative analysis showed that the proposed approach results in reasonable and consistent ranking results, and more flexible than the existing aggregation methods. A sensitivity analysis reveals that ranking results remain stable and do not significantly change with respect to changes in the parameter values, demonstrating the robustness and stability of the proposed operators. The results show that the BPN-GHHM and BPN-GWHHM operators are suitable AOs for incomplete and uncertain information in MCDM problems. The validated AOs established in this chapter provide the computational foundation for the practical application presented in the next chapter.

CHAPTER 5

APPLICATION ON EVALUATION OF ENGLISH VOCABULARY LEARNING STRATEGIES

5.1 Introduction

This chapter examines the English vocabulary learning strategy using the proposed ARIE method, BPN-GHHM operator, BPNN linguistic variable. Section 5.2 introduces two new linguistic variables under BPNN contexts, followed by their validation. Section 5.3 presents the numerical findings in evaluating English vocabulary learning strategies with detailed calculations and explanations. The comparative analysis in Section 5.4 followed by a sensitivity analysis in Section 5.5. Section 5.6 concludes this chapter.

5.2 Linguistic Variable for Bipolar Pythagorean Neutrosophic Number

A linguistic variable is a variable with a natural language value instead of an exact numeric value. Zadeh (1975) defines a linguistic variable as having five-tuple,

$$LV = \langle X, T(X), U, G, M \rangle$$

where X denotes the name of the variable, $T(X)$ represents the set of linguistic terms associated with X , U is the universe of discourse, G is a syntactic rule for generating linguistic terms, and M is a semantic rule that assigns a meaning to each linguistic term. Under conditions of uncertainty, experts may have to make judgments based on language terms such as Very Low, Low, Medium, High, Very High, rather than precise numbers. These linguistic evaluations support human reasoning while mitigating the complexity of giving accurate evaluations when faced with uncertainty or incomplete information.

Linguistic values that are assigned to a linguistic variable are usually represented in an ordered linguistic term set in order to make the computation and analysis of the DM easier. The linguistic term set offers a finite set of linguistic labels, ordered by

semantic preference, that can be systematically represented and manipulated to make qualitative assessments.

Definition 5.1 Linguistic term set (Zadeh, 1975)

Let

$$S = \{s_1, s_2, \dots, s_{2q+1}\},$$

where $q \in \mathbb{N}^+$. The linguistic labels in S are from lowest to highest preference value. For instance, if $q = 1$ then,

$$S = \{s_1, s_2, s_3\} = \{\text{low}, \text{fair}, \text{good}\}$$

The subscripts associated with the linguistic labels indicate their ranking and satisfy a strictly increasing order. To facilitate continuous operations, S is generalized to

$$\tilde{S} = \{s_\gamma \mid \gamma \in \mathbb{R}\},$$

where γ is a real-valued index and semantic ordering remains is still strictly increasing.

Linguistic term sets have proved to be the most useful method for structuring qualitative evaluations, but they are not necessarily sufficient to express the uncertainty, indeterminacy and bipolar nature of human judgments. In real life DM situations, experts compare options in terms of both positive support and negative opposition, and they show hesitation and uncertainty. To overcome such shortcomings, BPNSs enrich the conventional linguistic representations with positive and negative membership information under the Pythagorean constraints.

Based on BPNS theory, a BPNN can be used to describe an assessment by six independent components. BPNNs offer a more flexible and comprehensive representation of uncertain and bipolar information compared to the traditional linguistic term sets. A BPNN is generally expressed as,

$$\tilde{b} = \{\mu^+, \eta^+, \nu^+, \mu^-, \eta^-, \nu^-\},$$

where μ^+ , η^+ , and ν^+ denote the positive truth, indeterminacy, and falsity membership degrees, respectively, while μ^- , η^- , and ν^- represent the corresponding negative membership degrees. Therefore, a BPNN is assigned to each linguistic term in order to represent the semantic meaning of the corresponding linguistic term. The traditional BPNS linguistic scale, which is used for referring the linguistic evaluation in the study, is presented in the following sub-section.

5.2.1 Existing Bipolar Pythagorean Neutrosophic Scale

Table 5.1 presents the existing seven BPNS linguistic scale referred to in the literature. The scale is a group of ordered linguistic terms from the no influence to the very high influence level. A set of pre-defined BPNN for each linguistic term is defined reflecting its semantic meaning within the constraint of BPNS. The order of the terms provides a smooth progression from one language assessment to the next, and experts can make judgements in a consistent and understandable way.

Table 5.1
Bipolar Pythagorean Neutrosophic Representation of Linguistic Variable (Ahmad et al., 2024b)

Score	Linguistic expression	BPNN
1	No influence	[0.1,0.8,0.9, -0.93, -0.46, -0.31]
2	Low influence	[0.2,0.7,0.8, -0.45, -0.32, -0.15]
3	Medium low influence	[0.35,0.6,0.6, -0.38, -0.19, -0.12]
4	Medium influence	[0.5,0.4,0.45, -0.32, -0.16, -0.1]
5	Medium high influence	[0.65,0.3,0.3, -0.21, -0.1, -0.07]
6	High influence	[0.8,0.2,0.15, -0.12, -0.06, -0.04]
7	Very high influence	[0.9,0.1,0.1, -0.03, -0.015, -0.01]

The linguistic terms are ordered in Table 5.1, based on their preference levels, with unfavourable scale and favourable. The corresponding BPNNs are assigned so as to keep the same semantic order of the linguistic terms. Thus, the scale can be used to connect the qualitative expert judgments with quantitative BPNS computations. This linguistic scale has been used in past study, but the evaluation of BPNN values is determined in advance and is not necessarily a reflection of the semantic relationships

perceived by experts in this study. Therefore, a new linguistic scale is proposed in this study to represent the evaluation problem under consideration more appropriately.

5.2.2 Proposed 5-Point Bipolar Pythagorean Neutrosophic Linguistic Scale

Both linguistic variable, five and seven scale was newly defined for the proposed model and implemented in Microsoft Excel for Microsoft 365 (Version 2605). The computed values were verified within the same environment, while MATLAB R2024a was used to generate the graphical representations. The proposed five-point bipolar Pythagorean neutrosophic linguistic scale aims to offer a simple and user-friendly evaluation for decision makers. This scale is designed to reduce the cognitive load of the scale and make it easier to elicit the expert judgments by having fewer linguistic categories. This simple design is useful if respondents possess different levels of expertise, or if a fast evaluation is needed. In addition, the fewer words may help with consistency of assessments by reducing the ambiguities between successive categories.

Based on Definition 5.1, when $q = 2$ then, $S = \{s_1, s_2, s_3, s_4, s_5\} = \{\text{very poor, poor, medium, good, very good}\}$ is considered. Each linguistic term is mapped to a corresponding BPNN to enable qualitative evaluations to be represented in the BPNS framework. Table 5.2 represents the proposed five-scale linguistic variable conducive to BPNS contexts and graphically visualized in Figure 5.1.

Table 5.2
Five-Scale Linguistic Variable under BPNS

Score	Linguistic expression	BPNN
1	Very poor	[0.1,0.9,0.7, -0.7, -0.9, -0.7]
2	Poor	[0.2,0.6,0.6, -0.5, -0.6, -0.6]
3	Medium	[0.4,0.5,0.4, -0.4, -0.5, -0.4]
4	Good	[0.5,0.3,0.2, -0.2, -0.3, -0.1]
5	Very good	[0.7,0.2,0.1, -0.1, -0.2, -0.1]

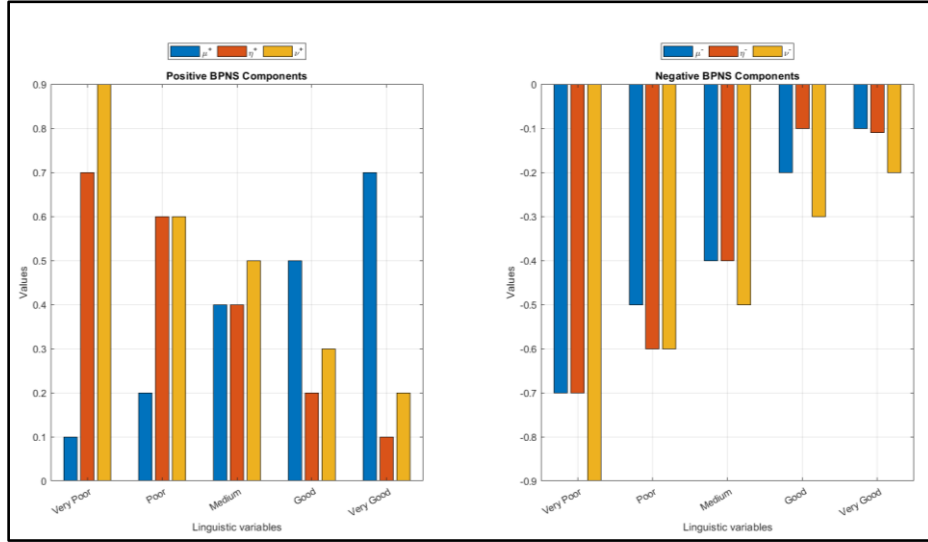


Figure 5.1 Visual Representation of the Five-scale BPNS Structure

Figure 5.1 presents the BPNS representation for a five-level linguistic scale, decomposed into positive components (μ^+, η^+, ν^+) and negative components (μ^-, η^-, ν^-) . On the positive side, the degree of positive membership μ^+ exhibits a clear monotonic increase from Very Poor to Very Good, indicating that as linguistic assessment improves, the affirmation strength of the evaluated concept consistently rises. Conversely, the positive indeterminacy η^+ and positive non-membership ν^+ decrease along the same progression, reflecting diminishing uncertainty and rejection as the evaluation moves toward more favourable linguistic terms. This reciprocal behaviour confirms the internal consistency of the BPNS framework, where higher-quality linguistic assessments are characterized by stronger support and reduced hesitation.

On the negative side, the magnitudes of μ^- , η^- , and ν^- are most pronounced in absolute value for Very Poor and gradually attenuate toward Very Good. This pattern indicates that negative evidence, negative uncertainty, and negative opposition are dominant at lower linguistic levels but weaken as assessments become more positive. This linguistic hypothesis of BPNS is referred from Ahmad, et al. (2024); Ismail, et al. (2023). Overall, the five-scale figure demonstrates that the BPNS model effectively captures bipolarity by simultaneously modelling positive reinforcement and negative opposition, yielding a balanced and interpretable semantic structure across coarse linguistic granularity. To ensure the mathematical validity of the proposed linguistic scales, the assigned BPNNs were verified against the fundamental properties of BPNS.

Specifically, the proposed scales were examined with respect to the Pythagorean constraints and complementary property.

5.2.2.1 Verification of the Pythagorean Constraints

Let $B = \{B_1, B_2, \dots, B_n\}$, where each B_i represents the class vector corresponding to the i -th row in Table 5.2 with membership degrees in the range of $[0, 1]$ for positive membership and $[-1, 0]$ for negative membership. Therefore, the set of BPNS linguistic variables for the five-scale is defined as follows:

$$B = \left\{ \begin{array}{l} \langle b_1, 0.1, 0.9, 0.7, -0.7, -0.9, -0.7 \rangle \\ \langle b_2, 0.2, 0.6, 0.6, -0.5, -0.6, -0.6 \rangle \\ \langle b_3, 0.4, 0.5, 0.4, -0.4, -0.5, -0.4 \rangle \\ \langle b_4, 0.5, 0.3, 0.2, -0.2, -0.3, -0.1 \rangle \\ \langle b_5, 0.7, 0.2, 0.1, -0.1, -0.2, -0.1 \rangle \end{array} \right\}$$

The linguistic variable of five scale validated according to (4.6)-(4.9). Assume two BPNNs from B ,

$$B_1 = \langle 0.1, 0.9, 0.7, -0.7, -0.9, -0.7 \rangle$$

and

$$B_2 = \langle 0.2, 0.6, 0.6, -0.5, -0.6, -0.6 \rangle$$

The validation is as follows,

$$\mu_{B_1}^+(x)^2 + \nu_{B_1}^+(x)^2 = (0.1)^2 + (0.7)^2 = 0.5 \leq 1$$

and

$$\mu_{B_2}^+(x)^2 + \nu_{B_2}^+(x)^2 = (0.2)^2 + (0.6)^2 = 0.4 \leq 1$$

Similarity, all linguistic terms in the proposed five-scale linguistic variable (Very Poor, Poor, Medium, Good, and Very Good) satisfy the Pythagorean constraints

$$0 \leq \mu_B^+(x)^2 + \nu_B^+(x)^2 \leq 1, \quad 0 \leq \mu_B^-(x)^2 + \nu_B^-(x)^2 \leq 1, \quad 0 \leq \mu_B^+(x)^2 + \eta_B^+(x)^2 + \nu_B^+(x)^2 \leq 2,$$

$0 \leq \mu_B^-(x)^2 + \eta_B^-(x)^2 + \nu_B^-(x)^2 \leq 2$ for all x in X . Therefore, the proposed linguistic scale is valid and can be employed.

5.2.2.2 Verification of the Complementary Property

The complement property of the bipolar Pythagorean neutrosophic linguistic variable is verified based on Definition 4.10. Let two BPNNs from B ,

$$B = \{b_1, b_2, b_3, b_4, b_5\}$$

The complement of B , denoted by B^C , is obtained as,

$$B = \left\{ \begin{array}{l} \langle b_1, 0.7, 0.9, 0.1, -0.7, -0.9, -0.7 \rangle \\ \langle b_2, 0.6, 0.6, 0.2, -0.6, -0.6, -0.5 \rangle \\ \langle b_3, 0.4, 0.5, 0.4, -0.4, -0.5, -0.4 \rangle \\ \langle b_4, 0.2, 0.3, 0.5, -0.1, -0.3, -0.2 \rangle \\ \langle b_5, 0.1, 0.2, 0.7, -0.1, -0.2, -0.1 \rangle \end{array} \right\}$$

for all x in X .

The verification results confirms that the proposed bipolar Pythagorean neutrosophic linguistic variable fulfils both the Pythagorean constraints as well as the complement property as specified in Definitions 4.7 and 4.10. These findings demonstrate that the proposed linguistic scale is mathematically consistent and adheres to the fundamental principles of the bipolar Pythagorean neutrosophic environment. Hence, it is deemed valid and suitable for use in constructing the bipolar Pythagorean neutrosophic decision matrix and performing subsequent evaluation and ranking processes.

5.2.3 Proposed 7-Point Bipolar Pythagorean Neutrosophic Linguistic Scale

A higher level of granularity is proposed for the assessment in the form of a seven-point bipolar neutrosophic linguistic scale. More linguistic terms allow decision-makers to make more subtle judgments and identify fine differences between alternatives. The increased discriminatory ability could lead to more precise evaluation process, especially when dealing with complex DM tasks. Besides, the use of a seven-point scale ensures compatibility with the other bipolar Pythagorean neutrosophic linguistic scales reported in the literature, making it easier to compare and validate the results with other studies. Below shown the proposed seven-scale linguistic variable for BPNS with different expressions in Table 5.3 and visualized in Figure 5.2.

