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COEFFICIENT ESTIMATES AND DETERMINANT INEQUALITIES FOR TILTED STARLIKE AND CONVEX COMBINATION FUNCTIONS ASSOCIATED WITH THE EPICYCLOID DOMAIN

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CONFIRMATION BY PANEL OF EXAMINERS

I certify that a Panel of Examiners has met on 8 April 2026 to conduct the final examination of Nur Athirah Hani binti Senin on her Masters of Science thesis entitled "Coefficient Estimates and Determinant Inequalities for Tilted Starlike and Convex Combination Functions Associated with the Epicycloid Domain" in accordance with Universiti Teknologi MARA Act 1976 (Akta 173). The Panel of Examiners recommends that the student be awarded the relevant degree. The Panel of Examiners was as follows:

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ABSTRACT

This thesis is set within the framework of Geometric Function Theory (GFT), a branch of complex analysis concerned with the geometric properties of analytic functions. In particular, it focuses on the study of subclasses of normalised analytic univalent functions in the open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$, where each function f has the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n .$$

The general class of normalised analytic univalent functions is denoted by A. In particular, this research considers two subclasses associated with the epicycloid domain. The first subclass is the tilted starlike subclass, associated with $n-1$ cusps epicycloid denoted as $S^*_{T,n-1}(\alpha)$, defined by the subordination condition

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-i\sin\alpha\right)\frac{1}{\cos\alpha}\prec 1+\frac{n}{n+1}z+\frac{1}{n+1}z^n,$$

where $n\geq 2$ (natural number) and $|\alpha|<\frac{\pi}{2}$. The second subclass is a convex combination of starlike and convex functions associated with the four-leaf-shaped epicycloid domain, denoted as $\mathrm{M}_{\gamma,4L}$. It is defined by subordination condition

$$(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\prec 1+\frac{5}{6}z+\frac{1}{6}z^5,$$

where $0\leq\gamma\leq 1$. This thesis establishes key results including coefficient estimates and coefficient bounds for functions belonging to both classes. The coefficient inequalities for the class $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$ are also discussed which comprise of the upper bounds for the Fekete–Szegö functional, $|a_3-\mu a_2^2|$ where μ can be complex or real numbers and the upper bounds for the second Hankel determinant, $|a_2a_4-a_3^2|$. In addition, this study establishes upper bounds for the Toeplitz determinants $T_2(2)$, $T_3(1)$ and $T_3(2)$ corresponding to functions in both subclasses. These findings not only extend existing results in the literature but also offer new insights into the geometric characteristics and coefficient structure of analytic functions associated with the epicycloid domains.

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Finally, this thesis is wholeheartedly dedicated to my family, friends, and loved ones who have always believed in me. This achievement is as much yours as it is mine.

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LIST OF SYMBOLS

Symbols

$\mathbb{C}$	Set of Complex Numbers
E	Open Unit Disk $\{z \in \mathbb{C} : \|z\| < 1\}$
$H(E)$	Collection of all Analytic Functions Defined in the Domain of E
D	Domain of the Univalence
A	Class of Analytic Functions in the Unit Disk E of the Form $f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n$
S	Subclass of A which are Univalent in E
S^*	The Class of Starlike Functions
C	The Class of Convex Functions
P	Collection of Analytic Functions $p \in P$ with Positive Real Part
$\omega(z)$	Schwarz Functions with $\omega(0) = 0$ and $\|\omega(z) < 1\|$
$S^*_{T,n-1}(\alpha)$	Tilted Starlike Functions Associated with Epicycloid Domain
$\mathrm{M}_{\gamma,4L}$	Convex Combinations of Starlike and Convex Functions Associated with the Four-Leaf Epicycloid Domain
α	Tilt Angle such that $\|\alpha\| < \frac{\pi}{2}$
γ	Convex Combination Parameter, $0 \le \gamma \le 1$
$\prec$	Subordination

LIST OF ABBREVIATIONS

Abbreviations

GFT	Geometric Function Theory

CHAPTER 1
INTRODUCTION

1.1 Introduction

The field of complex analysis involves the study of sets and functions within the complex plane, with particular emphasis on analytic functions of a complex variable and their associated conformal mappings. A significant branch of this field is Geometric Function Theory (GFT), which focuses on the geometric behavior and structural properties of analytic functions. One of the most important function classes in GFT is the class of normalised analytic univalent functions, commonly denoted by S, consisting of functions that are analytic and univalent in the open unit disk.

A cornerstone of this area is the Riemann Mapping Theorem, which guarantees the existence of a conformal map from any simply connected region other than the entire complex plane onto the unit disk. The study of univalent functions gained major traction following Koebe's seminal work in 1907. Later, the famous 'Bieberbach Conjecture' stated that the coefficients of functions in class S, the inequality $|a_n| \le n$ holds for all $n \ge 2$. This conjecture stimulated considerable mathematical inquiry for decades until it was ultimately proven by Louis De Branges in 1985 (De Branges, 1985).

A deeper understanding of such coefficient problems has been achieved through the introduction and study of many subclasses of S, including the subclasses of starlike, convex, close-to-convex, and typically real functions (Goodman, 1983; Pommerenke, 1975). These subclasses facilitate the derivation of sharper estimates and reveal further geometric and analytic properties of univalent functions.

This research focuses on a category of analytical univalent functions within the complex plane. Let $\mathbb{C}$ represents the set of complex numbers. A function f of a complex variable z is said to be differentiable at a point $z_0 \in \mathbb{C}$ if its derivative exists at z_0. The derivative of f at z_0 is defined as follows:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{1.1}$$

The function f is considered analytic at z_0 if it is differentiable at every point in the neighborhood of z_0. The terms analytic and holomorphic are used interchangeably in this context. An analytic function f is called univalent within a domain D if it is injective, meaning that it never takes the same value twice that is, if $f(z_1) = f(z_2)$, then $z_1 = z_2$, $(z_1, z_2 \in D)$. In other words, $f(z)$ is univalent, or "schlicht" (the German term for simple), if it establishes a one-to-one correspondence between points in the domain and their images.

The domain D of univalent functions is not constrained and can take any various shape. Refining the focus of the study involves replacing the general domain D with a specific, well-defined domain: the unit disk $E = \{z : |z| < 1\}$. This choice not only streamlines the computations but also results in more compact and manageable formulas. The univalence of function f in E is easy to see on geometric grounds (Goodman, 1983). Figure 1.1 provides a visual representation of the unit disk, E.

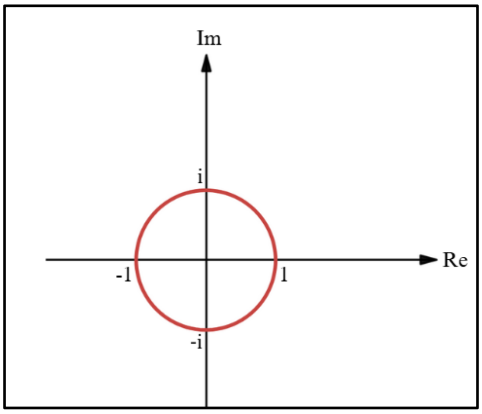

Figure 1.1 Region of the Unit Disk, E.

1.2 The Class of Functions with Positive Real Part

Consider the family $H(E)$ consisting of analytical functions within the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. Let A denote the class of analytic functions $f \in H(E)$ in the open unit disk E at the normalised form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

where $f(0) = 0$ and $f'(0) = 1$. If $f(z)$ is univalent and has the form (1.1), it is called a normalised univalent function and is denoted by S.

The univalence of functions in E is an important property, and several subclasses of analytic functions in the unit disk have been studied extensively. One such subclass is the Carathéodory class P, which consists of analytic functions in E that satisfy $\operatorname{Re} p(z) > 0$ for all z in E. These functions are often referred to as functions with a positive real part in E, and they exhibit significant geometric and analytic properties. More specifically, let P be a function in the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{1.3}$$

This condition ensures that the function is mapped to a region where the real part of $p(z)$ remains strictly positive. In 1907, Carathéodory (as referenced by MacGregor, 1962) established fundamental properties of functions with a positive real part. Let $f(z) \in P$, then the following results hold:

i. The coefficient c_1, c_2,... of f satisfy $|c_n| \le 2$ and for $0 < |z| = r < 1$.
ii. The function satisfies the inequality $\frac{1-r}{1+r} \le f(z) \le \frac{1+r}{1-r}$.
iii. The real part obeys $\operatorname{Re} f(z) \ge \frac{1-r}{1+r}$.
iv. The derivative satisfies $|f'(z)| \le \frac{2}{(1-r)^2}$.

1.3 The Classes of Univalent Functions

Building upon foundational insights in univalent function theory, particular attention is given to specific subclasses that exhibit rich geometric structures. Among these, the classes of convex and starlike functions are of significant interest due to their pivotal roles in GFT. The definitions and analytic characterisations of these classes are primarily based on the work of Duren (1983) and Goodman (1983). This research primarily focuses on subclasses of analytic functions that are starlike and convex in the open unit disk E.