Table 5.3
Seven-Scale Linguistic Variable under BPNS

Score	Linguistic expression	BPNN
1	Weak	[0.11,0.98,0.84, -0.60, -0.91, -0.78]
2	Not bad	[0.19,0.86,0.72, -0.51, -0.76, -0.76]
3	Preferable	[0.24,0.71,0.63, -0.44, -0.55, -0.64]
4	Good	[0.45,0.69,0.57, -0.32, -0.32, -0.42]
5	Very good	[0.66,0.54,0.49, -0.15, -0.28, -0.31]
6	Absolute	[0.87,0.36,0.38, -0.09, -0.10, -0.18]
7	Perfect	[0.92,0.20,0.21, -0.06, -0.04, -0.07]

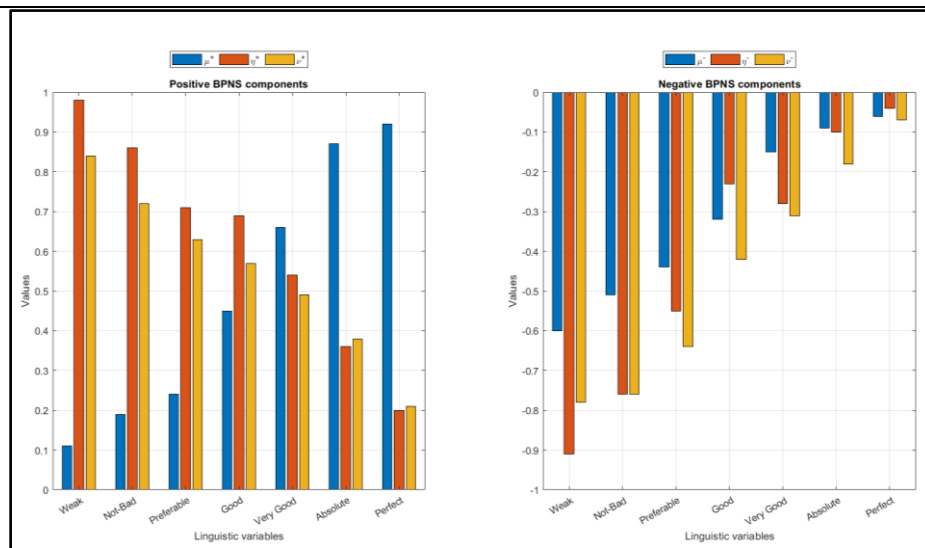


Figure 5.2 Visual Representation of the Seven-scale BPNS Structure

Figure 5.2 extends the BPNS analysis to a finer seven-level linguistic scale, enabling a more nuanced semantic discrimination. In the positive components, μ^+ increases steadily and more smoothly across the scale, culminating at Perfect, which reflects maximal positive affirmation. In contrast, η^+ and ν^+ display a systematic decline, signifying that higher linguistic precision is associated with reduced ambiguity and opposition. Compared to the five-scale case, this gradual transition highlights the advantage of a richer linguistic resolution, as subtle shifts in evaluative meaning are distinctly encoded within the BPNS parameters.

The negative components, μ^- , η^- , and ν^- show strong negative magnitudes at Weak and Not Bad, followed by a progressive reduction in absolute values toward Perfect. This indicates that negative belief and uncertainty are incrementally neutralized

as linguistic judgments approach duality. Collectively, the seven-scale figure illustrates the robustness of the BPNS model in handling refined linguistic assessments, demonstrating its capacity to model gradual semantic evolution while preserving bipolar information symmetry. In thesis terms, this figure substantiates that increasing linguistic granularity within the BPNS framework enhances expressive power without compromising logical coherence, making it particularly suitable for complex DM and evaluation problems.

The seven-scale BPNS linguistic variable was developed based on the same construction principles as the verified five-scale BPNS linguistic variable. The assigned BPNNs maintain the required monotonic ordering and complement relationships. Therefore, the Pythagorean constraints and complement property remain satisfied, and the detailed verification is not repeated. The next section applies the proposed models to evaluate the best English vocabulary learning strategies.

5.3 Application of the Proposed Models in Evaluation of English Vocabulary Learning Strategies

The proposed decision support model was applied to evaluate English vocabulary learning strategies. Microsoft Excel for Microsoft 365 (Version 2605) was used to compute and verify the evaluation scores, while MATLAB R2024a was employed to generate graphical visualizations of the results. The alternatives ranking was computed by the decision support system, InnoRank-ARIE system. Language acquisition and communication is an important aspect of vocabulary. Acquisition of new words is a complex process that engages numerous cognitive functions is facilitated with the help of different instructional methods and is developed as individuals acquire new knowledge. New vocabulary may however be challenging to learn, particularly among non-native speakers of English because of spelling, pronunciation, correct usage, contextual interpretation of meaning among other problems. The most important part of English as a second language course is vocabulary. Figure 5.3 the methodological framework adopted in this study for evaluating English vocabulary learning strategies.

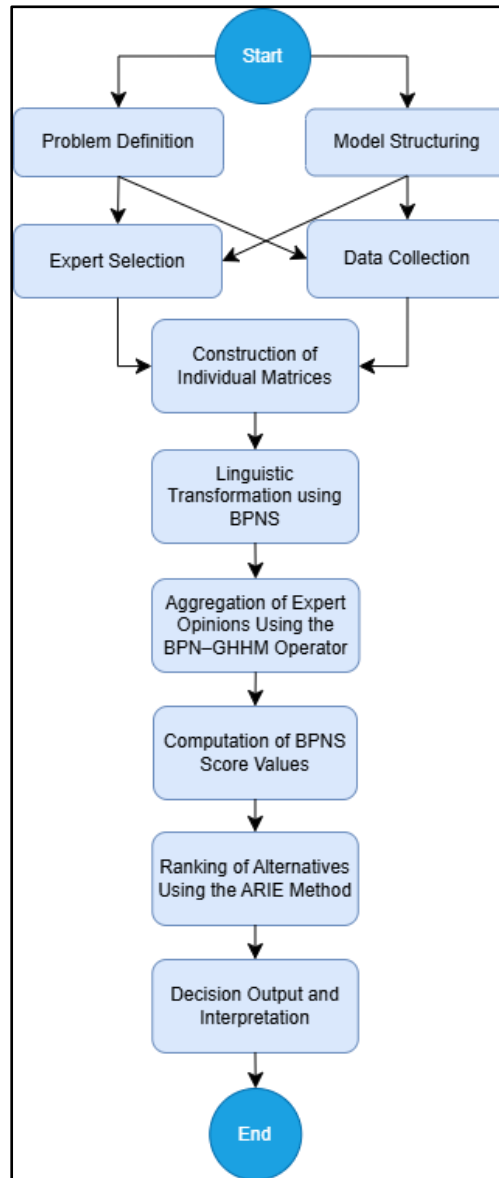


Figure 5.3 Research Framework for Evaluating English Vocabulary Learning Strategies

Figure 5.3 presents the overall workflow of the proposed evaluation framework for English vocabulary learning strategies. The proposed framework consists of a series of interconnected stages designed to ensure a systematic and reliable evaluation process. It begins with problem formulation, identification of evaluation criteria, and collection of expert judgments. These judgments are then represented using bipolar Pythagorean neutrosophic linguistic variables and aggregated into a collective decision matrix. Subsequently, the ARIE method is employed to calculate the ranking scores and determine the most appropriate English vocabulary learning strategy. The details of each stage are discussed below.

Step 1. Problem definition and model structuring

The initial step in this study is the definition of alternatives and criteria evaluation that can be used as the basis to apply the BPN-GHHM operator and ARIE method. The alternatives can be viewed as different English vocabulary learning strategies, whereas the criteria reveal the essential aspects according to which the methods can be rated in terms of the effectiveness and practicality. Seven vocabulary learning strategies mentioned below were selected using the former research, pedagogical significance, and their present application in the learning environments. These alternatives have been identified, analysed, and substantiated through a review of the relevant literature, ensuring their theoretical and practical relevance within the context of vocabulary acquisition.

- i) Traditional method (A_1) - the traditional classroom activities like memorizing, word lists, and writing exercises.
- ii) Contextual learning (A_2) - learning new words by reading, listening, or using words in contexts that make sense with the aim of getting better understanding and retention.
- iii) Gamification method (A_3) - integrates game-based elements into vocabulary learning to increase motivation and learner engagement.
- iv) Artificial Intelligence (A_4) - AI-based applications chatbots, personalized platforms, intelligent tutoring systems can be used in delivering personalized feedback and practice.
- v) Mnemonic technique (A_5) - uses memory aids techniques to aid in memorizing and long-term retention of vocabulary items. Mnemonics strategies are higher order mental processes and have been employed since the initial stages of learning vocabulary.
- vi) Multimedia (A_6) - using various media types which include audio, video and interactive visuals to help in acquiring multisensory vocabulary.
- vii) Peer (A_7) - group learning of vocabulary by means of group discussions, peer explanations and cooperative exercises that stimulate active use of language.

The seven options were chosen to assess a wide variety of traditional and technology enriched learning strategies that are widely embraced in contemporary teaching of the English language vocabulary. To evaluate and compare the efficiency of these alternatives, eight criteria were determined with references to previous research conducted on the topic of vocabulary acquisition and professional advice. These criteria reflect pedagogical as well as technological aspects of vocabulary acquisition:

- i. Long term recall (C_1): ability of learners to remember and use learned vocabulary over time.
- ii. Frequency (C_2): able to engage or experience new vocabulary elements in the learning process.
- iii. Motivation level (C_3): provokes interest with the learners and inspires them to participate regularly.
- iv. Interactivity (C_4): the level of learner engagement and interaction.
- v. Ease of understanding (C_5): learners easily understood without a lot of guidance.
- vi. Use in speaking & writing (C_6): efficiency in enhancing vocabulary application in productive language skills.
- vii. Availability on devices (C_7): use of digital gadgets like smartphones or computer.
- viii. Quality of feedback (C_8): the quality and the level of feedback given to learners to improve vocabulary.

Step 2. Expert selection and data collection

After the identification and analysis of alternatives and criteria, information was gathered to measure the English vocabulary learning strategies within the DM system. The procedure of data collection was to collect informed and valid judgments of the experts who has the pertinent experience in English language learning and teaching by interview (see APPENDIX 1). Ten experts in the Akademi Pengajian Bahasa (APB) Universiti Teknologi MARA (UiTM) Cawangan Negeri Sembilan, Kampus Kuala Pilah, were used in this study. These experts were chosen in a purposive way as they are professors with academic background and teaching experience on English language education. Their background and experience in the profession and the

knowledge of different methods of vocabulary teaching offered informed and credible reviews of the strategies identified. The professionalism offered by these experts to the research is an element that provides credibility and practical applicability to the research results. All the respondents were requested to rank the seven alternatives according to the eight criteria that were specified in Step 1. The tests were done on a seven-point linguistic scale in Table 5.3 which enabled the participants to judge qualitatively in lingo other than through strict numbers. This scale was made to reflect not just the extent of agreement but also the incomplete information that might exist in the judgment of people.

Step 3. Constructional of decision matrices

Based on (3.1), the decision matrix is constructed to reflect the evaluation of alternatives according to the defined criteria. Let $A_i (i=1,2,...,m)$ denote the set of alternatives and $C_j (j=1,2,...,n)$ denote the set of evaluation criteria. Each element $x_{ij}^{(k)}$ represents the assessment of alternative A_i under criterion C_j provided by the k -th expert. Accordingly, ten individual decision matrices $X^{(k)} = [x_{ij}^{(k)}]_{m \times n} (k=1,2,...,10)$ are constructed based on the opinions of ten experts (see APPENDIX 2).

Step 4. Linguistic transforming using BPNS information

This step involved conversion of the raw evaluation data of the ten respondents into linguistic variables against the seven-point BPNS scale. The conversion process would enable the qualitative judgments to be expressed in structured mathematical form of representation that can support uncertainty, hesitation and dual, positive and negative, viewpoints that exist within the MCDM. Let the BPNS evaluation of alternative A_i with respect to criterion C_j from respondent k be

$$x_{ij}^{(k)} = (\mu_{ij}^{+(k)}, \eta_{ij}^{+(k)}, \nu_{ij}^{+(k)}, \mu_{ij}^{-(k)}, \eta_{ij}^{-(k)}, \nu_{ij}^{-(k)})$$

Since the decision matrices obtained from all experts have the same structure, only one of the constructed decision matrices is presented in Table 5.4. The remaining matrices are processed similarly in the subsequent aggregation step.

Table 5.4
Seven-Level BPNS Linguistic Evaluation from Respondent 1

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	7	6	7	2	7	7	6	6
A_2	7	7	7	2	7	7	7	6
A_3	5	3	4	5	6	3	4	4
A_4	4	5	4	5	4	2	4	4
A_5	6	6	6	4	5	6	5	4
A_6	4	5	4	4	4	4	5	4
A_7	6	6	7	7	7	7	4	6

Table 5.4 presents the decision matrix obtained from Respondent 1, where the evaluation values represent the performance of each alternative with respect to the considered criteria. The decision matrices from Respondent 2 until Respondent 10 are not presented to avoid redundancy, as they follow the same structure. These individual decision matrices serve as the input for the aggregation process. Therefore, the BPN-GHHM operator is applied to combine the decision information from all respondents into an aggregated decision matrix.

Step 5. Aggregation using the BPN-GHHM operator

Each respondents made independent decisions, there was a need to bring the two or more decisions together to form one collective decision matrix which would reflect a unified group decision. Computation of the BPN-GHHM operator was computed that no single respondent's evaluation was influencing the outcome disproportionately, as opposed to BPN-GWHHM operator with $n=10$, $p=2$, $q=2$, and $\zeta=1$ to prevent the consequence of the weighted computation twice. To fuse the opinions of n respondents, each BPNS component is aggregated into

$$X_{ij}^g, g \in \{\mu^+, \eta^+, \nu^+, \mu^-, \eta^-, \nu^-\}$$

Define the set of respondent values for component g as

$$g_{ij}^{(k)} \in \begin{cases} [0,1], g \in \{\mu^+, \eta^+, \nu^+\} \\ [-1,0], g \in \{\mu^-, \eta^-, \nu^-\} \end{cases}, k=1,2,\dots,n.$$

Then the aggregated component X_{ij}^g using (4.27) is given below.

$$X_{ij}^{\mu^+} = \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1-\mu_i^+))^p (1+(\zeta-1)(1-\mu_j^+))^q \right]^{\frac{2(p+q)}{n(n+1)}} - (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n (\mu_i^+)^p (\mu_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}$$

The computation of X_{11}^g is also presented. Microsoft excel is used to compute the positive truth membership degree.

$$X_{11}^{\mu^+} = \frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (\mu_1^+)^2 (\mu_1^+)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1+(1-1)(1-\mu_1^+))^2 (1+(1-1)(1-\mu_1^+))^2 \right]^{\frac{2(2+2)}{10(10+1)}} - (1-1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} (\mu_1^+)^2 (\mu_1^+)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}$$

$$X_{11}^{\mu^+} = \frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (\mu_1^+)^2 (\mu_1^+)^2 \right]^{\frac{8}{110}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}}}$$

$$X_{11}^{\mu^+} = \frac{\left[\begin{aligned} &(\mu_1^+)^2 (\mu_1^+)^2 + (\mu_1^+)^2 (\mu_2^+)^2 + (\mu_1^+)^2 (\mu_3^+)^2 + \dots + (\mu_1^+)^2 (\mu_{10}^+)^2 + \\ &(\mu_2^+)^2 (\mu_2^+)^2 + (\mu_2^+)^2 (\mu_3^+)^2 + (\mu_2^+)^2 (\mu_4^+)^2 + \dots + (\mu_2^+)^2 (\mu_{10}^+)^2 + \\ &\dots \\ &+ (\mu_{10}^+)^2 (\mu_{10}^+)^2 \end{aligned} \right]^{\frac{8}{110}}}{\left[\begin{aligned} &(1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ &(1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ &\dots \\ &+ (1)^2 (1)^2 \end{aligned} \right]^{\frac{8}{110}}}$$

$$X_{11}^{\mu^+} = \frac{\left[\begin{array}{c} (0.92)^2 (0.92)^2 + (0.92)^2 (0.92)^2 + (0.92)^2 (0.92)^2 + \dots + (0.92)^2 (0.87)^2 + \\ (0.92)^2 (0.92)^2 + (0.92)^2 (0.92)^2 + (0.92)^2 (0.87)^2 + \dots + (0.92)^2 (0.92)^2 + \\ \dots \\ (0.87)^2 (0.87)^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}}$$

$$X_{11}^{\mu^+} = \frac{1.2507}{1.3384}$$

$$X_{11}^{\mu^+} = 0.9345$$

For the positive indeterminacy membership degree, the computation is as follows:

$$X_{ij}^{\eta^+} = \frac{\left[\sum_{i=1}^n \sum_{j=1}^n (1 + (\zeta - 1) \eta_i^+)^p (1 + (\zeta - 1) \eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n (1 - \eta_i^+)^p (1 - \eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1 + (\zeta - 1) \eta_i^+)^p (1 + (\zeta - 1) \eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n (1 - \eta_i^+)^p (1 - \eta_j^+)^q \right]^{\frac{2(p+q)}{n(n+1)}}}$$

$$X_{11}^{\eta^+} = \frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1 + (1 - 1) \eta_i^+)^2 (1 + (1 - 1) \eta_j^+)^2 \right]^{\frac{2(2+2)}{10(10+1)}} - \left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1 - \eta_i^+)^2 (1 - \eta_j^+)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1 + (1 - 1) \eta_i^+)^2 (1 + (1 - 1) \eta_j^+)^2 \right]^{\frac{2(2+2)}{10(10+1)}} + (1 - 1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1 - \eta_i^+)^2 (1 - \eta_j^+)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}$$

$$X_{11}^{\eta^+} = \frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}} - \left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1 - \eta_i^+)^2 (1 - \eta_j^+)^2 \right]^{\frac{8}{110}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}}}$$

$$\begin{aligned}
X_{11}^{\eta^+} &= \frac{\left[\begin{array}{c} (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}} - \left[\begin{array}{c} (1-\eta_1^+)^2 (1-\eta_1^+)^2 + (1-\eta_1^+)^2 (1-\eta_2^+)^2 + \\ (1-\eta_1^+)^2 (1-\eta_3^+)^2 + \dots + (1-\eta_1^+)^2 (1-\eta_{10}^+)^2 + \\ (1-\eta_2^+)^2 (1-\eta_2^+)^2 + (1-\eta_2^+)^2 (1-\eta_3^+)^2 + \\ (1-\eta_2^+)^2 (1-\eta_4^+)^2 + \dots + (1-\eta_2^+)^2 (1-\eta_{10}^+)^2 + \\ \dots \\ + (1-\eta_{10}^+)^2 (1-\eta_{10}^+)^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}} \\
X_{11}^{\eta^+} &= \frac{\left[\begin{array}{c} (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}} - \left[\begin{array}{c} (1-0.2)^2 (1-0.2)^2 + (1-0.2)^2 (1-0.2)^2 + \\ (1-0.2)^2 (1-0.2)^2 + \dots + (1-0.2)^2 (1-0.36)^2 + \\ (1-0.2)^2 (1-0.2)^2 + (1-0.2)^2 (1-0.2)^2 + \\ (1-0.2)^2 (1-0.36)^2 + \dots + (1-0.2)^2 (1-0.36)^2 + \\ \dots \\ + (1-0.36)^2 (1-0.36)^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}} \\
X_{11}^{\eta^+} &= \frac{1.3384 - 1.1679}{1.3384} \\
X_{11}^{\eta^+} &= 0.1273
\end{aligned}$$

The computation for the positive falsity membership degree is given below.

$$\begin{aligned}
X_{ij}^{\nu^+} &= \sqrt{\frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-\nu_i^+)^2 \right)^p \left(1 + (\zeta - 1)(-\nu_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (-\nu_i^+)^2 \right)^p \left(1 - (-\nu_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(-\nu_i^+)^2 \right)^p \left(1 + (\zeta - 1)(-\nu_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (-\nu_i^+)^2 \right)^p \left(1 - (-\nu_j^+)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}} \\
X_{11}^{\nu^+} &= \sqrt{\frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 + (1-1)(-\nu_1^+)^2 \right)^2 \left(1 + (1-1)(-\nu_1^+)^2 \right)^2 \right]^{\frac{2(2+2)}{10(10+1)}} - \left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 - (-\nu_1^+)^2 \right)^2 \left(1 - (-\nu_1^+)^2 \right)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 + (1-1)(-\nu_1^+)^2 \right)^2 \left(1 + (1-1)(-\nu_1^+)^2 \right)^2 \right]^{\frac{2(2+2)}{10(10+1)}} + (1-1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 - (-\nu_1^+)^2 \right)^2 \left(1 - (-\nu_1^+)^2 \right)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}}
\end{aligned}$$

$$X_{11}^{\nu^+} = \sqrt{\frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}} - \left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 - (-\nu_1^+)^2 \right)^2 \left(1 - (-\nu_1^+)^2 \right)^2 \right]^{\frac{8}{110}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}}}}$$

$$X_{11}^{\nu^+} = \sqrt{\frac{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}} - \left[\begin{array}{c} \left(1 - (-\nu_1^+)^2 \right)^2 \left(1 - (-\nu_1^+)^2 \right)^2 + \left(1 - (-\nu_1^+)^2 \right)^2 \left(1 - (-\nu_2^+)^2 \right)^2 + \\ \dots + \left(1 - (-\nu_1^+)^2 \right)^2 \left(1 - (-\nu_{10}^+)^2 \right)^2 + \left(1 - (-\nu_2^+)^2 \right)^2 \left(1 - (-\nu_2^+)^2 \right)^2 + \\ \left(1 - (-\nu_2^+)^2 \right)^2 \left(1 - (-\nu_3^+)^2 \right)^2 + \dots + \left(1 - (-\nu_2^+)^2 \right)^2 \left(1 - (-\nu_{10}^+)^2 \right)^2 + \\ \dots + \left(1 - (-\nu_{10}^+)^2 \right)^2 \left(1 - (-\nu_{10}^+)^2 \right)^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}}}$$

$$X_{11}^{\nu^+} = \sqrt{\frac{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}} - \left[\begin{array}{c} \left(1 - (-0.21)^2 \right)^2 \left(1 - (-0.21)^2 \right)^2 + \left(1 - (-0.21)^2 \right)^2 \left(1 - (-0.21)^2 \right)^2 + \\ \dots + \left(1 - (-0.21)^2 \right)^2 \left(1 - (-0.38)^2 \right)^2 + \left(1 - (-0.21)^2 \right)^2 \left(1 - (-0.21)^2 \right)^2 + \\ \left(1 - (-0.21)^2 \right)^2 \left(1 - (-0.21)^2 \right)^2 + \dots + \left(1 - (-0.21)^2 \right)^2 \left(1 - (-0.38)^2 \right)^2 + \\ \dots + \left(1 - (-0.38)^2 \right)^2 \left(1 - (-0.38)^2 \right)^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + \\ \dots \\ + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}}}$$

$$X_{11}^{\nu^+} = \sqrt{\frac{1.3384 - 1.2661}{1.3884}}$$

$$X_{11}^{\nu^+} = 0.2323$$

For the negative truth membership degree, the calculation proceeds as follows.

$$X_{ij}^{\mu^-} = - \sqrt{\frac{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (\mu_i^-)^2 \right)^p \left(1 + (\zeta - 1) (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} - \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2 \right)^p \left(1 - (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1) (-\mu_i^-)^2 \right)^p \left(1 + (\zeta - 1) (-\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n \left(1 - (\mu_i^-)^2 \right)^p \left(1 - (\mu_j^-)^2 \right)^q \right]^{\frac{2(p+q)}{n(n+1)}}}}$$

$$X_{11}^{\mu^-} = - \sqrt[10]{\frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 + (1-1)(\mu_1^-)^2\right)^2 \left(1 + (1-1)(\mu_1^-)^2\right)^2 \right]^{\frac{2(2+2)}{10(10+1)}} - \left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 - (\mu_1^-)^2\right)^2 \left(1 - (\mu_1^-)^2\right)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 + (1-1)(-\mu_1^-)^2\right)^2 \left(1 + (1-1)(-\mu_1^-)^2\right)^2 \right]^{\frac{2(2+2)}{10(10+1)}} + (1-1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 - (\mu_1^-)^2\right)^2 \left(1 - (\mu_1^-)^2\right)^2 \right]^{\frac{2(2+2)}{10(10+1)}}}}$$

$$X_{11}^{\mu^-} = - \sqrt[110]{\frac{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}} - \left[\sum_{i=1}^{10} \sum_{j=1}^{10} \left(1 - (\mu_1^-)^2\right)^2 \left(1 - (\mu_1^-)^2\right)^2 \right]^{\frac{8}{110}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}}}}$$

$$X_{11}^{\mu^-} = - \sqrt[110]{\frac{\left[(1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 \right]^{\frac{8}{110}} - \left[\begin{aligned} &\left(1 - (\mu_1^-)^2\right)^2 \left(1 - (\mu_1^-)^2\right)^2 + \left(1 - (\mu_1^-)^2\right)^2 \left(1 - (\mu_2^-)^2\right)^2 + \\ &\dots + \left(1 - (\mu_1^-)^2\right)^2 \left(1 - (\mu_{10}^-)^2\right)^2 + \left(1 - (\mu_2^-)^2\right)^2 \left(1 - (\mu_2^-)^2\right)^2 + \\ &\left(1 - (\mu_2^-)^2\right)^2 \left(1 - (\mu_3^-)^2\right)^2 + \dots + \left(1 - (\mu_2^-)^2\right)^2 \left(1 - (\mu_{10}^-)^2\right)^2 + \\ &\dots + \left(1 - (\mu_{10}^-)^2\right)^2 \left(1 - (\mu_{10}^-)^2\right)^2 \end{aligned} \right]^{\frac{8}{110}}}{\left[\begin{aligned} &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &\dots + (1)^2 (1)^2 \end{aligned} \right]^{\frac{8}{110}}}}$$

$$X_{11}^{\mu^-} = - \sqrt[110]{\frac{\left[(1)^2 (1)^2 + (1)^2 (1)^2 + \dots + (1)^2 (1)^2 \right]^{\frac{8}{110}} - \left[\begin{aligned} &\left(1 - (-0.06)^2\right)^2 \left(1 - (-0.06)^2\right)^2 + \left(1 - (-0.06)^2\right)^2 \left(1 - (-0.06)^2\right)^2 + \\ &\dots + \left(1 - (-0.06)^2\right)^2 \left(1 - (-0.09)^2\right)^2 + \left(1 - (-0.06)^2\right)^2 \left(1 - (-0.06)^2\right)^2 + \\ &\left(1 - (-0.06)^2\right)^2 \left(1 - (-0.06)^2\right)^2 + \dots + \left(1 - (-0.06)^2\right)^2 \left(1 - (-0.09)^2\right)^2 + \\ &\dots + \left(1 - (-0.09)^2\right)^2 \left(1 - (-0.09)^2\right)^2 \end{aligned} \right]^{\frac{8}{110}}}{\left[\begin{aligned} &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &\dots + (1)^2 (1)^2 \end{aligned} \right]^{\frac{8}{110}}}}$$

$$X_{11}^{\mu^-} = - \sqrt{\frac{1.3384 - 1.3201}{1.3384}}$$

$$X_{11}^{\mu^-} = -0.0828$$

For the negative indeterminacy membership degree, the computation is as follows:

$$X_{ij}^{\eta^-} = \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n \left(1 + (\zeta - 1)(1 + \eta_i^-)\right)^p \left(1 + (\zeta - 1)(1 + \eta_j^-)\right)^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta - 1) \left[\sum_{i=1}^n \sum_{j=1}^n |\eta_i^-|^p |\eta_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}$$

$$X_{11}^{\eta^-} = \frac{-(1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} |\eta_i^-|^2 |\eta_j^-|^2 \right]^{\frac{2(2+2)}{10(10+1)}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1+(1-1)(1+\eta_i^-))^2 (1+(1-1)(1+\eta_j^-))^2 \right]^{\frac{2(2+2)}{10(10+1)}} + (1-1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} |\eta_i^-|^2 |\eta_j^-|^2 \right]^{\frac{2(2+2)}{10(10+1)}}}$$

$$X_{11}^{\eta^-} = \frac{- \left[\sum_{i=1}^{10} \sum_{j=1}^{10} |\eta_i^-|^2 |\eta_j^-|^2 \right]^{\frac{8}{110}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}}}$$

$$X_{11}^{\eta^-} = \frac{- \left[\begin{array}{c} |\eta_1^-|^2 |\eta_1^-|^2 + |\eta_1^-|^2 |\eta_2^-|^2 + \\ \dots + |\eta_1^-|^2 |\eta_{10}^-|^2 + |\eta_2^-|^2 |\eta_2^-|^2 + \\ |\eta_2^-|^2 |\eta_3^-|^2 + \dots + |\eta_2^-|^2 |\eta_{10}^-|^2 + \\ \dots + |\eta_{10}^-|^2 |\eta_{10}^-|^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + \\ \dots + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}}$$

$$X_{11}^{\eta^-} = \frac{- \left[\begin{array}{c} |-0.04|^2 |-0.04|^2 + |-0.04|^2 |-0.04|^2 + \\ \dots + |-0.04|^2 |-0.10|^2 + |-0.04|^2 |-0.04|^2 + \\ |-0.04|^2 |-0.04|^2 + \dots + |-0.04|^2 |-0.10|^2 + \\ \dots + |-0.10|^2 |-0.10|^2 \end{array} \right]^{\frac{8}{110}}}{\left[\begin{array}{c} (1)^2 (1)^2 + (1)^2 (1)^2 + \\ (1)^2 (1)^2 + (1)^2 (1)^2 + \\ \dots + (1)^2 (1)^2 \end{array} \right]^{\frac{8}{110}}}$$

$$X_{11}^{\eta^-} = \frac{-0.9929}{1.3384}$$

$$X_{11}^{\eta^-} = -0.7419$$

For the negative falsity membership degree, the computation is as follows:

$$X_{ij}^{\nu^-} = \frac{-\zeta \left[\sum_{i=1}^n \sum_{j=1}^n |\nu_i^-|^p |\nu_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}{\left[\sum_{i=1}^n \sum_{j=1}^n (1+(\zeta-1)(1+\nu_i^-))^p (1+(\zeta-1)(1+\nu_j^-))^q \right]^{\frac{2(p+q)}{n(n+1)}} + (\zeta-1) \left[\sum_{i=1}^n \sum_{j=1}^n |\nu_i^-|^p |\nu_j^-|^q \right]^{\frac{2(p+q)}{n(n+1)}}}$$

$$\begin{aligned}
X_{11}^{v^-} &= \frac{-(1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} |v_1^-|^2 |v_1^-|^2 \right]^{\frac{2(2+2)}{10(10+1)}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1+(1-1)(1+v_1^-))^2 (1+(1-1)(1+v_1^-))^2 \right]^{\frac{2(2+2)}{10(10+1)}} + (1-1) \left[\sum_{i=1}^{10} \sum_{j=1}^{10} |v_1^-|^2 |v_1^-|^2 \right]^{\frac{2(2+2)}{10(10+1)}}} \\
X_{11}^{v^-} &= \frac{- \left[\sum_{i=1}^{10} \sum_{j=1}^{10} |v_1^-|^2 |v_1^-|^2 \right]^{\frac{8}{110}}}{\left[\sum_{i=1}^{10} \sum_{j=1}^{10} (1)^2 (1)^2 \right]^{\frac{8}{110}}} \\
X_{11}^{v^-} &= \frac{- \left[\begin{aligned} &|v_1^-|^2 |v_1^-|^2 + |v_1^-|^2 |v_2^-|^2 + \\ &|v_1^-|^2 |v_3^-|^2 + \dots + |v_1^-|^2 |v_{10}^-|^2 + \\ &|v_2^-|^2 |v_2^-|^2 + |v_2^-|^2 |v_3^-|^2 + \dots + \\ &|v_2^-|^2 |v_{10}^-|^2 + \dots + |v_{10}^-|^2 |v_{10}^-|^2 \end{aligned} \right]^{\frac{8}{110}}}{\left[\begin{aligned} &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &\dots + (1)^2 (1)^2 \end{aligned} \right]^{\frac{8}{110}}} \\
X_{11}^{v^-} &= \frac{- \left[\begin{aligned} &|-0.07|^2 |-0.07|^2 + |-0.07|^2 |-0.07|^2 + \\ &|-0.07|^2 |-0.07|^2 + \dots + |-0.07|^2 |-0.18|^2 + \\ &|-0.07|^2 |-0.07|^2 + |-0.07|^2 |-0.07|^2 + \dots + \\ &|-0.07|^2 |-0.18|^2 + \dots + |-0.18|^2 |-0.18|^2 \end{aligned} \right]^{\frac{8}{110}}}{\left[\begin{aligned} &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &(1)^2 (1)^2 + (1)^2 (1)^2 + \\ &\dots + (1)^2 (1)^2 \end{aligned} \right]^{\frac{8}{110}}} \\
X_{11}^{v^-} &= \frac{-0.9777}{1.3384} \\
X_{11}^{v^-} &= -0.7305
\end{aligned}$$

Table 5.5 shows the single aggregated BPNS decision matrix that is used to describe the overall assessment of ten experts.