1.3.1 The Class of Convex Functions

A domain $D \subset \mathbb{C}$ is said to be convex if, for any two points in D the straight-line segment connecting them lies entirely within D. A function f is said to be convex if it is univalent and maps the open unit disk E onto a convex domain. The class of convex functions is denoted by C. In particular, the normalised class of convex functions consists of the set of all $f \in S$ for which $f(E)$ is convex. Analytically, a function $f \in S$ is convex if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0, \quad z \in E. \tag{1.4}$$

The concept of convex functions was generalised by Ma and Minda (1994) through the use of the subordination principle, leading to a broader framework for studying geometric properties of analytic functions. They considered a function $\varphi(z)$ analytic in E, maps E onto a convex domain, possessing positive real part, symmetric about the real axis, and normalised by $\varphi(0)=1$ and $\varphi'(0)>1$. The associated class of Ma–Minda convex functions is then defined as

$$C(\varphi):=\left\{f \in A : 1+\frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\}, \tag{1.5}$$

where A denote the class of functions analytic in E with the standard normalisation $f(0)=0$ and $f'(0)=1$, and the symbols "$\prec$" denotes subordination.

1.3.2 The Class of Starlike Functions

A domain $D \subset \mathbb{C}$ is said to be starlike with respect to $z=0$ if $0 \in D$ and every line segment connecting the origin to any point in D lies entirely within D. A function f is called starlike if it is a conformal mapping from E onto a domain that is starlike with respect to the origin. The collection of all such functions is denoted by S^*. A function $f \in S$ belongs to S^* if and only if the following condition holds:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in E. \tag{1.6}$$

A significant contribution in this area was made by Ma and Minda (1994), who introduced a unified framework for these geometric function classes using the principle of subordination. In their formulation, the function $\varphi(z)$ is assumed to be analytic in the open unit disk E, possesses a positive real part, and maps E onto a domain that is starlike with respect to 1, symmetric with respect to the real axis, and satisfies the normalisation conditions $\varphi(0)=1$ and $\varphi'(0)>1$. Within this framework, they defined the generalised class of starlike functions as

$$S^*(\varphi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}. \tag{1.7}$$

This formulation extends many well-known subclasses of starlike functions and provides a unified approach for studying their analytic and geometric properties via subordination.

1.4 The Concept of Subordination

The concept of subordination was formalised by Miller and Mocanu (1981). Given two analytic functions $f(z)$ and $g(z)$ in E. The function $f(z)$ is subordinate to $g(z)$ (written as $f(z) \prec g(z)$), if there exists a function $\omega \in H$, that is analytic in

E with $\omega(0)=0$ and $|\omega(z)<1|$ for all $z\in E$, such that $f(z)=g(\omega(z))$. Moreover, if $g(z)$ is univalent in E, then subordination is equivalent to the condition that

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0),\ f(E)\subset g(E). \tag{1.8}$$

1.5 The Class Functions Associated with the Epicycloid Domain

Gandhi et al. (2022) introduced a generalised subclass of starlike functions, denoted $S^*_{n-1,\mathrm{L}}$, defined via subordination to a univalent function $\varphi_{n-1,\mathrm{L}}$ that maps E onto a domain bounded by an epicycloid with $n-1$ cusps.

Delving deeper into the topic, an epicycloid is defined as a plane curve created by following the path of a specified point on the circumference of a circle of radius b that rolls without slipping around a fixed circle of radius a (Gandhi et al., 2022). The parametric equation of an epicycloid is

$$\begin{aligned} x(t)&=m\cos t-b\cos\left(\frac{mt}{b}\right)\\ y(t)&=m\sin t-b\sin\left(\frac{mt}{b}\right), \end{aligned} \tag{1.9}$$

where $-\pi\le t\le\pi$ and $m=a+b$. If $\frac{m}{b}$ is an integer, then the curve has $\frac{m}{b}-1$ cusps.

Some of the epicycloid have unique names such as cardioid, where $a=b$ and only has one cusp. Another example is nephroid with two cusps for $a=2b$. Moreover, further investigation in this study is devoted to the epicycloid with $n-1$ cusps.

Gandhi et al. (2022) have considered a more general domain whose boundary has the following parametric form for $n\ge 2$ (natural number):

$$\begin{aligned} x(t)&=1+\frac{n}{n+1}\cos t+\frac{1}{n+1}\cos(nt)\\ y(t)&=\frac{n}{n+1}\sin t+\frac{1}{n+1}\sin(nt). \end{aligned} \tag{1.10}$$

The curve (1.10) represents a rotated and translated epicycloid with $n-1$ cusps for $a=\dfrac{n-1}{n+1}$ and $b=\dfrac{1}{n-1}$. It is an algebraic curve of order $2n$. It can be easily seen that $x'(t_k)=0$ and $y'(t_k)=0$ for $t_k=\dfrac{(2k-1)\pi}{n-1}$, where $k=1,2,\cdots,\dfrac{n-2}{2}$. Moreover, $x''(t_k)$ and $y''(t_k)$ are not zero together. By the definition of cusps, (1.10) has cusps at the points t_k. The general function of epicycloid with $n-1$ cusps was introduced by Gandhi et al. (2022) in the form

$$\varphi_{n-1,\mathrm{L}}(z)=1+\frac{n}{n+1}z+\frac{1}{n+1}z^n,\ \ z\in E. \tag{1.11}$$

that maps unit circle to this curve and the unit disk onto the region bounded by the curve. This function satisfies all the conditions established by Ma and Minda (1994) for the unified class of starlike and convex functions. Specifically, the function $\varphi_{n-1,\mathrm{L}}$ is univalent with a positive real part, the image domain $\varphi_{n-1,\mathrm{L}}(E)$ is symmetric with respect to the real axis, starlike with respect to $\varphi_{n-1,\mathrm{L}}(0)=1$ and $\varphi'_{n-1,\mathrm{L}}(0)>0$.

Notably, for $n\geq 2$, it maps the unit disk E onto a domain bounded by an epicycloid with $n-1$ cusps. Using the function $\varphi_{n-1,\mathrm{L}}(z)$, a subclass of starlike functions $S^*_{n-1,\mathrm{L}}\,(n\geq 2)$ is written in the following

$$S^*_{n-1,\mathrm{L}}:=\left\{f\in S:\frac{zf'(z)}{f(z)}\prec 1+\frac{n}{n+1}z+\frac{1}{n+1}z^n,\ \ z\in E\right\}. \tag{1.12}$$

Figures 1.2 and 1.3 illustrate the boundary curves corresponding to the four-leaf and three-leaf epicycloid shapes, respectively. By substituting $n=5$ (Shi et al., 2022) and $n=4$ (Gandhi, 2018), it is observed that the associated starlike function maps the unit disk onto a domain lying entirely in the right half-plane. This geometric property confirms that the real part of the quantifier remains positive, satisfying the analytic requirements for the class.

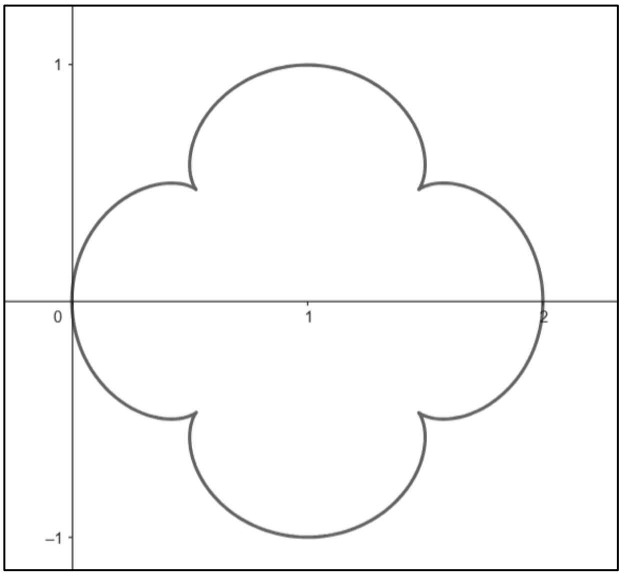

Figure 1.2 The Boundary Curve of $S^*_{n-1,\mathrm{L}}$ for $n=5$ that Lies to the Right of the Half Plane where $\varphi_{5,\mathrm{L}}(z)=1+\frac{5}{6}z+\frac{1}{6}z^5$.