Table 5.5
Aggregated Value using BPN-GHHM Operator

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	[0.9345, 0.1273, 0.2323, -0.0828, -0.7419, -0.7305]	[0.9313, 0.1355, 0.2306, -0.0646, -0.6701, -0.7051]	[0.8865, 0.1630, 0.2798, -0.1216, -0.8005, -0.8023]	[0.8444, 0.2109, 0.3265, -0.1479, -0.8367, -0.8797]	[0.8797, 0.1739, 0.2915, -0.1275, -0.8105, -0.8151]	[0.8887, 0.1761, 0.2797, -0.1078, -0.7639, -0.7887]	[0.8940, 0.1665, 0.2960, -0.1399, -0.8435, -0.8268]	[0.9007, 0.1630, 0.2683, -0.0992, -0.7694, -0.7798]
A_2	[0.9362, 0.1273, 0.2186, -0.0542, -0.6240, -0.6640]	[0.9408, 0.1238, 0.2134, -0.0427, -0.6178, -0.6487]	[0.9232, 0.1396, 0.2395, -0.0808, -0.7017, -0.7379]	[0.9039, 0.1571, 0.2615, -0.1000, -0.7439, -0.7687]	[0.9535, 0.1121, 0.1994, -0.0375, -0.5773, -0.6218]	[0.9429, 0.1259, 0.2185, -0.0609, -0.5979, -0.6620]	[0.9415, 0.1162, 0.2131, -0.0677, -0.7073, -0.7173]	[0.9152, 0.1436, 0.2449, -0.0864, -0.7240, -0.7433]
A_3	[0.9152, 0.1685, 0.2559, -0.0598,	[0.9263, 0.1392, 0.2357, -0.0730,	[0.9457, 0.1053, 0.1975, -0.0684,	[0.9527, 0.1020, 0.1878, -0.0488,	[0.9491, 0.1109, 0.1980, -0.0393,	[0.9019, 0.1646, 0.2641, -0.0962,	[0.9545, 0.0959, 0.1790, -0.0481,	[0.9102, 0.1454, 0.2446, -0.0887,

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
	-0.6550, -0.6923]	-0.6736, -0.7136]	-0.6506, -0.6925]	-0.5823, -0.6317]	-0.5975, -0.6308]	-0.7142, -0.7567]	-0.5798, -0.6264]	-0.6876, -0.7322]
A_4	[0.8961, 0.1821, 0.2788, -0.0957, -0.7482, -0.7708]	[0.9272, 0.1529, 0.2465, -0.0656, -0.6723, -0.7103]	[0.9492, 0.1153, 0.2049, -0.0501, -0.5871, -0.6418]	[0.9396, 0.1233, 0.2213, -0.0682, -0.7080, -0.7196]	[0.9455, 0.1252, 0.2147, -0.0410, -0.6018, -0.6413]	[0.9063, 0.1479, 0.2537, -0.0997, -0.7434, -0.7671]	[0.9570, 0.1032, 0.1893, -0.0473, -0.5581, -0.6231]	[0.9260, 0.1469, 0.2475, -0.0845, -0.7138, -0.7377]
A_5	[0.9500, 0.1265, 0.2161, -0.0392, -0.5825, -0.6332]	[0.9362, 0.1549, 0.2465, -0.0634, -0.6616, -0.7012]	[0.9288, 0.1460, 0.2439, -0.0708, -0.7154, -0.7293]	[0.9067, 0.1649, 0.2696, -0.0983, -0.7659, -0.7778]	[0.9322, 0.1435, 0.2348, -0.0555, -0.6274, -0.6718]	[0.9188, 0.1462, 0.2424, -0.0745, -0.6823, -0.7196]	[0.9166, 0.1553, 0.2503, -0.0750, -0.6833, -0.7220]	[0.8901, 0.1770, 0.2722, -0.0922, -0.7244, -0.7596]
A_6	[0.9190, 0.1520, 0.2488,	[0.9198, 0.1549, 0.2465,	[0.9409, 0.1287, 0.2200,	[0.9381, 0.1199, 0.2104,	[0.9271, 0.1419, 0.2334,	[0.9144, 0.1592, 0.2516,	[0.9611, 0.1002, 0.1836,	[0.9120, 0.1628, 0.2531,

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_7	-0.0811,	-0.0666,	-0.0529,	-0.0536,	-0.0568,	-0.0748,	-0.0336,	-0.0682,
	-0.6780,	-0.6442,	-0.6093,	-0.6223,	-0.6399,	-0.6509,	-0.5430,	-0.6580,
	-0.7240]	-0.6920]	-0.6578]	-0.6599]	-0.6772]	-0.7020]	-0.5971]	-0.6994]
	[0.9401,	[0.9381,	[0.9353,	[0.9447,	[0.9253,	[0.9401,	[0.8978,	[0.9595,
	0.1215,	0.1290,	0.1202,	0.1141,	0.1315,	0.1215,	0.1675,	0.1066,
	0.2157,	0.2239,	0.2144,	0.2035,	0.2316,	0.2157,	0.2731,	0.1923,
A_8	-0.0623,	-0.0629,	-0.0635,	-0.0516,	-0.0804,	-0.0623,	-0.1066,	-0.0346,
	-0.6594,	-0.6605,	-0.6680,	-0.6053,	-0.7009,	-0.6594,	-0.7596,	-0.5467,
	-0.6956]	-0.6984]	-0.6996]	-0.6490]	-0.7358]	-0.6956]	-0.7853]	-0.6046]

Table 5.5 presents the aggregated decision matrix obtained through the BPN-GHHM operator. The aggregated values represent the collective evaluation of all respondents for each alternative under the selected criteria. Subsequently, the score function is applied to convert these aggregated values into crisp scores for further evaluation.

Step 6. Computation of score function

After the process of aggregation of the individual decision matrices to one collective BPNS decision matrix and entailed the use of the score function in (4.14), a methodical process of attaining this change by computing crisp score that represents the performance of each option in general is presented below.

$$s(A_{11}) = \frac{0.9345 + 1 - 0.1273 + 1 - 0.2323 + 1 - (-0.0828) + (-0.7419) + (-0.7305)}{6} = 0.3642$$

$$s(A_{21}) = \frac{0.9345 + 1 - 0.1273 + 1 - 0.2186 + 1 - (-0.0542) + (-0.6240) + (-0.6640)}{6} = 0.3928$$

$$s(A_{31}) = \frac{0.9152 + 1 - 0.1685 + 1 - 0.2559 + 1 - (-0.0598) + (-0.6550) + (-0.6923)}{6} = 0.3672$$

The aggregated output score matrix is in the Table 5.6.

Table 5.6
Aggregated Value using Score Function

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.3642	0.3758	0.3271	0.2961	0.3194	0.3313	0.3168	0.3366
A_2	0.3928	0.3967	0.3642	0.3454	0.4134	0.3999	0.3759	0.3576
A_3	0.3672	0.3728	0.3947	0.4163	0.4163	0.3498	0.4203	0.3648
A_4	0.3353	0.3685	0.4084	0.3726	0.4006	0.3490	0.4218	0.3608
A_5	0.4051	0.3780	0.3608	0.3378	0.3851	0.3671	0.3634	0.3415
A_6	0.3662	0.3745	0.3964	0.3965	0.3819	0.3709	0.4285	0.3678
A_7	0.3850	0.3815	0.3828	0.4041	0.3677	0.3850	0.3365	0.4240

Table 5.6 presents the score values obtained from the aggregated decision matrix. The score function transforms the bipolar Pythagorean neutrosophic values into crisp values, which can be further processed in the ARIE method in the next step.

Step 7. Ranking of alternatives using ARIE method

To facilitate efficient computation and visualization, ARIE was implemented using the InnoRank–ARIE system (see APPENDIX 3), an interactive application developed with Streamlit and hosted on GitHub. This system automates calculation process and offers a user-friendly interface for inputting decision matrices, executing computations, and generating ranking outputs. Since all the criteria consider as benefit type, then maximum values for each criterion based on Table 5.6 are,

$$x_1^{\max} = 0.4051, x_2^{\max} = 0.3967, x_3^{\max} = 0.4084, \dots, x_8^{\max} = 0.4240$$

Followed the (3.2)-(3.4), the normalize values are calculated as below,

$$\begin{aligned} n_{11} &= \frac{0.3642}{0.4051} = 0.8990, & n_{12} &= \frac{0.3758}{0.3967} = 0.9473, \\ n_{21} &= \frac{0.3928}{0.4051} = 0.9694, & n_{22} &= \frac{0.3967}{0.3967} = 1.0000, \\ n_{31} &= \frac{0.3672}{0.4051} = 0.9064, & n_{32} &= \frac{0.3728}{0.3967} = 0.9400 \end{aligned}$$

The calculation process is similar for each criterion shown in Table 5.7.

Table 5.7
Normalized Decision Matrix of the Alternatives

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.8990	0.9473	0.8009	0.7112	0.7726	0.8285	0.7395	0.7938
A_2	0.9694	1.0000	0.8919	0.8298	1.0000	1.0000	0.8773	0.8436
A_3	0.9064	0.9400	0.9665	1.0000	0.9882	0.8746	0.9808	0.8606
A_4	0.8277	0.9289	1.0000	0.8951	0.9690	0.8726	0.9844	0.8509
A_5	1.0000	0.9529	0.8836	0.8115	0.9314	0.9180	0.8483	0.8056
A_6	0.9040	0.9441	0.9706	0.9526	0.9239	0.9275	1.0000	0.8676
A_7	0.9504	0.9619	0.9373	0.9707	0.8894	0.9628	0.7853	1.0000

This study utilizes equal weight, $w_j = 0.125$, the calculation is as, $v_{11} = 0.9006(0.125) = 0.1126$, $v_{21} = 0.9694(0.125) = 0.1212$, $v_{31} = 0.9064(0.125) = 0.1133$. The weighted normalized decision matrix presented in Table 5.8.

Table 5.8
Weighted Normalized Decision Matrix Across Criteria

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.1126	0.1184	0.1001	0.0889	0.0966	0.1036	0.0924	0.0992
A_2	0.1212	0.1250	0.1115	0.1037	0.1250	0.1250	0.1097	0.1054
A_3	0.1133	0.1175	0.1208	0.1250	0.1235	0.1093	0.1226	0.1076
A_4	0.1035	0.1161	0.1250	0.1119	0.1211	0.1091	0.1226	0.1064
A_5	0.1250	0.1191	0.1104	0.1014	0.1164	0.1148	0.1060	0.1007
A_6	0.1130	0.1180	0.1213	0.1191	0.1155	0.1159	0.1250	0.1084
A_7	0.1188	0.1202	0.1172	0.1213	0.1112	0.1203	0.0982	0.1250

Maximum and minimum values are obtained according to Table 5.8.

$$\begin{aligned} \nu_1^{\max} &= 0.1250, \nu_2^{\max} = 0.1250, \nu_3^{\max} = 0.1250, \dots, \nu_8^{\max} = 0.1250, \\ \nu_1^{\min} &= 0.1035, \nu_2^{\min} = 0.1161, \nu_3^{\min} = 0.1001, \dots, \nu_8^{\min} = 0.0992, \end{aligned}$$

Based on the (3.6) and (3.7), let $\gamma = 0.5$,

$$\begin{aligned} Sim_1^{best} &= \left(\frac{0.1124}{0.1250}\right)^{0.5} + \left(\frac{0.1184}{0.1250}\right)^{0.5} + \left(\frac{0.1001}{0.1250}\right)^{0.5} + \left(\frac{0.0913}{0.1250}\right)^{0.5} + \left(\frac{0.0889}{0.1250}\right)^{0.5} + \left(\frac{0.1036}{0.1250}\right)^{0.5} \\ &\quad + \left(\frac{0.0924}{0.1250}\right)^{0.5} + \left(\frac{0.0992}{0.1250}\right)^{0.5} \\ &= 7.1998 \\ Sim_1^{worst} &= \left(\frac{0.1035}{0.1124}\right)^{0.5} + \left(\frac{0.1161}{0.1184}\right)^{0.5} + \left(\frac{0.1001}{0.100}\right)^{0.5} + \left(\frac{0.0889}{0.0889}\right)^{0.5} + \left(\frac{0.0966}{0.0966}\right)^{0.5} + \left(\frac{0.1036}{0.1036}\right)^{0.5} \\ &\quad + \left(\frac{0.0924}{0.0924}\right)^{0.5} + \left(\frac{0.0992}{0.0992}\right)^{0.5} \\ &= 7.9498 \end{aligned}$$

Let $\kappa = 0.5$ according to the (3.8),

$$RC_1 = \frac{0.5(7.1998)}{0.5(7.1998) + (1-0.5)(7.9498)} = 0.4752$$

Step 8. Decision output and interpretation

The highest RC value indicates effective learning strategy shown in Figure 5.4.

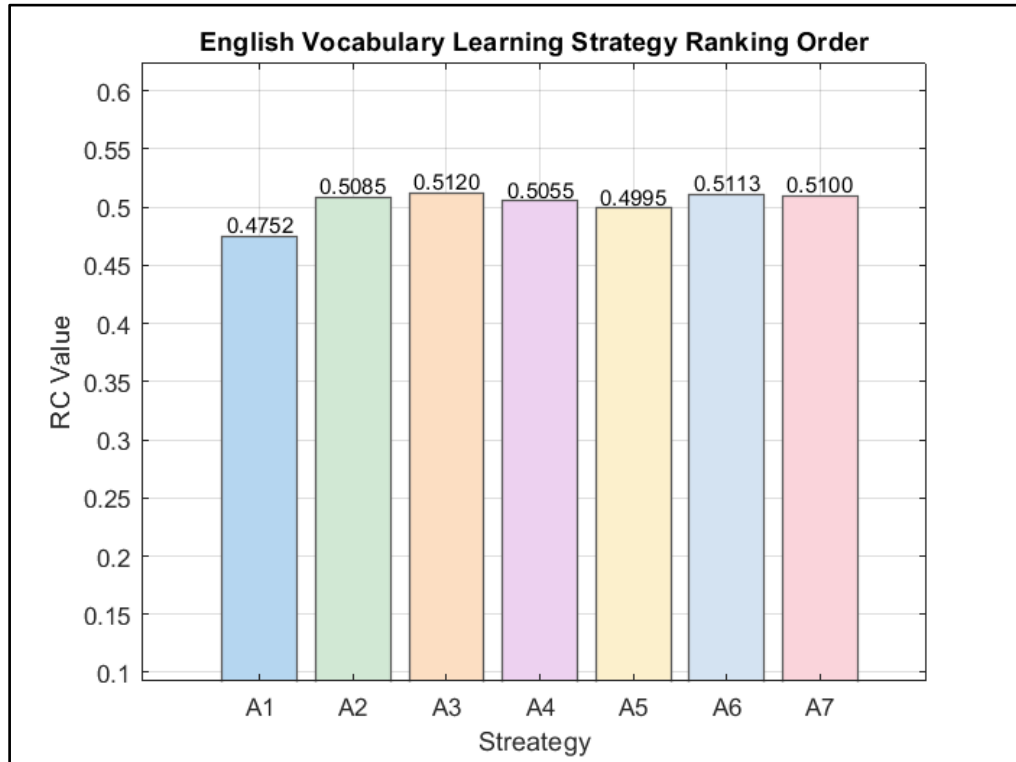


Figure 5.4 Final Ranking of English Vocabulary Learning Strategies

Based on the Figure 5.4, the x-axis represents the vocabulary learning strategies, while the y-axis indicates their respective RC values derived from the ARIE method. Among the seven strategies, A_3 , gamification-based learning achieved the highest RC value, identifying it as the most suitable method for enhancing English vocabulary acquisition. This result suggests that students tend to learn more effectively when learning activities are designed to resemble games, incorporating elements of enjoyment, small rewards, and immediate feedback. The observed results are supported by prior studies. In particular, Slamet & Meng (2025) reported that gamification has a positive influence on both cognitive and attitudinal learning outcomes when integrated into collaborative learning contexts, which aligns with the effectiveness patterns identified in this study.

Closely following A_3 , A_6 , multimedia, demonstrated nearly identical performance. This finding highlights the strong potential of multimedia resources such as videos, images, and audio materials in supporting vocabulary development. The other strategies, A_2 , contextual learning, A_7 , peer learning, and A_4 , AI, showed very close RC values. These methods remain highly effective, as learning vocabulary through real-life contexts, A_2 , collaboration with peers, A_7 , or intelligent digital tools, A_4 , all yielded comparable and positive outcomes.

Meanwhile, A_5 , mnemonic, ranked lower, suggesting that while mnemonic strategies can aid memory retention, they may be less effective as a primary instructional approach. The lowest-ranking strategy was A_1 , traditional method, which primarily involves rote memorization, repetitive reading, and mechanical copying of word lists. This indicates that conventional drill-based techniques, when used alone, provide the weakest outcomes for vocabulary learning.

5.4 Comparative Analysis

The comparative analysis was conducted using MATLAB R2024a including the calculation process, and the generation of figures. Comparative evaluation of the ranking outcomes produced using ARIE method and several MCDM approaches is presented. This comparison is intended to assess the consistency, strength and stability of the ARIE method ranking results. In order to achieve a wholesome evaluation, seven of the developed MCDM techniques were chosen to be compared, which are, ARAS, CRADIS, MABAC, MOORA, SAW, TOPSIS, and VIKOR.

The outcome of the ranking using each approach was compared to determine any patterns, stability, similarities and deviations in assessing the seven strategies of learning English vocabulary. To make the comparison more readable, it is presented in figures and graphical representation to show the ranking orders produced by each method. Also, the correlation coefficients were calculated among the ARIE rankings with the rest of the MCDM methods to measure in quantitative terms the extent to which the various ranking methods agreed with each other. The results of the comparative analysis presented in Table 5.9.

Table 5.9
Company Ranking Results

Alternative	ARAS	ARIE	CRADIS	MABAC	MOORA	SAW	TOPSIS	VIKOR
A_1	0.8623	0.4752	0.2315	-0.4574	0.3366	0.0841	0.1111	1.0000
A_2	0.9844	0.5085	1.0000	0.1710	0.3842	0.6825	0.5681	0.0000
A_3	1.0000	0.5120	0.8602	0.1020	0.3902	0.6135	0.6819	0.2403
A_4	0.9747	0.5055	0.6796	-0.0146	0.3804	0.4969	0.5893	0.6551
A_5	0.9498	0.4995	0.6492	-0.0374	0.3707	0.4740	0.4558	0.5575
A_6	0.9964	0.5113	0.8680	0.1063	0.3888	0.6178	0.6792	0.1113
A_7	0.9925	0.5100	0.9374	0.1422	0.3873	0.6537	0.6201	0.1570

According to the findings in Table 5.9, A_3 , gamification method, ranked the first in the ARAS, ARIE, MOORA, and TOPSIS methods, which means that it is always the best strategy to consider in learning English vocabulary. On the other hand, A_2 , contextual learning, ranked best under the CRADIS, MABAC, SAW and VIKOR methods. Although there are slight differences in the middle-ranking options, there is considerable coherence in the approaches among all the methods, with the Traditional method, A_1 consistently in the seventh position. This consistent result brings out a common conclusion that the usual rote memorizing methods are the least effective in terms of vocabulary development.

The comparative ranking order of all the MCDM methods are illustrated in Figure 5.5 and Figure 5.6. Each evaluation method is indicated on the x-axis, and the ranking order is indicated on the y-axis. The associated alternatives are marked on the right to facilitate in the interpretation of the rank change across methods.

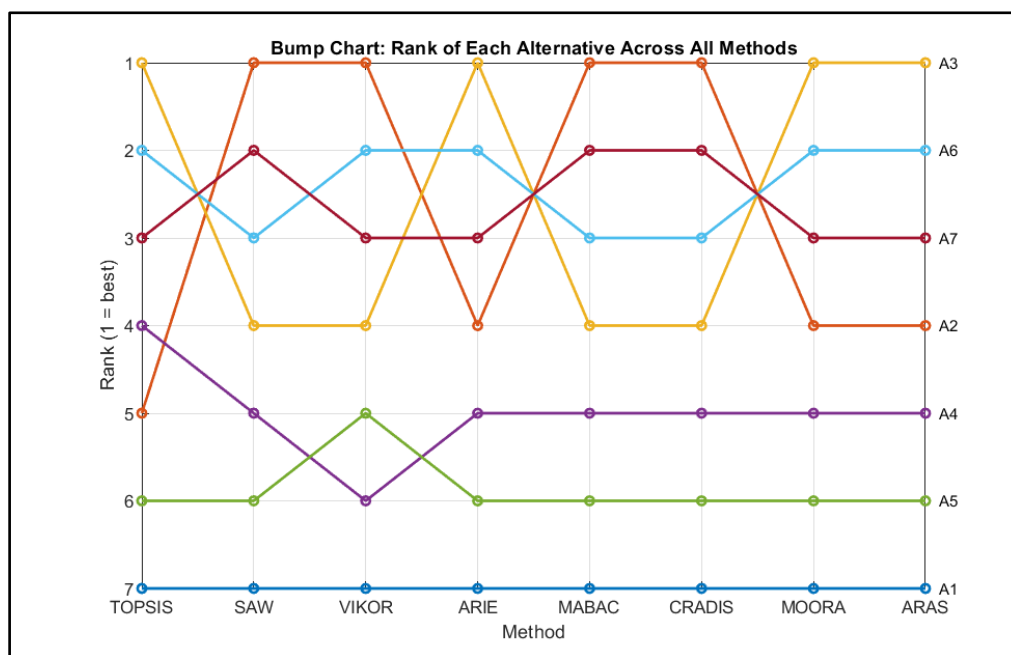


Figure 5.5 Visualization of English Vocabulary Learning Strategies Ranking Across Different MCDM Models

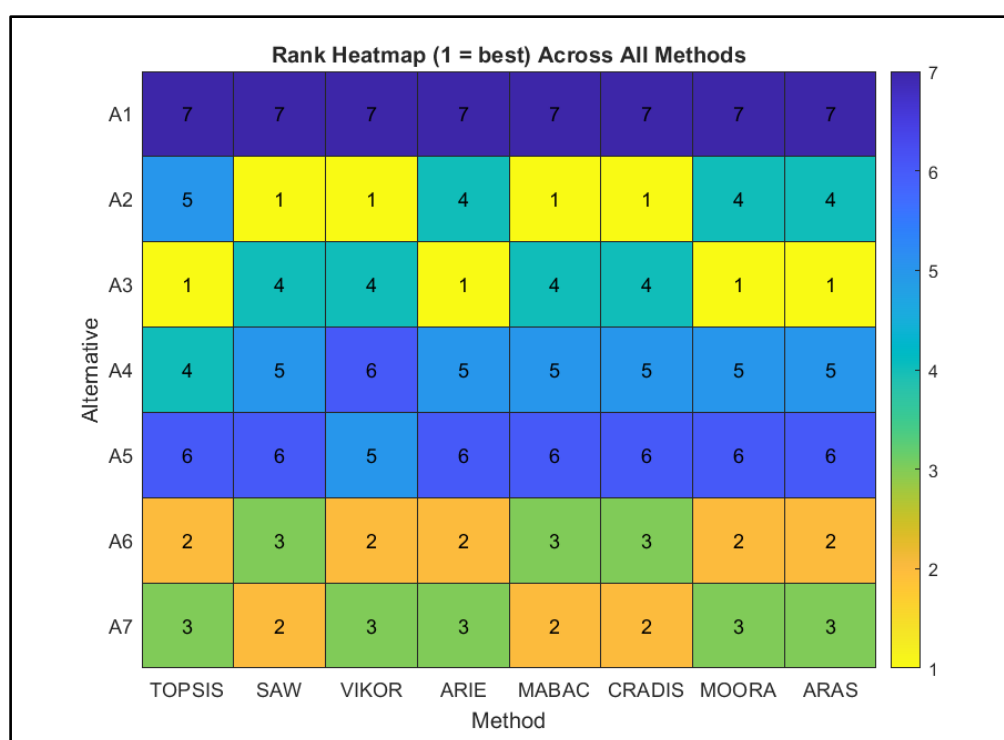


Figure 5.6 Visual Representation of Ranking Pattern for English Vocabulary Learning Strategies Across Evaluation Methods

Figure 5.5 and Figure 5.6 demonstrate that the comparative performance of the seven MCDM approaches. The bump chart shows the position of each of the vocabulary

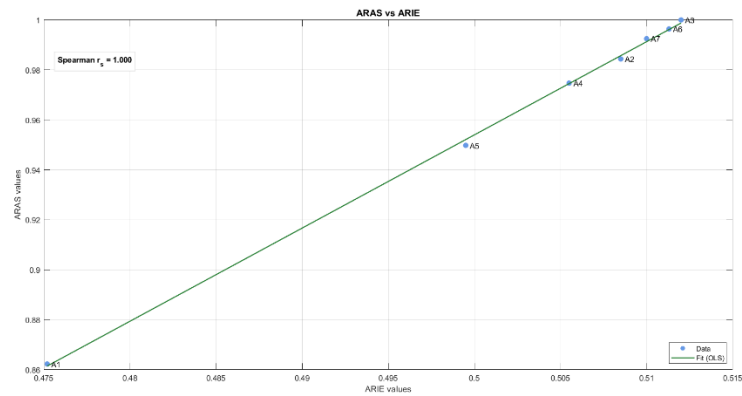
learning strategies in all the methods, and a colour coded in Figure 5.6 depict the ranking strength, where lighter colours mean better performance. In most of the approaches, ARIE, MABAC, CRADIS, MOORA, and ARAS, the gamification method, A_3 , remain the first place, which is indicative of its strong image as the most effective vocabulary learning approach. Multimedia, A_6 being next in line, with very steady second-place performance in nearly all methods. Contextual learning, A_2 and Peer, A_7 switched the second and third positions in some approaches, in particular, SAW and TOPSIS methods, and it shows that both methods are credible alternatives with slight differences in positions based on the evaluation model utilized. The bottom-ranked alternatives were found to have higher consistency. Mnemonic technique, A_5 and artificial intelligence, A_4 were also at positions five and six in almost all models, while the traditional method, A_1 continued to hold the last position in all models, an indication of widespread consensus about its relatively low effectiveness as a primary approach to vocabulary learning.

5.4.1 Spearman Rank Correlation Coefficient

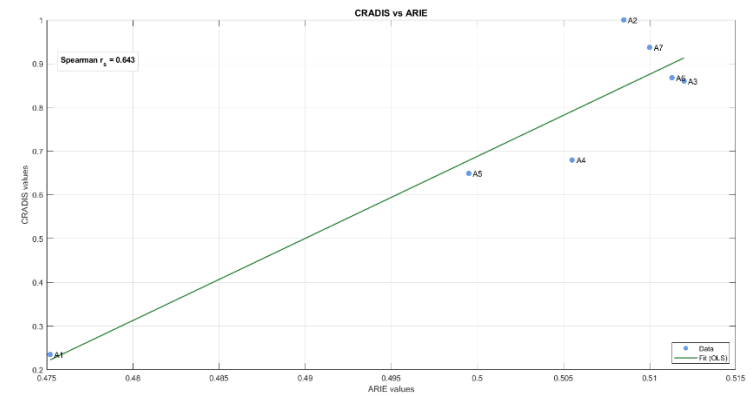
The validation of comparative analysis was computed by spearman correlation coefficient using MATLAB R2024a. Spearman's rank correlation coefficient is a measurement of the strength and direction of a monotonic relationship between two ranked variables that is nonparametric that is more suitable than the parametric correlation measures for MCDM problems. The coefficient is defined as,

$$\rho = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)}$$

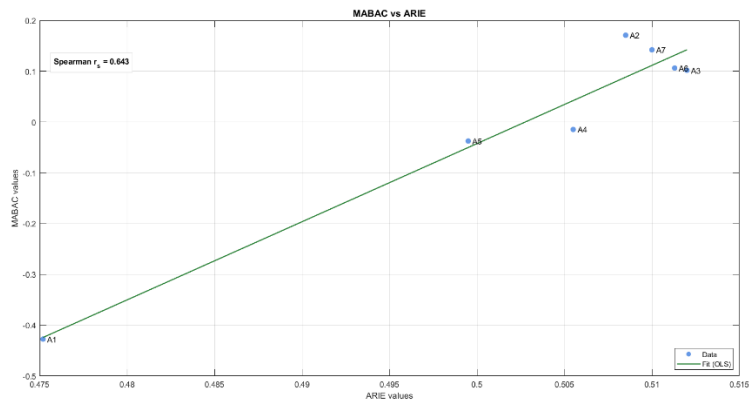
where d_i is the difference between the ranks of the i -th alternative for two methods, and n is the number of alternatives. Spearman's rank correlation coefficient is used in this study to evaluate the consistency of the rank ordering of the ARIE method and benchmark MCDM methods. Figure 5.7 illustrates the scatter plot relationships between ARIE and the selected MCDM methods, together with the corresponding Spearman rank correlation coefficients.