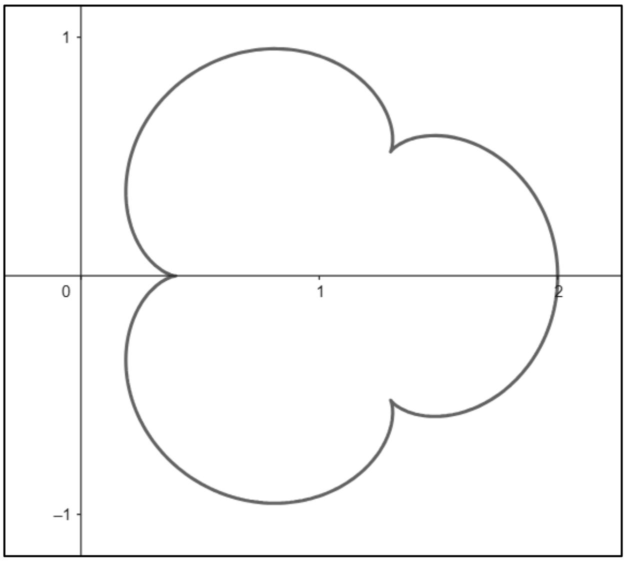

Figure 1.3 The Boundary Curve of $S^*_{n-1,\mathrm{L}}$ for $n=4$ that Lies to the Right of the Half Plane where $\varphi_{4,\mathrm{L}}(z)=1+\frac{4}{5}z+\frac{1}{5}z^4$.

1.6 Research Motivation

Building upon the geometric framework of the epicycloid domain established by Gandhi et al. (2022), who introduced a generalised subclass of starlike functions $S^*_{n-1,\mathrm{L}}$, defined using subordination to a mapping onto domains bounded by epicycloid with $n-1$ cusps, this study extends the concept by incorporating a tilted function into the analytic setting. Specifically, a multiplicative factor of the form $e^{i\alpha}$, where $|\alpha| < \frac{\pi}{2}$ is introduced to rotate the starlike mapping direction within the complex plane. This modification gives rise to a tilted subclass of generalised starlike functions associated with the epicycloid domain, enabling the analysis of geometric behavior under directional rotation. The formulation draws upon the foundation of geometric function theory and subordination principles, which have been central to the study of univalent functions in the open unit disk E.

In a complementary direction, Wahid et al. (2015) proposed the concept of tilted starlike functions with respect to conjugate points, incorporating a tilted factor into the subordination condition. This modification adjusts the orientation of starlikeness within the complex plane and allows for the analysis of functions relative to a rotated direction, an approach particularly useful in capturing more general geometric behavior.

Motivated by these developments, this research defines and investigates a new subclass of tilted starlike functions associated with the epicycloid domain. The study merges the tilted approach introduced by Wahid et al. (2015) with the geometric structure developed in the framework of Gandhi et al. (2022). This work aims to establish foundational results for the proposed class, including coefficient bounds, Fekete–Szegö inequalities, second Hankel determinant and Toeplitz determinants.

More recently, Shi et al. (2022) investigated the properties of functions belonging to the class introduced by Gandhi et al. (2022) in a specific case, such as $n=5$, denoted as $S^*_{4,\mathrm{L}}$. They derived sharp coefficient estimates and related inequalities. These results highlight the analytical richness of function classes defined on epicycloid domains and provide motivation for further investigation in this area.

Consequently, in addition to the subclass of tilted starlike functions, this study defines a new subclass, denoted by $\mathrm{M}_{\gamma,4L}$, that represents a convex combination of starlike and convex functions associated with a four-leaf-shaped epicycloid domain, as considered by Shi et al. (2022). This inclusion is justified by the fact that the four-leaf epicycloid domain maintains univalence while enabling the simultaneous investigation of starlikeness and convexity within a unified analytic framework. Such a formulation facilitates the derivation of refined coefficient bounds and functional inequalities, thereby extending the GFT in epicycloid domains.

1.7 Problem Statement

Gandhi et al. (2022) explored a specific subfamily of starlike functions, denoted as $S^{*}_{n-1,\mathrm{L}}$ linked with domains bounded by epicycloids possessing $n-1$ cusps. Further, Shi et al. (2022) studied the class function $S^{*}_{4,\mathrm{L}}$ associated with a domain shaped like a four-leaf clover, they derived coefficient inequalities and determined the sharp upper bounds for the second and third Hankel determinants.

Building upon the pioneering work of Gandhi, this study defines a new subclass of tilted starlike functions associated with the epicycloid domain by utilising the class $S^{*}_{n-1,\mathrm{L}}$, incorporating a generalised approach through the inclusion of a tilted factor $e^{i\alpha}$ where $|\alpha|<\frac{\pi}{2}$. These generalisations aim to extend the geometric framework of analytic function theory by incorporating the directional effects introduced by the tilt within the unit disk. The inclusion of tilted factor is inspired by the work of Wahid et al. (2015) where they introduced a new subclass of titled starlike functions concerning the conjugate points.

The notable findings by Shi et al. (2022) provide further motivation and confidence in examining the analytical properties of function classes associated with epicycloid-shaped domains. Building upon this, the present study defines a new subclass of analytic functions, denoted by $\mathrm{M}_{\gamma,4L}$, which represents a convex combination of starlike and convex functions associated with a four-leaf-shaped epicycloid domain. This subclass is formulated using the function class $S^{*}_{4,\mathrm{L}}$, with the parameter $0\le\gamma\le 1$, that merges the geometric characteristics of both starlikeness and convexity.

This research addresses a gap in the existing literature regarding the generalisation of starlike functions within the specific domain shape of the epicycloid by incorporating a tilted factor, and a structural combination of classical function classes. These dual generalisations introduce substantial analytical complexity. The presence of the tilted factor alters the orientation of the mapping in the complex plane, while the hybrid nature of the subclass $\mathrm{M}_{\gamma,4L}$ introduces non-trivial interactions between the starlike and convex differential operators.

However, exploring properties such as coefficient bounds, coefficient inequalities including the Fekete–Szegö functional and the second Hankel determinant as well as the Toeplitz determinants becomes particularly challenging. The generalised structure of these classes, influenced by the rotation introduced through the factor $e^{i\alpha}$, adds layers of complexity, which modifies the standard mapping into a rotated version. This directional tilt requires more refined analytical techniques and a deeper examination of the underlying geometry to derive meaningful results and accurate estimates. Simultaneously, the newly introduced subclass, $\mathrm{M}_{\gamma,4L}$ poses its own challenges due to the combined nature of its defining operator. The interaction between starlikeness and convexity, combined with the influence of the epicycloid mapping, adds further layers of complexity in deriving sharp coefficient estimates and functional inequalities. Together, these challenges require careful formulation, deeper geometric insights, and more advanced analytical tools to investigate the fundamental properties of the proposed function classes.

The exploration of these dual generalisations is essential, as neglecting the interplay between geometric deformation and operator interaction would result in an incomplete understanding of analytic function behaviour. This limitation restricts further theoretical development within GFT and may hinder the discovery of more general function classes with richer structural and geometric properties. Therefore, it is necessary to investigate these subclasses in order to advance the theory and establish a more comprehensive framework for analysing analytic functions associated with complex geometric domains.

1.8 Research Objectives

The objectives of this research are:

a) To define a new subclass of tilted starlike functions associated with epicycloid domain, denoted by $S^{*}_{T,n-1}(\alpha)$, where $f \in A$ and satisfies the subordination condition

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^{n}, \tag{1.13}$$

where $n \geq 2$ and $|\alpha| < \dfrac{\pi}{2}$.

b) To define a new subclass of a convex combination of starlike and convex functions associated with a four-leaf-shaped epicycloid domain, denoted by $\mathrm{M}_{\gamma,4L}$, where $f \in A$ and satisfies the subordination condition

$$(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + \frac{5}{6}z + \frac{1}{6}z^5, \tag{1.14}$$

where $0 \leq \gamma \leq 1$.

c) To establish coefficient bounds and derive coefficient inequalities for functions in the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$, including Fekete–Szegö functional and second Hankel determinant.

d) To compute the Toeplitz determinant, $T_2(2)$, $T_3(1)$ and $T_3(2)$ for the functions belonging to the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$.

1.9 Research Questions

The present study is guided by the following research questions, which are formulated to address the primary objectives of the investigation:

i) How can a new subclass of tilted starlike functions associated with the epicycloid domain be defined using subordination?

ii) How can a convex combination of starlike and convex functions be formulated through subordination involving a four-leaf-shaped epicycloid mapping?

iii) What analytical properties can be established for the newly defined subclasses?

iv) What further functional results can be computed for these subclasses based on their analytic form?

1.10 Significance of Study

This study contributes to the advancement of GFT by defining a novel subclass of starlike functions associated with the epicycloid-shaped domains, incorporating a tilted factor into the subordination framework. Although epicycloid domains have been explored in the context of starlike functions, the incorporation of a tilted factor into such domains remains largely unexamined. Motivated by the work of Wahid et al. (2015), this study introduces a new geometric perspective by combining the concept of tilted starlike and convex functions with the function-theoretic framework of epicycloid-shaped domains.

In addition to the tilted subclass, this study also defines a new analytic class formed by a convex combination of starlike and convex functions associated with a four-leaf-shaped epicycloid domain. This hybrid subclass extends the existing structure by enabling the simultaneous investigation of starlike and convex properties within a unified framework. The inclusion of this class addresses another important gap in the literature, particularly concerning the analytic behavior of functions mapped onto epicycloid regions.

The significance of this research lies in its generalisation of existing function classes and its comprehensive examination of key analytic properties, including coefficient bounds, Fekete–Szegö inequalities, and higher-order determinants such as the second Hankel and various Toeplitz determinants. The results obtained contribute to a deeper understanding of GFT in specialised domains and extend the prior work of Gandhi et al. (2022) and Shi et al. (2022), particularly for domains characterised by epicycloids with $n-1$ cusps.

Moreover, the findings offer valuable insights not only for pure mathematics but also for applied fields such as engineering, physics, and computer graphics, where complex-valued functions and curved geometries are used in modelling and simulation. The analytical tools and bounds developed in this work could potentially enhance mathematical modelling in these domains. This study not only enriches mathematical studies but also contributes to advancements in theory and practical applications.

Finally, this study opens several avenues for future research, including the investigation of logarithmic coefficients, higher-order Hankel determinants, and further geometric characterisations of tilted function classes. Overall, this work establishes a solid foundation for deeper exploration of starlike functions in non-classical domains, highlighting both its originality and broader mathematical significance.

1.11 Limitation

This study is theoretical in nature and is limited to the class of analytic functions defined on the open unit disk E. Specifically, it is limited to functions that are subordinate to a mapping function that transforms the unit disk into a region bounded by an epicycloid curve with $n-1$ cusps. The primary focus lies on two subclasses: $S^*_{T,n-1}(\alpha)$, which incorporates a tilted factor, $e^{i\alpha}$ and $\mathrm{M}_{\gamma,4L}$, a class formed by a convex combination of starlike and convex functions associated with a four-leaf-shaped epicycloid domain.

The inclusion of the tilted factor introduces additional angular complexity and rotational symmetry, which makes the derivation of results as coefficient estimates, the Fekete–Szegö inequality, the second Hankel determinant and Toeplitz determinants more analytically challenging. While the hybrid class of $\mathrm{M}_{\gamma,4L}$ does not involve directional rotation, the combination of two distinct geometric behaviors under a non-circular epicycloid mapping still results in lengthy and intricate calculations. Compared to the tilted class, the analysis for this subclass is relatively more straightforward, though it remains technically involved due to the structure of the defining operator and the complexity of the image domain.

Moreover, several classical lemmas and techniques in GFT, which are often formulated for real-valued functions or standard univalent classes, do not directly apply to the tilted starlike subclass. Since this study focuses on complex-valued functions within the tilted framework, lemmas designed for real variables or standard domains require careful adaptation.

Additionally, this study focuses on a selected set of properties, including coefficient bounds, coefficient inequalities, and Toeplitz determinants. Other notable geometric properties, such as the growth and distortion theorems, convolution structures, covering theorems, and logarithmic coefficients, are not investigated in this work and are proposed as directions for future research.

Although analytical results are the focus, graphical representations are provided only for established classes, and the newly defined subclass is not visualized. This may restrict the geometric intuition for the new class, especially for readers interested in geometric interpretations.

Finally, results are developed only for integer values $n \geq 2$ in the case of the tilted subclass $S^*_{T,n-1}(\alpha)$ and the analytical framework depends heavily on the constraint $|\alpha| < \frac{\pi}{2}$, which may restrict broader generalisations or applications. As for the subclass of $\mathrm{M}_{\gamma,4L}$, the parameter γ is confined to the interval $0 \leq \gamma \leq 1$, which has been fixed from the outset to maintain a consistent balance between starlikeness and convexity. These constraints, while essential for maintaining analytical tractability, also set the limits within which the study's findings are valid.

1.12 Assumption

In this study, all functions considered are assumed to be analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$, and belong to the class A of normalised analytic functions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, where $f(0) = 0$ and $f'(0) = 1$.

The new subclass defined in this research is characterised by a subordination condition involving a tilted factor, specifically, $\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} \prec \varphi_{n-1,L}(z)$ where $\varphi_{n-1,L}(z) = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n$ with $|\alpha| < \frac{\pi}{2}$ and $n \geq 2$. Next, another new subclass is defined using subordination principle as $(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi_{4,L}(z)$, where $\varphi_{4,L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ and $0 \leq \gamma \leq 1$.

The functions $\varphi_{n-1,L}(z)$ and $\varphi_{4,L}(z)$ are assumed to be univalent has a positive real part, mapping the unit disk E onto a domain bounded by an epicycloid with $n-1$ cusps. This setup aligns with the conditions established by Ma and Minda for the unified class of starlike and convex functions.

Furthermore, the sharpness of the derived bounds is not assumed unless explicitly demonstrated. However, in certain special cases, such as when $\alpha = 0$ for the tilted parameter, the results reduce to known outcomes from established subclasses, thereby validating the generalisation introduced in this study. Similarly, for the convex combination subclass associated with the four-leaf-shaped epicycloid domain, the parameter $\gamma \in [0,1]$ governs the balance between starlikeness and convexity. In particular, when $\gamma = 0$, the class reduces to the corresponding starlike subclass, and when $\gamma = 1$, it reduces to the convex subclass.

These limiting cases help verify the correctness of the proposed subclasses and support its interpretation as a natural generalisation that smoothly transitions between the well-known starlike and convex function classes.

1.13 Ethical Committee

This research is theoretical in nature and does not involve any human participants, animal testing, or sensitive data. Therefore, ethical approval was not required. All mathematical derivations and references to existing literature have been properly cited and acknowledged to maintain academic integrity.

1.14 Thesis Scope

The focus of this research is to define a new subclass of tilted starlike functions associated with the epicycloid domain, $S^*_{T,n-1}(\alpha)$ and characterised by subordination to epicycloid mapping function with $n-1$ cusps for $n \geq 2$. This study incorporates a tilted factor $e^{i\alpha}$, with $|\alpha| < \frac{\pi}{2}$ into the subordination condition, which introduces a generalised structure to the classical class.

In addition to the tilted subclass, this research also defines a second subclass formed by a convex combination of starlike and convex functions, $\mathrm{M}_{\gamma,4L}$ associated with a four-leaf-shaped epicycloid domain. This class is defined using the parameter $\gamma \in [0,1]$, serves as a unifying framework that interpolates between starlike and convex behaviors, and is considered under the specific case $n = 5$.

This study confines to the unit disk E and the class of normalised analytic functions A. Within this framework, several geometric function properties are investigated, including coefficient bounds, Fekete–Szegö inequalities, second Hankel determinants, and Toeplitz determinants. The analysis is primarily theoretical and focuses on deriving estimates, establishing functional inequalities, and verifying consistency with known results when the tilted factor becomes neutral where $\alpha = 0$ and when the combination class reduces to classical starlike ($\gamma = 0$) or convex ($\gamma = 1$) subclasses.

While grounded in complex analysis and GFT, this study does not extend to numerical simulations, experimental validations, or interdisciplinary applications in fields like engineering or physics.

1.15 Thesis Outline

This thesis begins with Chapter 1, which provides an introduction to the background and motivation of the study, followed by the problem statement, research objectives, and an outline of the scope and limitations of the research. Chapter 2 presents a comprehensive survey of the literature relevant to the analytic function classes studied in this thesis. It begins with key lemmas and foundational results in univalent function theory, establishing the necessary mathematical tools and principles. The review then explores various analytic function classes defined via non-circular geometric domains, followed by a detailed discussion on subclasses associated with epicycloid domains. Additional attention is given to tilted functions, which incorporate rotational parameters, and to convex-combination classes that blend starlike and convex behaviors. Chapter 3 focuses on the preliminaries, defining analytic functions, subordination, and epicycloid mapping functions, as well as defining the new subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$. This chapter also presents the lemmas used to substantiate the main results. In Chapter 4, the analytical results of this research are outlined, focusing on coefficient inequalities and determinants, including the derivation of coefficient bounds, the Fekete–Szegö inequality, the second Hankel determinant, and the Toeplitz determinants for the new subclasses of tilted starlike and convex combinations functions. Chapter 5 concludes the thesis by summarizing the results obtained and offering insights into potential future research directions.

CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

The theory of univalent functions has long served as a cornerstone of GFT due to its deep connections with complex analysis and its applications in areas such as mathematical modeling, fluid dynamics, and conformal mapping. Within this framework, the class of starlike and convex functions has attracted considerable attention. These functions, often defined via subordination to the Koebe function or related geometric conditions, exhibit rich analytical and geometric behavior.

Over time, numerous generalisations of starlike and convex functions have emerged by incorporating geometric constraints, rotational (tilted) factors, and alternative functional transformations. These variations broaden the scope of classical starlike and convex functions theory and enable the study of more intricate mapping behaviors (Ma and Minda, 1994).

One area of particular interest involves functions associated with epicycloid domains, where the image region is bounded by an epicycloid curve. The aim of this literature review is to provide a comprehensive overview of existing works relevant to function classes linked to such domains. It encompasses the development of classical starlike and convex functions, the incorporation of tilted factors to introduce rotational behavior, and the foundational role of subordination principles in defining these classes.

Moreover, it highlights key results in the literature related to initial coefficient estimates, coefficient bounds, the Fekete–Szegö inequality, Hankel determinants, and Toeplitz determinants, which form the basis of the analytic properties explored in this thesis. With this background in place, the next section begins by presenting essential lemmas and preliminary results relevant to the study of univalent function theory.