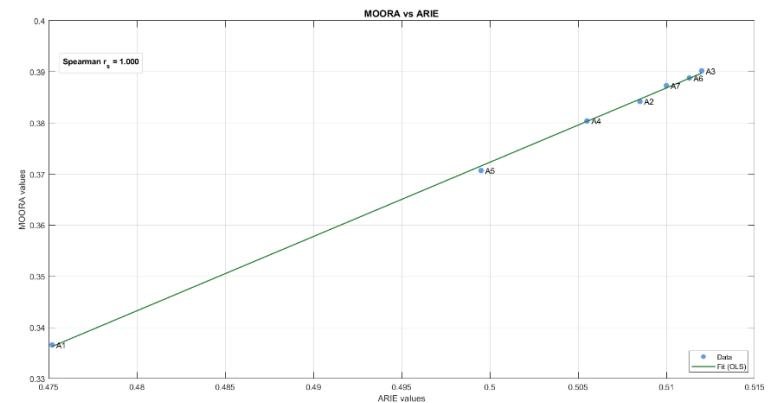
a) Correlation between ARAS and ARIE



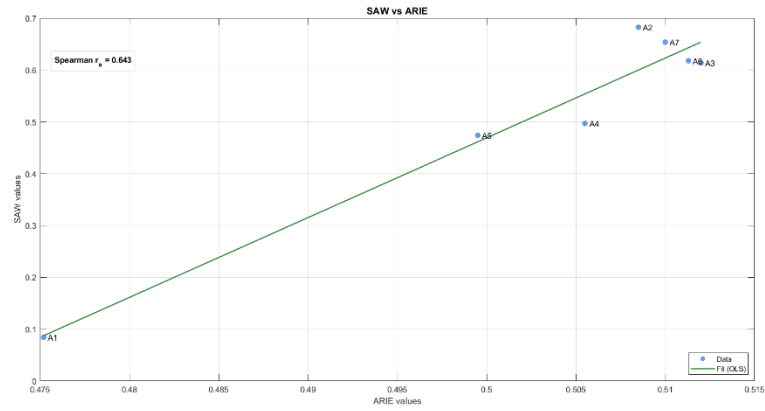
b) Correlation between CRADIS and ARIE



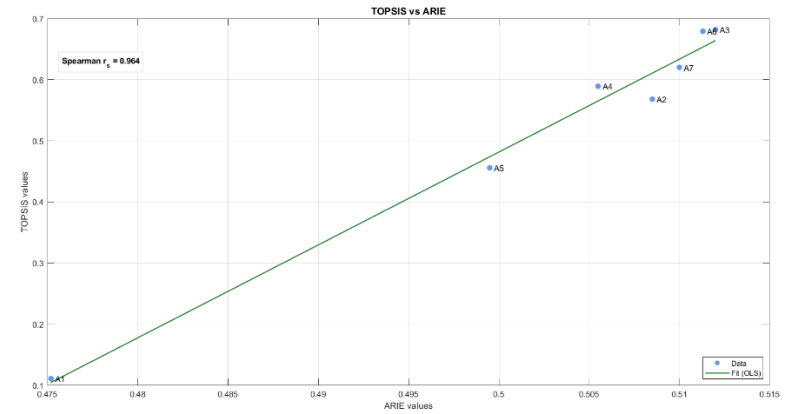
c) Correlation between MABAC and ARIE



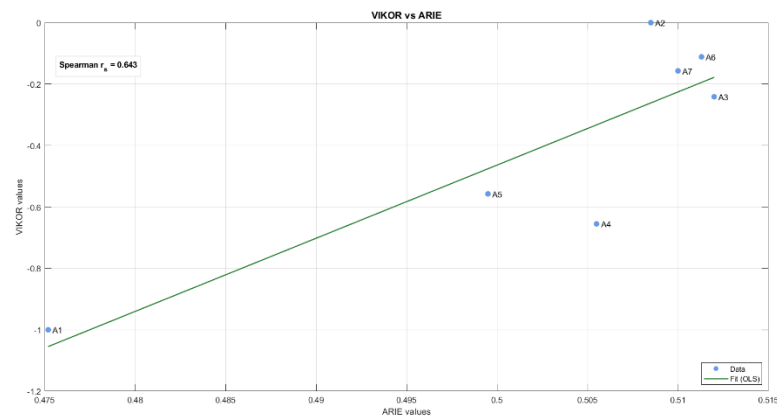
d) Correlation between MOORA and ARIE



e) Correlation between SAW and ARIE



f) Correlation between TOPSIS and ARIE



g) Correlation between VIKOR and ARIE

Figure 5.7 Correlation Analysis Between ARIE and Selected MCDM Approaches

Based on Figure 5.7, the Spearman rank correlation coefficient for both ARAS and MOORA methods with the ARIE method is 1.000. This reflects complete monotonic agreement between ARIE, ARAS and MOORA in ranking order across all the alternatives. In other words, both methods maintain the same order of alternatives, with each alternative having the same position or rank under all three approaches. The numerical scores are different, but there are no deviations in the ranking or rank reversals. The values obtained by ARIE are clustered in a relatively smaller interval, 0.475 until 0.512, compared to those obtained by ARAS and MOORA which are wider, 0.862 until 1.000. This implies that ARAS and MOORA are scale expanding transformations that increase the difference in numbers between alternatives without altering the preference structure. Therefore, these methods do not change the decisions but only shift the extent and variation of the scoring scale. The stability of the rankings suggests that the most successful English learning strategies are consistently identified in both ARIE, ARAS, and MOORA, and that the least successful alternative strategy is constant. There are no rank inversions or structural anomalies found, which suggests a stable ordering and a deterministic process in all three methods. Thus, although the numerical representations differ, the results of the decisions are fully equivalent in their practical interpretation, only differing in scale, and not in their preference structure.

The other scatter plots between CRADIS, MABAC, SAW, TOPSIS and VIKOR and ARIE show strong and consistent positive monotonic relationships, but with varying degrees of rank agreement between the methods. Spearman Rank correlation coefficients indicate that there is a range of consistency with ARIE, from moderate to very high. The method that is most consistent with ARIE is TOPSIS, which shows Spearman coefficient of around 0.964, showing a good consistency of the ranking's structure and consistent alignment of the positions of the alternatives. The rank agreement of the remaining methods, CRADIS, MABAC, SAW, and VIKOR, on the other hand, is moderate and the Spearman rank correlation coefficient is in the range of 0.64 with ARIE showing rank order only to a certain extent. The discrepancies are mostly found in the mid-ranked alternatives: small differences in aggregation and normalization result in rank reordering effects. Highest and lowest alternatives are consistent across methods, but the intermediate alternatives are more sensitive to methodological differences, leading to more localized rank shifts. Overall, the results indicate that all comparative methods are consistent in terms of general performance patterns of English learning strategies as identified by ARIE; however, the underlying

decision aggregation frameworks of these methods yield differences in rank stability. The structural alignment of TOPSIS with ARIE is the highest while with SAW, VIKOR, MABAC and CRADIS is moderate with some rank changes in the second half alternatives. The next section examines the effect of sensitivity parameters on the decision analysis.

5.5 Sensitivity Analysis

This section analysed the influence of parameter variations in the BPN-GHHM operator and ARIE method. It was performed using Microsoft Excel (Version 2605) and the figures were generated by using MATLAB R2024a. The sensitivity results across different parameter sets (p, q, ζ) , as well as γ and κ values illustrated in Figure 5.8-Figure 5.10.

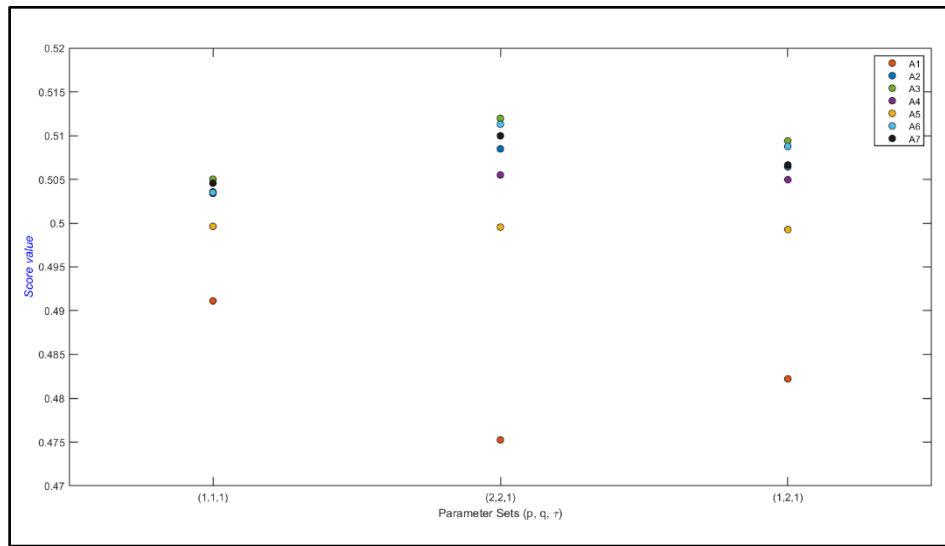


Figure 5.8 Sensitivity of Ranking Order to the Changes in Parameter Settings

Figure 5.8 illustrates the sensitivity analysis of seven alternatives to evaluate English vocabulary learning strategies using three parameter combinations of p , q , and ζ , $(1,1,1)$, $(2,2,1)$ and $(1, 2, 1)$ respectively. The variation of the parameters does not have a significant effect on the score of the y-axis, which means the rank of the alternatives is not sensitive to the change in parameter. In all parameter sets, the strategy A_3 , gamification method, has the best score value, which proves that it is more effective and reliable strategy. A_1 , traditional method, on the other hand, is the one that always

performs the worst, implying that it is less adaptable and has less impact as compared to modern and interactive methods. The other options, A_2 , A_4 , A_5 , A_6 , and A_7 , exhibit small variations though they tend to maintain their relative order, which supports the overall consistency of the decision model.

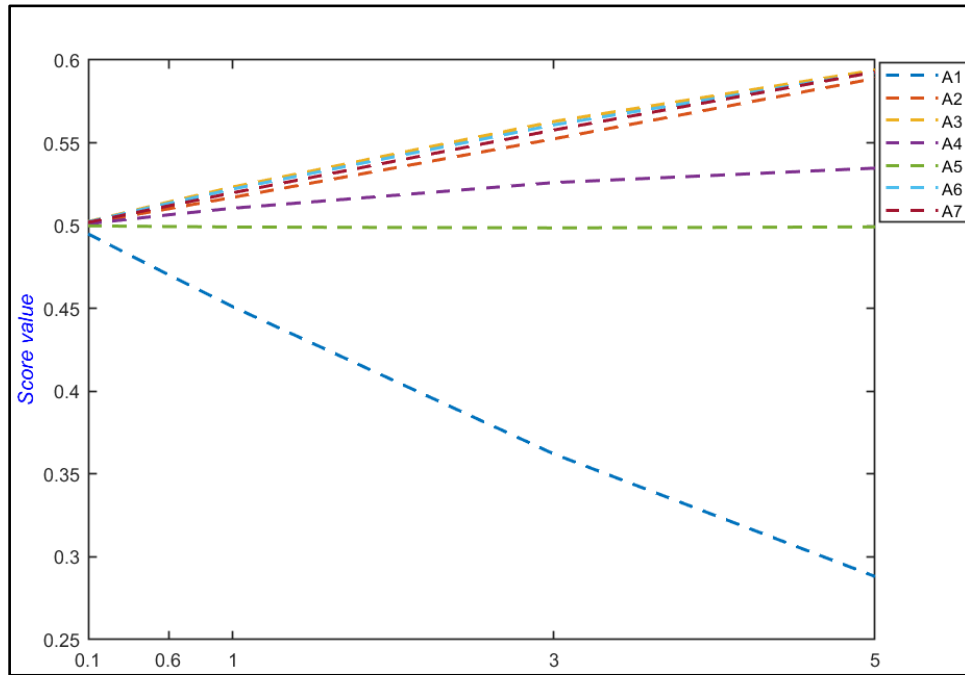


Figure 5.9 Sensitivity of Ranking Outcomes to Different γ Values

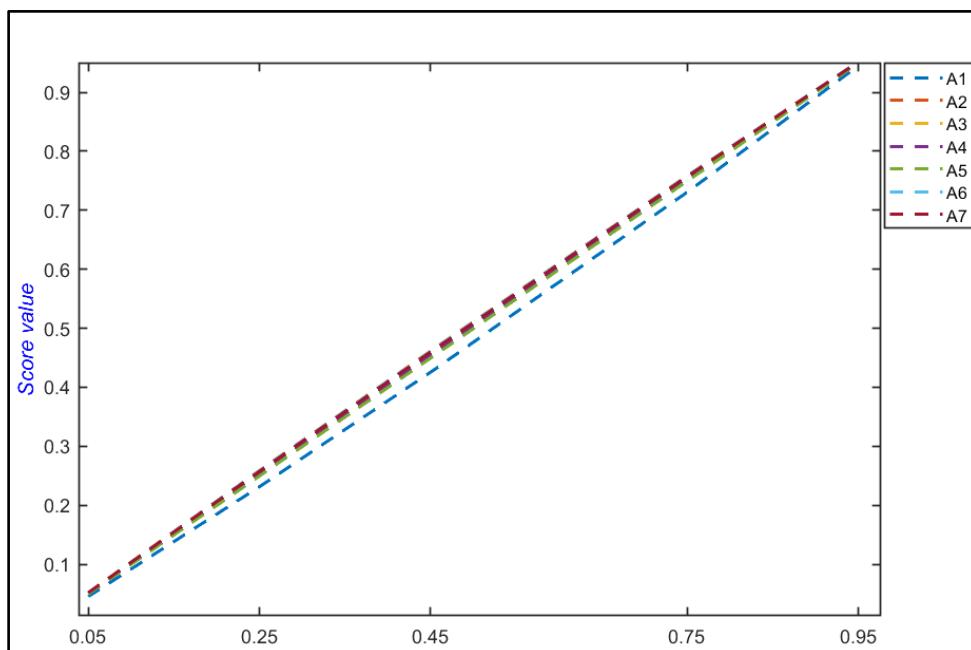


Figure 5.10 Sensitivity of Ranking Outcomes to Different κ Values

Figure 5.9 and Figure 5.10 display the sensitivity evaluation of the vocabulary learning strategies as per the γ and κ parameters. The γ values graph indicates that most of the alternatives score values increase gradually as γ changes its level between 0.1 and 5 except for the strategy A_1 , which indicates more stability in performance to the dynamic conditions. A_3 , gamification method, remained on the first place all the time with the upward movement whereas A_1 , traditional method, has suffered significant performance decline, which reflects its inability to flex according to the changes in parameters. A_5 is almost non-sensitive to the changes in γ indicating that it is an independent variable in performance.

Conversely, the plot of the κ values depicts that all the alternatives follow almost the same linear upward trend, meaning the scores are proportionally scaled to κ parameter. This consistency implies that κ has no effect on the ranking of methods. Based on the results of the sensitivity analysis, the proposed ARIE framework demonstrates stable ranking performance under varying parameter settings. This provides a final validation for the reliability of the application results presented in this chapter.

5.6 Conclusion

English vocabulary learning strategies were evaluated using the integrated BPN-GHHM operator and proposed ARIE method. The results suggest that gamification-based learning strategies rank the highest among other examined alternatives, showing it is an effective strategy for improving English vocabulary learning. Comparative analysis and Spearman rank correlation also indicate a high level of agreement between the results of the ARIE and the existing MCDM methods. Although the methodological structures in MCDM approaches are different, the ranking results are similar. To sum up, the application outcomes confirm the ability of the ARIE method and BPN-GHHM operator together with the new BPNN linguistic variable in creating accurate and stable rankings. Based on the results, it can be concluded that gamification is the best alternative of the evaluated strategies for learning English words, with the possibility of adopting it as a practical method in language learning contexts.

CHAPTER 6

CONCLUSION

6.1 Introduction

The conclusion of the study is presented in this chapter, which includes a summary of the major findings, the achievement of the research objectives, a discussion of the contributions and implications of the proposed methodologies, and the limitations and future directions of the study. The study was conducted with the aim of developing, formulating, and validating an effective MCDM model, namely the ARIE method, together with advanced AOs within the BPNS environment.

The main goal of this research was to develop a stable, flexible, and reliable DM tool that can deal with uncertainty, vagueness, and incompleteness in qualitative expert judgments. In this regard, the proposed methodologies were theoretically developed and then tested by applying it to a study of English vocabulary learning strategy. The empirical results and the comparative analysis show the effectiveness, robustness and applicability of the proposed framework in complex situations of DM

6.2 Objectives Achievement

This study successfully achieved all three research objectives. The first goal was to create an efficient MCDM model, which is the ARIE model. The reason of this objective was due to the drawbacks of the distance-based ranking methods, including rank reversal issues in the TOPSIS. To address this issue, the ARIE method was developed, which also involves adaptive mechanisms to evaluate ideal and anti-ideal reference solutions. The proposed method was shown to be able to rank satisfactorily, with a stable and consistent ranking results, while also accommodating uncertainty of subjective expert evaluations. In addition, its flexibility enables it to be applied in different DM situations, which helps in both the theoretical aspects and the application of the MCDM methodologies.

The second aim was to suggest new aggregation methods in the BPNS environment using the Hamacher and HM operator. The development of the BPN-GHHM and BPN-GWHHM operators was a method used to achieve this objective. This

development is driven by the motivation of providing a solution to the information that is sometimes incomplete, uncertain and contradictory, which may not be sufficiently catered for by the ARIE method alone for complex DM problems. The proposed operators are able to properly integrate the mathematical features of both Hamacher and HM operators in a bipolar neutrosophic setting, thus allowing the representation of indeterminacy, hesitation, and inconsistency of experts' judgments. Therefore, the proposed AOs are more comprehensive and realistic to aggregate the linguistic and uncertain information, which can make the DM process more robust and reliable.

The third target was to implement and validate the proposed ARIE method and BPN-GHHM operator by using the evaluation of English vocabulary learning strategies. This goal was realized by conducting an application with the help of expert evaluations in uncertain and incomplete information conditions. The application showed the feasibility of the proposed framework for educational DM problems. Furthermore, comparative analysis was performed with some famous MCDM methods such as ARAS, CRADIS, MABAC, MOORA, SAW, TOPSIS and VIKOR. The results showed a high level of consistency in the rankings obtained and good correlation with those obtained by these methods. The validation results showed that the proposed models in this research are effective, reliable, and robust, and that they are suitable to be used in the management of qualitative, linguistic and uncertain information in professional DM environments. These successful results in meeting all the research goals validate the proposed framework as an approach to ranking, aggregating and evaluating under uncertainty. The developed methods provide valuable advancements in the field of MCDM and provide practical tools to support decisions in education and other DM domains.

6.3 Contributions of the Study

In this study, it is found that a modified MCDM framework has been developed for MCDM techniques to overcome the drawbacks of the existing methods, which are based on the ARIE method and advanced bipolar neutrosophic AOs. Methodologically, the proposed ARIE method proposes a new ranking method that can deal with a new criterion type and contains two parameters: sensitivity and dual sensitivity, so it has a higher flexibility and adaptability in DM analysis. This enhancement allows decision

makers to simulate different risk attitudes and assessment styles, which is more effective than traditional ranking methods.