2.2 Key Lemmas and Foundational Results in the Study of Univalent Functions

The study of univalent functions, analytic and injective functions defined on the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ has been significantly shaped by foundational lemmas and classical results. These tools serve as the basis for deriving coefficient estimates, studying functional inequalities, and investigating determinant-based problems such as those involving the Fekete–Szegö functional, Hankel determinants, and Toeplitz determinants.

One of the most influential results is the Bieberbach conjecture, proposed in 1916 and proved by De Branges in 1985, which states that for any function $f \in S$, the class of normalised univalent functions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, the sharp bound $|a_n| \leq n$ holds for all $n \geq 2$, with equality only for the Koebe function. This result laid the groundwork for estimating the size of coefficients in various subclasses of univalent functions.

The Schwarz Lemma and its generalisations are often employed to handle the function $f(z)$ are analytic in E satisfying $f(0) = 0$ and $|f(z)| \leq 1$. This lemma is particularly useful in the context of subordination, where a function f is subordinate to a function g written as $f \prec g$ which means that there exists a Schwarz function $\omega(z)$, analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. This concept plays a central role in the definition and analysis of various generalised subclasses of starlike functions (Miller and Mocanu, 1981).

In the study of such subclasses, coefficient-based functionals provide essential tools for understanding the structural behavior of functions. One such functional is the Hankel determinant, which captures the relationship among successive coefficients. Pommerenke (1966, 1967) introduced the concept of q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$, denoted as $H_q(n)$. These determinants are expressed in matrix form using the coefficients of the function as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{2.1}$$

A particularly significant case is the second Hankel determinant, obtained by substituting $n = q = 2$, which takes the form

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2. \tag{2.2}$$

Closely related to Hankel determinants is the Fekete–Szegö functional, which was originally studied in the context of univalent functions (Fekete and Szegö, 1933). It is defined by $\left|a_3-\mu a_2^2\right|$, where μ is real or complex parameter. The well-known problem solved by Fekete and Szegö involves determining the greatest value of the coefficient functional.

Efforts to establish upper bounds for these determinants have been ongoing since the year 2000. For instance, Janteng et al. (2007) calculated the sharp bound of $\left|H_2(2)\right|$ within the class of convex and starlike functions.

In recent years, there has been a growing interest in establishing sharp bounds for various orders of Hankel determinants in GFT. Cho et al. (2023) obtained sharp estimates for the Hankel determinants of starlike functions with respect to symmetrical points. Mohamad and Wahid (2023) derived upper bounds for the second Hankel determinant of logarithmic coefficients for the class of tilted starlike functions with respect to conjugate points. Zaprawa (2024) investigated the bounds of $H_2(3)$ for two subclasses consisting of starlike and convex univalent functions and established sharp upper bounds in the case of real coefficients. More recently, Raza et al. (2025) provided sharp estimates for the third Hankel determinant for starlike and convex functions associated with the exponential function. In addition, Hussen et al. (2025) addressed both the Fekete–Szegö functional problem and the bounds of the second Hankel determinant for a certain subclass of bi-univalent functions associated with Lucas–Balancing polynomials. These studies collectively demonstrate the continued development and expanding scope of Hankel determinant analysis across a wide range of function subclasses.

In addition to Hankel determinants, Toeplitz determinants serve as another important class of functionals employed in the study of analytic and univalent functions. While Hankel matrices are characterised by constant values along their anti-diagonals, Toeplitz matrices are defined by having constant elements along each diagonal (Ali et al, 2018). Although the symmetric Toeplitz determinant has been discussed in earlier literature, some of the initial sources are no longer considered reliable. This study adopts a formulation consistent with that used in later works, such as Ali et al. (2018). Let a function be defined in (1.1). The Toeplitz determinant $T_q(n)$ is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}, \tag{2.3}$$

where the matrix entries satisfy the Toeplitz property of constant values along each diagonal. Toeplitz determinants, such as $T_2(2)$, $T_3(1)$ and $T_3(2)$ have gained prominence due to their applications in deriving coefficient inequalities and bounds related to the coefficients of various subclasses of univalent functions. Their structured form facilitates the study of complex interdependencies among coefficients and complements the insights provided by Hankel determinants.

Recent progress on Toeplitz determinants has also gained attention in the study of analytic and univalent functions. Wanas et al. (2025) derived coefficient estimates for the first four determinants of symmetric Toeplitz matrices, contributing an important step toward understanding Toeplitz structures in coefficient theory. Hadi et al. (2025) further contributed by establishing upper and lower bounds for Toeplitz determinants associated with inverse holomorphic functions, using techniques based on the parametrisation of Carathéodory functions. In another recent development, Bareh et al. (2025) obtained sharp bounds for the Toeplitz determinants of several orders, specifically $T_2(1)$, $T_2(2)$, $T_3(1)$ and $T_4(1)$ for functions associated with the lemniscate of Bernoulli, further expanding the scope of Toeplitz determinant analysis in GFT.

The derivation of coefficient estimates and determinant functionals is often supported by several classical lemmas found in the literature. These results provide bounds and structural characterisations for functions belonging to the Carathéodory class P in the form of (1.2), with positive real part and normalised by $p(0)=1$. Duren (1983) provides sharp bounds for coefficients of functions $|a_n|$. Efraimidis (2016) established coefficient inequalities involving a complex parameter μ, with sharpness attained for extremal functions. Bounds for combinations of the second and third coefficients were provided by Pommerenke (1975) and Libera and Zlotkiewicz (1982), while Prokhorov and Szynal (1981) and Ali (2003) extend these results to more general coefficient functionals involving additional parameters. These lemmas form the foundation for the coefficient estimates, Fekete–Szegö inequalities, and determinant bounds studied in this study. These foundational lemmas constitute the analytic framework required to investigate the properties of the newly defined subclasses of tilted starlike and convex combination functions associated with the epicycloid domain. Given their central role in the calculations, their formal presentation and application are reserved for the appropriate sections in Chapter 3.

2.3 Analytic Function Classes Defined via Non-Circular Geometric Domains

The theory of univalent functions has evolved significantly through the introduction of geometric mappings that extend beyond the classical unit disk. Among the most notable are mappings onto non-circular domains, such as epicycloid, cardioid and nephroid. These specialised image regions provide refined geometric constraints, allowing for the definition of subclasses of starlike and convex functions with enriched analytical properties. This section presents the original forms of these domain mappings and outlines how each has been generalised within the framework of GFT.

Cardioid-shaped domains, which form heart-shaped curves, offer a classical geometric mapping framework in univalent function theory. Sharma et al. (2016) introduced a subclass of starlike functions associated with such cardioid domains as follows:

$$S_C^*(z) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}. \tag{2.4}$$

They derived the structural formula, coefficient bounds, growth estimates, and various radius constants for the class, with all results shown to be sharp.

Building upon this class, Shi et al. (2019) investigated the estimates for the third Hankel determinant of functions associated with the cardioid domain. Their analysis extended to subclasses with two-fold and three-fold symmetry, offering insight into more specialised geometric behaviors.

Furthermore, Kumar and Kamaljeet (2021) considered a related class of starlike functions defined via mappings into contracted cardioid regions represented by the function

$$S^{*}(\wp,\alpha):=\left\{f\in A:\frac{zf'(z)}{f(z)}\prec 1+\alpha ze^{z}\right\}. \tag{2.5}$$

They analysed the radius of convexity and established inclusion relationships between this class and other classical subclasses. The study also derived sharp coefficient-related results and precise radius constants.

A generalised version of the cardioid domain was later explored by Yunus et al. (2020), who introduced a class of analytic functions lying within the interior of a generalised cardioid region in the right half-plane as

$$S_{GC}^{*}(k):=\left\{f\in A:\frac{zf'(z)}{f(z)}\prec 1+2kz+kz^{2}\right\}. \tag{2.6}$$

They obtained bounds for initial coefficients, established the Fekete–Szegö inequality, derived estimates for the second Hankel determinant and calculated sharp estimate for the Toeplitz determinant, $T_2(2)$.

The nephroid domain, defined by a two-cusped kidney-shaped curve, also serves as a foundation for generalised function classes. Wani and Swaminathan (2019) introduced Ma-Minda-type subclasses, S_{Ne}^{*} and C_{Ne}^{*} associated with the nephroid domain as

$$\varphi_{Ne}(z)=1+z-\frac{1}{3}z^{3}. \tag{2.7}$$

They analysed the structural properties, extremal functions, growth and distortion estimates, inclusion relationships, and sharp coefficient bounds. In particular, sharp estimates for the Fekete–Szegö functional were derived.

In a subsequent study, Wani and Swaminathan (2020) extended their analysis to include radii problems associated with starlike and convex functions mapped onto nephroid domains. They determined sharp radii for several subclasses and addressed radii problems involving function ratios, with results shown to be sharp.

More recently, Ullah et al. (2025) introduced a new subclass of analytic functions associated with the nephroid domain, denoted by

$$R_n := \{ f \in A : f'(z) \prec \varphi_{Ne}(z) \}, \tag{2.8}$$

where $\varphi_{Ne}(z)$ represents a mapping onto a nephroid-shaped region. The authors focused on establishing sharp analytic results for this class, including upper bounds for the second and third Hankel determinants. Additionally, they derived sharp estimates for logarithmic coefficients, further enriching the analytical framework surrounding function classes defined via non-circular geometric domains.

In addition to cardioid and nephroid domains, the epicycloid also represents a prominent example of a non-circular geometric region employed in the study of univalent functions. A detailed discussion on function classes associated with epicycloid domains is provided in Section 2.4. It is worth highlighting here that epicycloids, multi-cusped curves generated by the rolling motion of a circle around the exterior of another have contributed significantly to the development of geometric function theory, offering deeper insights into the structural behavior of analytic and univalent functions.

2.4 Subclass of Starlike Functions Associated with Epicycloid and Its Properties

As briefly outlined in the preceding section, the epicycloid is a non-circular geometric domain that has proven to be particularly effective in defining meaningful subclasses of univalent functions. Epicycloid is a plane curve traced by a point on the circumference of a circle rolling around the outside of a fixed circle (Gandhi et al., 2022). Due to its cusped boundary and symmetrical structure, the epicycloid offers a flexible and geometrically rich framework for exploring complex mapping behaviors within the unit disk. This section provides a detailed review of the analytic function classes specifically associated with epicycloid domains, with a focus on their foundational definitions, coefficient estimates, functional inequalities, and structural properties.

The foundation of the study on subclasses of starlike functions associated with epicycloid domains can be traced back to the work of Gandhi (2018), who introduced the class of three-leaf-type starlike functions defined as

$$S_{3,\mathrm{L}}^{*} := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^{4},\ z \in E \right\}. \tag{2.9}$$

In this work, Gandhi focused on the radii problem for functions within this class, motivated by domains obtained from convex combinations of linear and exponential mappings. Building upon this, Shi et al. (2021) extended the study by deriving sharp estimates for the first four Taylor coefficients of functions in $S_{3,L}^{*}$. Their analysis also included a sharp Fekete–Szegö inequality, sharp upper bounds for the second Hankel determinant, upper bounds for the third Hankel determinant, and sharp bounds for logarithmic coefficients. Moreover, they explored the third-order Hankel determinant for functions exhibiting two-fold and three-fold symmetries, thereby deepening the analytical understanding of the class.

In a related development, Arif et al. (2022) introduced a new subclass of bounded turning functions associated with the three-leaf-type starlike functions as

$$BT_{3l} := \left\{ f \in A : f'(z) \prec 1 + \frac{4}{5}z + \frac{1}{5}z^{4},\ z \in E \right\}. \tag{2.10}$$

Recognising that Shi et al. (2021) did not obtain sharp bounds for the third Hankel determinant in $S_{3,L}^{*}$, Arif et al. proposed a novel approach to address this gap by reformulating the coefficient estimation problem as a multivariable optimization problem over a cuboidal domain. This method led to sharp bounds for the third Hankel determinant for the class $S_{3,L}^{*}$, thereby improving existing results.

Tang et al. (2023) further extended this line of research by studying a family of starlike functions defined with respect to symmetric points, linked to a three-leaf-shaped image domain initially proposed by Gandhi (2018). They defined the class

$$S^*_{3\mathrm{L,s}} := \left\{ g \in S : \frac{2zg'(z)}{g(z)-g(-z)} \prec 1+\frac{4}{5}z+\frac{1}{5}z^4,\ z \in E \right\}, \tag{2.11}$$

and obtained sharp estimates for the first five coefficients, as well as bounds for the third-order Hankel determinant. Their study also addressed Zalcman and Krushkal inequalities for this symmetric subclass.

Expanding the theory further, Gandhi et al. (2022) introduced a generalised class of starlike functions associated with epicycloid domains, given by

$$S^*_{n-1,\mathrm{L}} := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec 1+\frac{n}{n+1}z+\frac{1}{n+1}z^n,\ z \in E \right\}, \tag{2.12}$$

where $n \geq 2$, which maps the open unit disk onto domains bounded by epicycloids with $n-1$ cusps. Their study presented sharp coefficient estimates, inclusion results, and solutions to various radii problems, offering a unified framework that extends the classical theory of starlike functions.

Building on these results, Shi et al. (2022) focused on the specific case $n = 5$, corresponding to a four-leaf-shaped epicycloid domain. They defined the subclass

$$S^*_{4,\mathrm{L}} := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec 1+\frac{5}{6}z+\frac{1}{6}z^5,\ z \in E \right\}, \tag{2.13}$$

and analysed its analytic properties. Their work included the derivation of a sharp Fekete–Szegö inequality and upper bounds for the second Hankel determinant for both the general $S^*_{n-1,\mathrm{L}}$ class and the specific $S^*_{4,\mathrm{L}}$ case. The sharp bound for the third Hankel determinant, however, was obtained only for the four-leaf subclass $S^*_{4,\mathrm{L}}$.

2.5 A Comprehensive Review of Tilted Starlike Functions and Related Classes

In addition to constructions based on specific geometric domains, another distinct approach in the study of starlike functions involves the incorporation of rotational parameters, giving rise to the concept of tilted functions. These function classes introduce a directional tilt in the mapping behavior by embedding a rotation factor into the analytic formulation, thereby allowing for a more generalised treatment of geometric features within the unit disk.

One of the earliest contributions to this concept was made by Mohamad (2000), who introduced a class of normalised analytic functions denoted by $G(\alpha,\delta)$. This class is defined by the condition $\operatorname{Re}\left(e^{i\alpha}f'(z)>\delta\right)$, where $|\alpha|\leq\pi$ and $\cos\alpha-\delta>0$. The condition can be rewritten using the transformation

$$p(z)=\frac{e^{i\alpha}f'(z)-i\sin\alpha-\delta}{\cos\alpha-\delta}, \tag{2.14}$$

such that $f\in G(\alpha,\delta)$ if and only if $p\in P$, the class of functions with positive real parts. Mohamad explored various properties of this class, including extremal points, coefficient bounds, distortion theorems for the derivatives of these functions, and the argument of functions in $G(\alpha,\delta)$. Based on Mohamad (2000), earlier and related forms of this function class were also examined by several researchers such as MacGregors (1962) for $G(0,0)$, Goel and Mehrok (1983) for $G(\alpha,\delta)$ with $\delta>0$ and Silverman and Silvia (1996) for $S^*(\theta,\varepsilon,s,t)$.

Moreover, Yahya et al. (2022) proposed a generalised class of tilted analytic univalent functions, denoted by $S^*(\theta,\varepsilon,s,t)$, defined through the inequality

$$\operatorname{Re}\left(e^{i\theta}f'(z)\frac{z}{m(z)}\right)>\varepsilon, \tag{2.15}$$

where $\cos\theta>\varepsilon$, $|\theta|<\pi$, $0\leq\varepsilon\leq1$ and $m(z)=\dfrac{z}{(1-sz)(1-tz)}$. This class captures a broader framework of tilted behavior and general multiplier transformations. Using the Herglotz Representation Theorem, the authors derived representation formulas and coefficient bounds for the class.

Further developments in tilted function theory were introduced by Wahid et al. (2015), inspired by earlier works of El-Ashwah and Thomas (1987), Halim (1991), and Dahhar and Janteng (2009). They investigated functions satisfying, $S_c^*(\alpha,\delta)$ satisfying

$$\operatorname{Re}\left(e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > \delta, \tag{2.16}$$

where $g(z) = \dfrac{f(z)+\overline{f(\bar{z})}}{2}$, $\cos\alpha - \delta > 0$, $0 \le \delta \le 1$ and $|\alpha| < \dfrac{\pi}{2}$. This formulation led to the definition of a subclass of tilted starlike functions associated with conjugate points, denoted as $S_c^*(\alpha,\delta,A,B)$, characterised by the subordination

$$\left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} - \delta - i\sin\alpha \right\} \frac{1}{t_{\alpha\delta}} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \tag{2.17}$$

where $t_{\alpha\delta} = \cos\alpha - \delta > 0$. The study provided upper and lower bounds for the real and imaginary parts of the ratio $\dfrac{zf'(z)}{g(z)}$, using the subordination principle. It was also shown that this framework reduces to several previously known classes investigated by El-Ashwah and Thomas (1987), Halim (1991), and Dahhar and Janteng (2009) when specific values of α, δ, A and B are considered.

Wahid and Mohamad (2021) extended this formulation by replacing $f(z)$ with a function $g(z)$ where $g(z) = \dfrac{f(z)+\overline{f(\bar{z})}}{2}$ for the class $S_c^*(\alpha,\delta,A,B)$. This modification incorporates the concept of conjugate symmetry in the analytic formulation, allowing the class to capture functions that are symmetric with respect to the real axis. The authors further analysed the coefficient bounds of this generalised class and investigated their applications in evaluating Toeplitz determinants $T_2(2)$, $T_3(1)$, and $T_3(2)$. They demonstrated how the results generalise various existing subclasses by selecting appropriate parameter values.