In addition, the BPNS based AOs, known as the BPN-GHHM and BPN-GWHHM are presented in this study, which are extended to the existing AOs in the BPNS environment. These operators can model interdependencies between the criteria and can handle bipolar, uncertain, incomplete and inconsistent information. This makes them more representative of the real-world decisions involving ambiguity and uncertainty. On theoretical level, the proposed framework extends the current knowledge base of neutrosophic DM, integrating both adaptive ranking mechanisms and sophisticated aggregation process into a common framework. Incorporating these methods expands the potential use of MCDM methods and presents new concepts for methodological advances.

The framework has been shown to be effective from a practical perspective in assessing the use of English vocabulary learning strategies, offering a reliable and systematic method for decision-makers to prioritize strategies when making decisions under uncertainty. The framework proposed is flexible, robust, and can be easily adapted to different fields such as engineering, management, education, healthcare, and more generally to other areas of complex DM. Hence, the results of this study will provide not only theoretical progress but also practical solutions to make these MCDM models more powerful to solve the more complex problems in the real world applications.

6.4 Limitations and Recommendations for Future Work

Although the goals of this study were successfully achieved, several limitations should be acknowledged. First, ARIE method is fundamentally designed as a ranking and evaluation mechanism and does not inherently address uncertainty, incompleteness, indeterminacy, or vagueness in decision information. In practical DM environments, especially within complex field, experts frequently express their judgments using subjective, imprecise, and sometimes incomplete information. While the introduction of new BPNS linguistic variables allows the representation of such uncertainty prior to the ranking stage, ARIE method itself operates on aggregated numerical information and therefore may not fully preserve all uncertainty characteristics in the evaluation process. Thus, part of the nuanced information contained in the expert assessments may

be reduced during aggregation and transformation procedures before the final ranking is produced.

Furthermore, ARIE method assumes the evaluation criteria contribute independently to the overall assessment and may not fully capture complex interdependencies, which may affect the relationships or feedback among criteria values. In real-world educational DM problems, criteria such as learner motivation, engagement, learning environment, and instructional effectiveness influence one another in a dynamic manner. As a result, the ranking outcomes generated by ARIE may be less representative in situations where strong interactions among criteria exist. In addition, the quality of the final rankings remains dependent on the accuracy, consistency, and completeness of the input evaluations. Any bias, inconsistency, or judgmental error introduced during the elicitation of expert opinions may propagate through the aggregation and scoring process, therefore affect the reliability of the final decision outcomes.

The proposed AOs also possess relatively complex computational structures. Although the BPN-GHHM and BPN-GWHHM operators effectively capture interactions among criteria and accommodate uncertainty in bipolar neutrosophic environments, the computational burden increases as the numbers of criteria, alternatives, and decision-makers grow. This may affect scalability and implementation efficiency in large-scale datasets DM applications. Moreover, the parameter selection process and criteria weighting procedures may introduce additional subjectivity, potentially influence the final ranking results if it is not carefully justified.

Finally, the proposed frameworks were evaluated in the context of English vocabulary learning and involved a relatively limited number of experts. Therefore, caution should be exercised when generalizing the findings to other educational contexts or application domains. Although ARIE method, BPNS linguistic variables, and the proposed AOs demonstrate promising performance, further empirical validation across diverse datasets, institutions, and decision environments is necessary to establish their broader robustness, reliability, and practical applicability.

6.5 Implications on English Vocabulary Learning Strategies

The findings of this study provide several important implications for English vocabulary learning strategies. After the evaluation was carried out with the two

proposed models, gamification was found to be the most effective vocabulary learning strategy, followed using multimedia learning, contextual learning, peer learning, AI based learning, mnemonic learning, and traditional learning. Based on these findings, there seems to be a preference, and an effectiveness change for the learners to acquire vocabulary using more interactive, engaging and ICT enhanced methods.

First, the importance of gamification is clear, as it shows that the engagement of the learners is very important in vocabulary development. The use of gamified learning environments involves various features like rewards, challenges, points, badges and immediate feedback (Wulantari et al., 2023), which increases motivation and interest of the vocabulary learning activities. The result suggests that learning vocabulary should go beyond simple rote repetition and should also be infused with elements of a game, to increase the level of active participation and continuous repetition in learning. Language teachers and educational institutions can thus take advantage of gamification elements in classroom activities, mobile applications and online learning platforms to improve the vocabulary acquisition results.

The lowest ranked traditional method of learning suggests that traditional ways of learning like learning lists of vocabulary and repetition drills, may not be sufficient to serve the learning goals and preferences of modern learners. The results indicate that traditional methods can still yield a basic knowledge and reinforcement, but when students depend solely on these methods, they might be unable to engage in the learning process and remember some of the vocabulary. So, teaching and learning vocabulary should be more learner centred with the use of technology to increase the effectiveness of vocabulary learning.

In general, the results suggest that appropriate strategies for learning English are more closely linked to active involvement, multimodal learning, meaningful context, and learner involvement. The findings offer significant insights into vocabulary learning strategy selection and implementation for educators, curriculum developers, language learning institutions, and technology designers to improve selecting and applying vocabulary learning strategies for today's learners

6.6 Conclusion

Last but not least, this study has fulfilled all the research objectives through the introduction of ARIE method, BPN-GHHM operator, BPN-GWHHM operator, and the

validation of the proposed integrated framework by applying them to practical examples. The results prove that the proposed methodologies yield stable and reliable decision results in uncertain, incomplete and linguistically described environments.

This study contributes to the development of MCDM which offers a comprehensive framework that combines adaptive ranking mechanisms with advanced bipolar neutrosophic aggregation techniques in evaluating English vocabulary learning strategies. The framework proposed here not only improves decision-makers' ability to solve complex evaluation problems but also acquires more flexibility in dealing with various expressions of uncertainty information. The application results also show that the methods proposed in this paper are effective, robust and applicable in supporting informed DM processes.

In short, the results of this study provide a solid base for further research on MCDM modelling and continue the knowledge base in neutrosophic and bipolar DM environment. The proposed methodologies are expected to stimulate additional theoretical developments and applications in other fields of activity, such as medical, management, engineering especially under complex and uncertain MCDM problems

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APPENDICES

APPENDIX 1

English Vocabulary Learning Strategies Questionnaire

8/12/25, 4:23 PM

Evaluation of English Vocabulary Learning Strategies

Evaluation of English Vocabulary Learning Strategies

Hi, my name is **Nur Fathiah Fatin Mohamad Fauzi**, and I'm a postgraduate student conducting this study to evaluate and rank English vocabulary learning strategies.

This survey is part of my academic research which applies the **Bipolar Pythagorean Neutrosophic-Generalized Hamacher Heronian Mean (BPN-GHHM)** and **Adaptive Ranking with Ideal Evaluation (ARIE)** models. The goal is to gain insights from stakeholders to enhance English strategies quality and relevance to current academic and industry needs.

Please rate each strategies (**Traditional Method, Contextual Learning, Gamification Method, Artificial Intelligence, Mnemonic, Multimedia, Peer**) based on your experience, using a scale from **1 (Very Poor)** to **7 (Very Good)**. Your responses will be kept confidential and used solely for research and academic improvement purposes.

Thank you sincerely for your time and valuable feedback!

Hai, nama saya **Nur Fathiah Fatin Mohamad Fauzi**, dan saya ialah seorang pelajar pascasiswazah yang sedang menjalankan kajian ini untuk menilai dan menyusun strategi pembelajaran perbendaharaan kata Bahasa Inggeris.

Tinjauan ini adalah sebahagian daripada penyelidikan akademik saya yang menggunakan model **Bipolar Pythagorean Neutrosophic-Generalized Hamacher Heronian Mean (BPN-GHHM)** dan **Adaptive Ranking with Ideal Evaluation (ARIE)**. Tujuannya adalah untuk mendapatkan pandangan daripada pihak berkepentingan bagi meningkatkan kualiti dan kesesuaian strategi Bahasa Inggeris dengan keperluan akademik dan industri semasa.

Sila nilaikan setiap strategi (**Kaedah Tradisional, Pembelajaran Kontekstual, Kaedah Gamifikasi, Kecerdasan Buatan, Mnemonik, Multimedia, Rakan Sebaya**) berdasarkan pengalaman anda, menggunakan skala daripada **1 (Sangat Lemah)** hingga **7 (Sangat Baik)**. Maklum balas anda akan dirahsiakan dan digunakan hanya untuk tujuan penyelidikan dan penambahbaikan akademik.

Terima kasih yang ikhlas atas masa dan maklum balas berharga anda!

* Indicates required question

<https://docs.google.com/forms/d/1PEyJ5wqDlCyk2oJ7dlnU8pU83z685d1Ue4KGHncOg/edit>

1/20

1. Name>Nama *

2. Gender/Jantina *

Mark only one oval.

- ☐ Male/Lelaki
☐ Female/Perempuan

3. Age/Umur *

Mark only one oval.

- ☐ 23 & Below/Kebawah
☐ 24-27
☐ 28-32
☐ 33 & Above/Keatas

4. Academic/Pendidikan *

Mark only one oval.

- ☐ SPM
☐ Matriculation/Diploma/STPM
☐ Bachelor Degree/Ijazah Sarjana Muda
☐ Master Degree/Ijazah Sarjana
☐ PhD

5. Years of Learning English/Tahun Pembelajaran Bahasa Inggris *

Mark only one oval.

- ☐ Less than 5 year/Kurang dari 5 tahun
- ☐ 5-15 years
- ☐ More than 15 years/Lebih dari 5 tahun

Traditional method/ Kaedah tradisional

6. Does this method help you recall key concepts after several weeks or months? *

Adakah kaedah ini membantu anda mengingati konsep utama selepas beberapa minggu atau bulan?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

7. How regularly do you use or practice this method? *

Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

8. Does this method make you feel more excited or willing to learn? *

Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

9. Does it involve discussions, group work, games, or real-time feedback?
Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas nyata?

*

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

10. How easy is it to understand the lessons or materials using this learning method?

*

Sejauh manakah kemudahan untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

11. How useful is this learning method in improving your speaking and writing skills?

*

Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

12. How easy is it to access this learning method using your devices?
Adakah mudah untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

13. How would you rate the quality of feedback provided by this learning method? *
- Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?*

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

Contextual Learning Method/ Kaedah Pembelajaran Kontekstual

14. Does this method help you recall key concepts after several weeks or months? *
- Adakah kaedah ini membantu anda mengingat konsep utama selepas beberapa minggu atau bulan?*

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

15. How regularly do you use or practice this method *
Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

16. Does it make you feel more excited or willing to learn *
Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

17. Does it involve discussions, group work, games, or real-time feedback? *
Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas nyata?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

18. How easy is it to understand the lessons or materials using this learning method? *

Adakah mudah untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

19. How useful is this learning method in improving your speaking and writing skills? *

Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

20. How easy is it to access this learning method using your devices? *

Adakah mudah untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

21. How would you rate the quality of feedback provided by this learning method? *

Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

Gamification Method/ Kaedah Gamifikasi

22. Does this method help you recall key concepts after several weeks or months? *

Adakah kaedah ini membantu anda mengingati konsep utama selepas beberapa minggu atau bulan?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

23. How regularly do you use or practice this method *

Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

24. Does it make you feel more excited or willing to learn? *
Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

25. Does it involve discussions, group work, games, or real-time feedback? *
Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas nyata?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

26. How easy is it to understand the lessons or materials using this learning method? *
Adakah mudah untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

27. How useful is this learning method in improving your speaking and writing skills? *

Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

28. How easy is it to access this learning method using your devices? *

Adakah mudah untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

29. How would you rate the quality of feedback provided by this learning method? *

Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

Artificial Intelligence (Assisted Learning)/ Kecerdasan Buatan (Bantuan Pembelajaran)

30. Does this method help you recall key concepts after several weeks or months? *

Adakah kaedah ini membantu anda mengingati konsep utama selepas beberapa minggu atau bulan?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

31. How regularly do you use or practice this method? *

Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

32. Does it make you feel more excited or willing to learn? *

Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

33. Does it involve discussions, group work, games, or real-time feedback? *

Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas nyata?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

34. How easy is it to understand the lessons or materials using this learning method? *

Adakah mudah untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Verj	☐	☐	☐	☐	☐	☐	☐	Very Good

35. How useful is this learning method in improving your speaking and writing skills? *

Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Verj	☐	☐	☐	☐	☐	☐	☐	Very Good

36. How easy is it to access this learning method using your devices? *

Sejauh manakah kemudahan untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Verj	☐	☐	☐	☐	☐	☐	☐	Very Good

37. How would you rate the quality of feedback provided by this learning method? *
Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?

Mark only one oval.

1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	Very Good

Mnemonic Techniques/ Teknik Mnemonik

38. Does this method help you recall key concepts after several weeks or months? *
Adakah kaedah ini membantu anda mengingati konsep utama selepas beberapa minggu atau bulan?

Mark only one oval.

1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	Very Good

39. How regularly do you use or practice this method? *Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?* *

Mark only one oval.

1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	Very Good

40. Does it make you feel more excited or willing to learn? *Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?* *

Mark only one oval.

1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	Very Good

41. Does it involve discussions, group work, games, or real-time feedback? *
Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas masa nyata?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

42. How easy is it to understand the lessons or materials using this learning method? *
Adakah mudah untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

43. How useful is this learning method in improving your speaking and writing skills? *
Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

44. How easy is it to access this learning method using your devices? *
- Sejauh manakah kemudahan untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

45. How would you rate the quality of feedback provided by this learning method? *
- Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

Multimedia (Assisted Learning)/ Multimedia (Bantuan Pembelajaran)

46. Does it help you recall key concepts after several weeks or months? *
- Adakah kaedah ini membantu anda mengingat konsep utama selepas beberapa minggu atau bulan?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

47. How regularly do you use or practice this method? *
- Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

48. Does it make you feel more excited or willing to learn? *
- Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

49. Does it involve discussions, group work, games, or real-time feedback? *
- Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas nyata?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

50. How easy is it to understand the lessons or materials using this learning method? *

Adakah mudah untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

51. How useful is this learning method in improving your speaking and writing skills? *

Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

52. How easy is it to access this learning method using your devices? *

Sejauh manakah kemudahan untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

53. How would you rate the quality of feedback provided by this learning method? *
- Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

Peer (Assisted Learning)/ Rakan Sebaya (Bantuan Pembelajaran)

54. Does this method help you recall key concepts after several weeks or months? *
- Adakah kaedah ini membantu anda mengingati konsep utama selepas beberapa minggu atau bulan?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

55. How regularly do you use or practice this method? *
- Seberapa kerap anda menggunakan atau mempraktikkan kaedah ini?*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

56. Does it make you feel more excited or willing to learn?
Adakah kaedah ini membuatkan anda berasa lebih teruja atau lebih bersemangat untuk belajar?

*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

57. Does it involve discussions, group work, games, or real-time feedback?
Adakah ia melibatkan perbincangan, kerja berkumpulan, permainan, atau maklum balas nyata?

*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

58. How easy is it to understand the lessons or materials using this learning method?
Adakah mudah untuk memahami pelajaran atau bahan pembelajaran menggunakan kaedah ini?

*

Mark only one oval.

	1	2	3	4	5	6	7	
Very	☐	☐	☐	☐	☐	☐	☐	Very Good

59. How useful is this learning method in improving your speaking and writing skills? *

Sejauh manakah kaedah pembelajaran ini berguna dalam meningkatkan kemahiran bertutur dan menulis anda?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

60. How easy is it to access this learning method using your devices? *

Sejauh manakah kemudahan untuk mengakses kaedah pembelajaran ini menggunakan peranti anda?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

61. How would you rate the quality of feedback provided by this learning method? *

Bagaimanakah anda menilai kualiti maklum balas yang diberikan oleh kaedah pembelajaran ini?

Mark only one oval.