2.6 Convex Combination of Starlike and Convex Functions in Analytic Function Theory

Beyond geometric deformations and directional tilts, another classical approach in GFT involves forming convex combinations of starlike and convex functions. These convex-combination classes act as a bridge between purely starlike and purely convex behaviors, enabling analytic functions to exhibit a continuum of intermediate geometric characteristics.

The earliest formal study of such a convex-combination subclass was conducted by Mocanu (1969). In his formulation, a function $f \in A$ is said to be α-starlike if it satisfies the inequality

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>0, \tag{2.18}$$

holds in unit disk E, with $\left(f(z)f'(z)/z\right)\neq 0$ and α is a real number. This condition represents a convex combination of the classical starlikeness and convexity conditions. These functions are univalent and naturally unify the classes of starlike ($\alpha=0$) and convex functions ($\alpha=1$).

A more influential study was carried out by Miller et al. (1973), who introduced the terminology α-convex functions, proving that this class is univalent and starlike in the open unit disk E. In particular, they demonstrated that all α-convex functions are starlike and for $\alpha\geq 1$, these functions are also convex.

In parallel, Miller (1973) studied the same class under the alternative nomenclature α-starlike functions, denoted as M_α, and focused on their distortion theorems and coefficient estimates. Despite the differing terminology, α-convex emphasising convexity and α-starlike emphasising starlikeness. Both describe the same function class defined by the above convex-combination condition.

Further generalising this framework, Sizhuk and Chernikov (1978) introduced the class $M(\alpha,\beta)$ of α-convex functions of order β, defined by the condition:

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>\beta,\ 0\le\alpha,\beta<1. \tag{2.19}$$

They determined extremal functions and the sharp radius of α-convexity for this class. Complementing this, Prokhorov and Szynal (1981) investigated the inverse coefficient estimates for functions in this class, introducing the inverse class

$$\hat{M}(\alpha,\beta):=\left\{F:F=f^{-1},f\in M(\alpha,\beta)\right\}. \tag{2.20}$$

The authors derive explicit, sharp upper bounds for the coefficients of functions in the class $M(\alpha,\beta)$, as well as sharp estimates for the first few inverse coefficients of functions belonging to the inverse class $\hat{M}(\alpha,\beta)$.

In a related direction, Shanmugam et al. (2014) introduced and studied the class M_α of α-starlike functions, which incorporate the rotational parameter α in the starlikeness condition thereby interpolating between classical starlike and convex functions. Their analysis included sharp bounds for the second and third Hankel determinants for the class M_α.

Subsequently, Sokół and Thomas (2018) determined the bounds of the second Hankel determinant for the class M_α, thereby extending several well-known results corresponding to the special cases, $\alpha=0$ and $\alpha=1$. In addition, they investigated the second Hankel determinant for the more general class of α-convex functions of order β, denoted as $M(\alpha,\beta)$. Sharp upper bounds were obtained in particular when $\alpha=1$ and $\beta=0$, confirming consistency with classical convex function theory.

2.7 Framework Connecting Tilted, Convex-Combination, and Epicycloid-Based Classes

This section presents a conceptual framework that organises the major subclasses of univalent functions reviewed in this chapter and illustrates how the present study extends them through meaningful generalisation. The framework integrates structural, directional, and geometric aspects of function theory, highlighting how existing subclasses are expanded to form the new classes introduced in this research.

The development of univalent function theory has evolved through several key directions, including domain generalisation, operator modification, and structural extension of analytic functions. Classical studies initially focused on starlike and convex functions defined in the open unit disk, typically associated with circular symmetry. More recent developments have extended these classes to non-circular domains, such as the epicycloid domain, allowing for the investigation of more intricate geometric behaviour. These advancements commonly utilise the principle of subordination to analyse coefficient estimates, geometric properties, and analytic inequalities.

In parallel, directional generalisations, such as tilted starlike functions introduced by Wahid et al. (2015), incorporate rotational deformation through an angular parameter, thereby relaxing the rigid symmetry of classical starlikeness. On the other hand, structural generalisations based on convex combinations, initiated by Mocanu (1969), provide a flexible framework for blending geometric properties of different function classes. However, these approaches have largely been developed independently in the literature.

Motivated by these developments, the present study combines three distinct generalisation strategies, namely geometric domain extension based on the epicycloid, directional deformation through tilting, and structural flexibility through convex combination, into a unified framework. This combination is not arbitrary but represents a natural progression in GFT since it captures multiple dimensions of generalisation that have previously been treated separately. The resulting subclasses provide a more comprehensive characterisation of analytic functions and offer a richer setting for the investigation of coefficient problems and functional inequalities by integrating these aspects.

The first proposed subclass, the tilted starlike functions associated with the epicycloid domain having $n-1$ cusps, is inspired by the work of Gandhi et al. (2022) and incorporates the tilted parameter introduced by Wahid et al. (2015). This integration combines geometric domain transformation with rotational adjustment, yielding a more flexible analytic structure. The second proposed subclass, the convex-combination of starlike and convex functions associated with the four-leaf-shaped epicycloid domain, is inspired by the work of Shi et al. (2022). This subclass further extends the theory by generalising the structural properties of univalent functions through a parameterised convex combination that blends starlikeness and convexity.

Together, these subclasses unify existing ideas from classical, tilted, and convex-combination frameworks within the epicycloid setting, thereby addressing a gap in the literature where these generalisation techniques have not been studied in a unified manner. This integrated approach not only extends existing theory but also facilitates deeper analysis of coefficient bounds, the Fekete–Szegö problem, and higher-order determinants such as the Hankel and Toeplitz determinants. The conceptual framework in Figure 2.1 illustrates how the newly defined subclasses are derived from and extend previously established function classes.

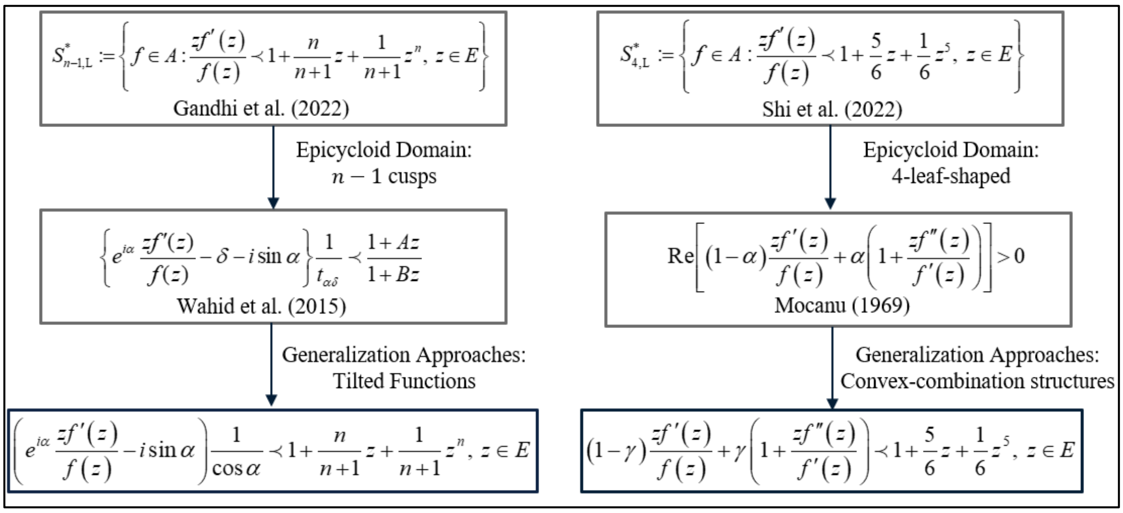

Figure 2.1 Conceptual Framework of the Proposed Subclasses in Relation to Existing Univalent Function Classes

CHAPTER 3
PRELIMINARIES

3.1 Introduction

This chapter establishes the foundational framework, definitions, and subordination principles required for the analytical investigation of two subclasses of univalent functions associated with the epicycloid domain. The first subclass, denoted by $S^{*}_{T,n-1}(\alpha)$, represents the tilted starlike functions defined through a generalised subordination condition involving a tilted argument and an epicycloid with $n-1$ cusps. The second subclass comprises the convex combinations of starlike and convex functions, $\mathrm{M}_{\gamma,4L}$ associated with the four-leaf epicycloid domain, characterised through standard geometric subordination conditions. The chapter opens with a review of the definitions of analytic, univalent, starlike, and convex functions, establishing the groundwork for the analysis that follows. It then introduces the proposed function classes together with the epicycloid mappings on which they are based.

3.2 Preliminaries and Notation

Throughout this research, several fundamental classes and standard notations in GFT are used. Consider the family $H(E)$ consisting of analytical functions within the open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$. Let A denote the class of functions $f\in H(E)$ that are analytic in the open unit disk E, and satisfy the normalisation conditions $f(0)=0$ and $f'(0)=1$. For every A has the Taylor-Maclaurin series expansion of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ z\in E. \tag{3.1}$$

Let S be the subclass of $f\in A$ consisting of univalent functions in E. In terms of subordination, the class of starlike and convex functions can be written as

$$S^*(\varphi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \tag{3.2}$$

and

$$C(\varphi) := \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}, \tag{3.3}$$

that are univalent and satisfy all the conditions provided by Ma and Minda (1994) on the properties of the unified class of starlike and convex functions.