1 2 3 4 5 6 7

Very ☐ ☐ ☐ ☐ ☐ ☐ ☐ Very Good

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APPENDIX 2

Raw Data of English Vocabulary Learning Strategies

RESPONDENT	AGE	GENDER	ACADEMIC	YEARS LEARN ENGLISH	RETENTION														
					LONG TERM RECALL						FREQUENCY								
					TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	
1	33+	M	MA	MORE 15 YRS	7	7	5	4	6	4	6	6	7	3		5	6	5	6
2	33+	F	MA	MORE 15 YRS	7	7	6	5	6	4	5	7	7	5		5	5	4	5
3	33+	F	MA	MORE 15 YRS	7	6	6	6	7	6	7	7	6	7		6	7	6	7
4	33+	F	MA	MORE 15 YRS	6	5	6	5	6	6	7	5	5	6		6	6	6	6
5	28-32	F	MA	MORE 15 YRS	4	6	4	6	6	6	7	3	5	4		6	6	6	6
6	33+	F	MA	MORE 15 YRS	5	4	5	6	5	6	6	6	5	6		6	3	4	7
7	28-32	F	MA	MORE 15 YRS	6	5	6	5	6	6	6	5	6	7		6	6	6	5
8	33+	F	MA	MORE 15 YRS	1	6	5	6	6	7	6	5	6	5		6	6	7	6
9	33+	F	PHD	MORE 15 YRS	6	7	5	2	6	5	3	6	7	6		2	5	5	3
10	33+	F	PHD	MORE 15 YRS	6	5	5	3	5	3	5	6	5	5		6	5	6	6

LEARNING ENGAGEMENT														COGNITIVE LOAD							
MOTIVATION LEVEL							INTERACTIVITY							EASE OF UNDERSTANDING							
TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	
7	7	4	4	6	4	7	2	2	5	5	4	4	7	7	7	6	4	5	4	7	
7	7	7	5	6	5	5	6	7	7	6	6	5	5	6	7	7	6	6	5	5	
7	6	7	7	7	6	7	7	6	7	7	6	7	6	7	6	7	7	7	6	7	
5	3	7	7	5	7	7	3	3	6	7	5	7	7	5	6	7	7	6	7	7	
4	4	6	6	6	6	7	3	7	6	7	6	6	7	3	4	4	6	6	6	6	
6	6	3	6	6	5	6	6	5	7	6	2	5	6	4	6	5	5	5	3	6	
4	5	7	6	5	6	5	4	4	7	6	6	7	5	3	6	7	6	6	4	3	
1	6	6	6	6	7	6	1	6	7	6	6	7	6	1	6	6	6	6	7	6	
2	6	7	7	5	6	4	2	6	6	2	2	7	6	2	7	6	6	3	6	3	
4	5	6	6	6	6	5	4	4	4	5	5	5	4	6	5	5	5	5	5	5	

APPLICATION IN CONTEXT							ACCESSIBILITY							FEEDBACK MECHANISM						
USE IN SPEAKING & WRITING							AVAILABILITY ON DEVICES							QUALITY OF FEEDBACK						
TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER	TRAD.	CONT.	GAMI.	AI	MNE	MAL	PEER
7	7	3	2	6	4	7	6	7	4	4	5	5	4	6	6	4	4	4	4	6
6	6	6	5	6	6	6	7	7	7	6	5	6	5	4	5	7	6	5	5	6
6	6	6	7	7	6	7	7	7	7	7	7	6	7	6	7	7	7	7	6	7
5	6	6	7	7	5	6	6	7	7	7	6	7	6	5	4	6	6	6	5	7
3	4	5	6	5	6	7	1	5	5	6	5	6	6	7	7	4	6	5	6	7
4	4	4	4	5	4	6	5	6	6	6	4	6	2	2	2	5	6	3	6	6
5	6	7	7	5	5	5	2	2	7	7	6	7	3	7	4	7	6	5	5	5
2	6	6	6	6	7	6	1	6	7	6	6	7	6	2	6	6	6	6	7	6
3	7	3	3	3	5	3	2	6	7	7	3	7	3	5	7	3	2	3	5	6
6	6	4	4	4	4	6	6	5	6	6	6	6	6	5	5	4	4	5	4	6

APPENDIX 3

Implementation of English Vocabulary Learning Strategies using InnoRank-ARIE System

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
1	Alternative	C1	C2	C3	C4	C5	C6	C7	C8														
2	A1	0.3642	0.3758	0.3271	0.2961	0.3194	0.3313	0.3168	0.3366														
3	A2	0.3928	0.3967	0.3642	0.3454	0.4134	0.3999	0.3759	0.3576														
4	A3	0.3672	0.3728	0.3947	0.4163	0.4085	0.3498	0.4203	0.3648														
5	A4	0.3353	0.3685	0.4084	0.3726	0.4006	0.3490	0.4218	0.3608														
6	A5	0.4051	0.3780	0.3608	0.3378	0.3851	0.3671	0.3634	0.3415														
7	A6	0.3662	0.3745	0.3964	0.3965	0.3819	0.3709	0.4285	0.3678														
8	A7	0.3850	0.3815	0.3828	0.4041	0.3677	0.3850	0.3365	0.4240														

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Number

Conditional Formatting

Format as Table

Cell Styles

Insert

Delete

Format

Editing

Sort & Filter

Find & Select

Sensitivity

Add-ins

Analyze Data

Copilot

Create a PDF

Adobe Acrobat

D6

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
1	Criterion	Weight																					
2	C1	0.125																					
3	C2	0.125																					
4	C3	0.125																					
5	C4	0.125																					
6	C5	0.125																					
7	C6	0.125																					
8	C7	0.125																					
9	C8	0.125																					
10																							
11																							
12																							
13																							
14																							
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<

Decision_Matrix

Manual_Weights

Notes

+

Ready Accessibility: Good to go

InnoRank-ARIE - Streamlit

2nd-arie-method-mmayoe4ltuks5ywuuvysa.streamlit.app

Verify it's you

Configuration

Upload a decision matrix (rows = alternatives, columns = criteria).

Upload CSV/Excel

Drag and drop file here

Limit 200MB per file • CSV, ...

Browse files

InnoRank_ARIE...
11.3KB

☐ Use demo dataset

Weights

Equal weights

Sensitivity γ

0.50

Balance κ

0.50

InnoRank-ARIE - Adaptive Ranking with Ideal Evaluation

Intelligent MCDM Ranking System

Upload your decision matrix, set criterion types & weights, and compute ARIE rankings with sensitivity controls (γ , κ).

Input Decision Matrix

Alterna	C1	C2	C3	C4	C5	C6	C7	C8
A1	0.3642	0.3758	0.3271	0.2961	0.3194	0.3313	0.3168	0.3366
A2	0.3928	0.3967	0.3642	0.3454	0.4134	0.3999	0.3759	0.3576
A3	0.3672	0.3728	0.3947	0.4163	0.4085	0.3498	0.4203	0.3648
A4	0.3353	0.3685	0.4084	0.3726	0.4006	0.349	0.4218	0.3608
A5	0.4051	0.378	0.3608	0.3378	0.3851	0.3671	0.3634	0.3415
A6	0.3662	0.3745	0.3964	0.3965	0.3819	0.3709	0.4285	0.3678
A7	0.385	0.3815	0.3828	0.4041	0.3677	0.385	0.3365	0.424

Criterion Settings

C1 type

benefit

C2 type

benefit

C3 type

benefit

C4 type

benefit

C5 type

benefit

C6 type

benefit

C7 type

benefit

C8 type

benefit

InnoRank-ARIE - Streamlit

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Verify it's you

Configuration

Upload a decision matrix (rows = alternatives, columns = criteria).

Upload CSV/Excel

Drag and drop file here

Limit 200MB per file • CSV, ...

Browse files

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11.3KB

☐ Use demo dataset

Weights

Equal weights

Sensitivity γ

0.50

Balance κ

0.50

Run ARIE Ranking

ARIE computed successfully. Explore results below.

Normalized (R)

Weighted (V)

Similarities

Ranking

Alternative	C1	C2	C3	C4	C5	C6	C7	C8
A1	0.898972	0.947326	0.800944	0.711199	0.772567	0.828523	0.739504	0.793844
A2	0.969431	1.000000	0.891884	0.829833	1.000000	1.000000	0.877312	0.843578
A3	0.906417	0.939980	0.966512	1.000000	0.988216	0.874592	0.980847	0.860572
A4	0.827710	0.928934	1.000000	0.895105	0.968992	0.872643	0.984398	0.850931
A5	1.000000	0.952925	0.883579	0.811519	0.931434	0.918036	0.848259	0.805586
A6	0.903969	0.944087	0.970581	0.952607	0.923886	0.927529	1.000000	0.867587
A7	0.950373	0.961880	0.937306	0.970672	0.889352	0.962787	0.785341	1.000000

Download Results

Download Excel (all sheets)

Download Ranking (CSV)

Download Config (JSON)

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InnoRank-ARIE - Streamlit

2nd-arie-method-mmayoe4ltuks5ywuvytsa.streamlit.app

ARIE computed successfully. Explore results below.

Configuration

Upload a decision matrix (rows = alternatives, columns = criteria).

Upload CSV/Excel

Drag and drop file here
Limit 200MB per file • CSV, ...

Browse files

InnoRank_ARIE...
11.3KB

☐ Use demo dataset

Weights

Equal weights

Sensitivity y
0.50

Balance k
0.50

Normalized (R) **Weighted (V)** Similarities Ranking

Alternative	C1	C2	C3	C4	C5	C6	C7	C8
A1	0.112371	0.118416	0.100118	0.088900	0.096571	0.103565	0.092438	0.099230
A2	0.121179	0.125000	0.111485	0.103729	0.125000	0.125000	0.109664	0.105447
A3	0.113302	0.117498	0.120814	0.125000	0.123527	0.109324	0.122606	0.107571
A4	0.103464	0.116117	0.125000	0.111888	0.121124	0.109080	0.123050	0.106366
A5	0.125000	0.119116	0.110447	0.101440	0.116429	0.114754	0.106032	0.100698
A6	0.112996	0.118011	0.121323	0.119076	0.115486	0.115941	0.125000	0.108448
A7	0.118797	0.120236	0.117163	0.121334	0.111169	0.120348	0.098168	0.125000

$v_{\max}$

	C1	C2	C3	C4	C5	C6	C7	C8
0	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

$v_{\min}$

	C1	C2	C3	C4	C5	C6	C7	C8
0	0.1035	0.1161	0.1001	0.0889	0.0966	0.1036	0.0924	0.0992

Download Results

Download Excel (all sheets)

Download Ranking (CSV)

InnoRank-ARIE - Streamlit

2nd-arie-method-mmayoe4ltuks5ywuvytsa.streamlit.app

ARIE computed successfully. Explore results below.

Configuration

Upload a decision matrix (rows = alternatives, columns = criteria).

Upload CSV/Excel

Drag and drop file here
Limit 200MB per file • CSV, ...

Browse files

InnoRank_ARIE...
11.3KB

☐ Use demo dataset

Weights

Equal weights

Sensitivity y
0.50

Balance k
0.50

Normalized (R) Weighted (V) **Similarities** Ranking

Alternative	Sim_best	Sim_worst	RC
A1	7.199842	7.949792	0.475249
A2	7.695059	7.438613	0.508473
A3	7.752033	7.389595	0.511968
A4	7.652853	7.486235	0.505503
A5	7.558815	7.572475	0.499549
A6	7.739327	7.397372	0.511296
A7	7.719469	7.417376	0.509979

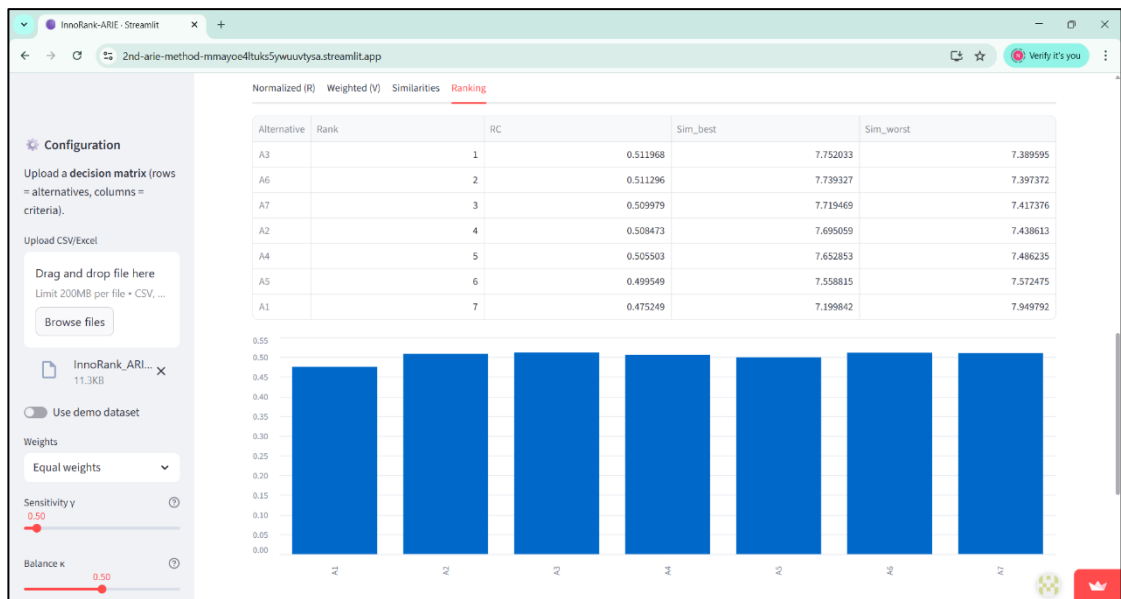
Download Results

Download Excel (all sheets)

Download Ranking (CSV)

Download Config (JSON)

> Method Notes (Formulas)



APPENDIX 4

Bronze Award Certificate from the Invention, Innovation & Design Exposition (iidx 2025) for the InnoRank-ARIE Decision Support System



iindex2025

Invention, Innovation &
Design Exposition



This certificate is presented to

NUR FATHIAH FATIN BINTI MOHAMAD FAUZI
ZAHARI MAD RODZI
NORZIEHA MUSTAPHA
AHMAD BAZLI KHAIRUDDIN
MUHAMMAD MUKHLIS KAMARUL ZAMAN
WAN SYAIDATUL IZZATI WAN ABDUL RAHMAN

sincere appreciation for your support during

INVENTION, INNOVATION & DESIGN EXPOSITION (iindex) 2025

28-30 October 2025
Dewan Agung Tuanku Canselor (DATC)
UiTM Shah Alam, Selangor, Malaysia

**Title: INNORANK-ARIE: INTELLIGENT
RANKING SYSTEM FOR MULTI-
CRITERIA DECISION MAKING**

PROF. TS. DR. NORAZAH ABD RAHMAN
Deputy Vice-Chancellor (Research & Innovation)
Universiti Teknologi MARA

Organised by
Department of Research & Innovation, UiTM

Certificate of
PARTICIPATION



AUTHOR'S PROFILE



Nur Fathiah Fatin Mohamad holds Bachelor of Science (Hons.) (Mathematics) in 2024 at Universiti Teknologi MARA (UiTM), Negeri Sembilan Branch, Seremban 3 Campus. Her research interests centre on the area of MCDM, which began during her final year project and continued to develop throughout her internship experience. This growing passion motivated her to pursue a Master of Science in Applied Science (Mathematics) at UiTM Seremban. She has presented research papers at the International Conference on Artificial Intelligence and Data Sciences (AiDAS) in both 2024 and 2025, further strengthening her academic and research foundation in the field.

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- Mohamad Fauzi, N. F. F., Md Rodzi, Z., & Mustapha, N. (2025). Bipolar Pythagorean neutrosophic Hamacher Heronian mean operators: a framework in MCDM. *Mathematical Sciences and Informatics Journal (MIJ)*, 6(1), 13-25. **[Indexed by MyCite]**
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Mohamad Fauzi, N. F. F., Md Rodzi, Z., & Mustapha, N. (2026). An Adaptive Ranking Model under Bipolar Pythagorean Neutrosophic Environment for Teaching Quality Evaluation. *Jurnal Resti (Rekayasa Sistem dan Teknologi Informasi)*. **[Indexed by Scopus]**

Mohamad Fauzi, N. F. F., Md Rodzi, Z., & Mustapha, N. (2026). An Intelligent Fuzzy MCDM Framework Using BPNS Aggregation and ARIE for Evaluating Vocabulary Learning Strategies. *International Conference on Intelligent and Fuzzy Systems (INFUS)*. **[Indexed by Scopus]**