3.3 The Epicycloid Mapping Function

Let the function $\varphi_{n-1,\mathrm{L}} : E \to \mathbb{C}$ be given by

$$\varphi_{n-1,\mathrm{L}}(z) = 1 + \frac{n}{n+1} z + \frac{1}{n+1} z^n, \tag{3.4}$$

where $n \geq 2$ (natural number). The function $\varphi_{n-1,\mathrm{L}}$ maps the unit disk onto a region bounded by an epicycloid-shaped curve with $n-1$ cusps. For the specific case $n = 5$, let the function $\varphi_{4,\mathrm{L}} : E \to \mathbb{C}$ be given by

$$\varphi_{n-1,\mathrm{L}}(z) = 1 + \frac{5}{6} z + \frac{1}{6} z^5. \tag{3.5}$$

The functions $\varphi_{n-1,\mathrm{L}}$ and $\varphi_{4,\mathrm{L}}$ are known to satisfy the geometric conditions required by Ma and Minda (1994) where the functions satisfy the univalence, positivity of the real part, symmetry with respect to the real axis, and starlike with respect to 1.

The subclass of starlike function subordinate to the epicycloid mapping $\varphi_{n-1,\mathrm{L}}$ is defined as

$$S^*_{n-1,L} := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi_{n-1,L}(z) \right\}, \tag{3.6}$$

where A denotes the class of analytic functions in E with $f(0)=0$ and $f'(0)=1$, and "$\prec$" denotes subordination. In particular, for $n=5$, the subclasses of starlike and convex functions associated with the epicycloid domain can be written as

$$S_{4,L}^{*} := \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \varphi_{4,L}(z) \right\}, \tag{3.7}$$

and

$$C_{4,L} := \left\{ f \in S : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_{4,L}(z) \right\}. \tag{3.8}$$

3.4 Preliminary Results

This section presents a collection of fundamental lemmas that are essential to the analytical methodology employed in this research. These lemmas provide crucial tools for deriving coefficient estimates and establishing coefficient inequalities related to the class of analytic and univalent functions investigated herein. The results are particularly relevant for functions subordinate to a specific analytic function with positive real part and will be frequently used in the subsequent chapters.

Lemma 3.1 (Duren, 1983). For a function $p \in P$ of the form given by (1.3), the sharp inequality $|c_n| \le 2$ holds for each $n \ge 1$. Equality holds for the function $p(z) = \dfrac{1+z}{1-z}$.

Lemma 3.2 (Efraimidis, 2016). Let $p \in P$ of the form (1.3) and let $\mu \in \mathbb{C}$, where μ may be real or complex. Then

$$|c_n - \mu c_k c_{n-k}| \le 2\max\{1, |2\mu - 1|\}, \ 1 \le k \le n-1. \tag{3.9}$$

If $|2\mu - 1| \ge 1$, then the inequality is sharp for the function $p(z) = \dfrac{1+z}{1-z}$ or its rotations.

If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \dfrac{1+z^n}{1-z^n}$ or its rotations.

Lemma 3.3 (Libera and Zlotkiewicz, 1982). If $p \in P$ with the series expansion of the form (1.3) with the values of x and δ satisfying $|x| \leq 1$ and $|\delta| \leq 1$ and $c_1 \in [0,2]$, then

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x, \tag{3.10}$$

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(1 - |x|^2\right)\left(4 - c_1^2\right)\delta. \tag{3.11}$$

Lemma 3.4 (Ali, 2003). Let $p \in P$ of the form (1.3). If $0 \leq \beta \leq 1$ and $\beta(2\beta - 1) \leq \delta^* \leq \beta$, then

$$\left|c_3 - 2\beta c_1 c_2 + \delta^* c_1^3\right| \leq 2. \tag{3.12}$$

3.5 Tilted Starlike Subclass Defined by Epicycloid Mapping and Its Subordination Representation

Let $S_T^*(\alpha)$ denote the class of functions $f \in A$ that are univalent in E and satisfy

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \ |\alpha| < \frac{\pi}{2}. \tag{3.13}$$

The following theorem connects this class with the Carathéodory class P.

Theorem 3.1.
Let $f \in S$. Then $f \in S_{T,n-1}^*(\alpha)$ if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} \in P. \tag{3.14}$$

Proof.

Let $f \in S^{*}_{T,n-1}(\alpha)$, and suppose

$$\frac{zf'(z)}{f(z)} = p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{3.15}$$

Then, from equation (1.1), the following is derived

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = e^{i\alpha}\left(1 + \sum_{n=1}^{\infty} b_n z^n\right)$$

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = (\cos\alpha + i\sin\alpha) + e^{i\alpha}\sum_{n=1}^{\infty} b_n z^n$$

$$e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha = \cos\alpha + e^{i\alpha}\sum_{n=1}^{\infty} b_n z^n$$

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = 1 + \frac{e^{i\alpha}}{\cos\alpha}\sum_{n=1}^{\infty} b_n z^n. \tag{3.16}$$

Hence,

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{3.17}$$

where $c_n = \dfrac{e^{i\alpha} b_n}{\cos\alpha}$.

Thus, for any $f \in S$, let

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = p(z), \tag{3.18}$$

so that $f \in S^{*}_{T,n-1}(\alpha)$ if and only if $p(z) \in P$. Note that $\cos\alpha$ must always greater than 0 so that (3.18) is valid.

Conversely, assume that

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = p(z) = 1 + \sum_{n=1}^{\infty}\frac{e^{i\alpha}b_n}{\cos\alpha}z^n \,. \tag{3.19}$$

Multiplying both sides by $\cos\alpha$ yields

$$e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha = \cos\alpha\left(1 + \sum_{n=1}^{\infty}\frac{e^{i\alpha}b_n}{\cos\alpha}z^n\right). \tag{3.20}$$

Simplifying the right-hand side gives

$$e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha = \cos\alpha + \sum_{n=1}^{\infty}e^{i\alpha}b_n z^n \,. \tag{3.21}$$

Next, adding $i\sin\alpha$ to both sides and get

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = \cos\alpha + i\sin\alpha + \sum_{n=1}^{\infty}e^{i\alpha}b_n z^n$$

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = e^{i\alpha} + \sum_{n=1}^{\infty}e^{i\alpha}b_n z^n$$

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = e^{i\alpha}\left(1 + \sum_{n=1}^{\infty}b_n z^n\right). \tag{3.22}$$

Dividing both sides by $e^{i\alpha}$ yields

$$\frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n . \tag{3.23}$$

Since $p(z) \in P$, it satisfies the defining condition of $S^*_{T,n-1}(\alpha)$. Hence, $f \in S^*_{T,n-1}(\alpha)$.

Proof of Theorem 3.1 completes.

The class $S^*_{T,n-1}(\alpha)$ is now defined in terms of subordination.

Definition 3.1

A function $f \in S$ is said to belong to the class $S^*_{T,n-1}(\alpha)$ if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n,\ z \in E, \tag{3.24}$$

where $n \geq 2$ and $|\alpha| < \frac{\pi}{2}$.

By the definition of subordination, this is equivalent to the existence of a Schwarz function $\omega(z)$, analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)} - i\sin\alpha\right)\frac{1}{\cos\alpha} = 1 + \frac{n}{n+1}\omega(z) + \frac{1}{n+1}\left[\omega(z)\right]^n = p(z). \tag{3.25}$$

If $\alpha = 0$, the class reduces to

$$S^*_{n-1,L} = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n \right\}, \tag{3.26}$$

as introduced by Gandhi et al. (2022). Furthermore, if $\alpha = 0$ and $n = 5$, the class reduces to

$$S_{4,L}^{*}=\left\{f\in S:\frac{zf'(z)}{f(z)}\prec 1+\frac{5}{6}z+\frac{1}{6}z^{5}\right\}, \tag{3.27}$$

as studied by Shi et al. (2022).

3.6 The Classical Starlike and Convex Functions Defined by Epicycloid Mapping and Its Subordination Representation

In addition to the subclass of starlike functions, this study defines a new subclass of classical starlike and convex functions associated with the epicycloid domain. Unlike the tilted subclass, which incorporate a rotational parameter $e^{i\alpha}$, the classical subclass is defined using standard differential expressions and subordinated directly to the epicycloid mapping function. The following class of analytic functions is defined by subordination to $\varphi_{4,\mathrm{L}}$, denoted by $\mathrm{M}_{\gamma,4L}$, consisting of functions f that is analytic and univalent in the open unit disk E.

A function f is given by (1.1) with $\left(f(z)f'(z)/z\right)\neq 0$ for $0<|z|<1$ and suppose γ is real numbers. Then, f is said to belong to the class $\mathrm{M}_{\gamma,4L}$ if and only if

$$\operatorname{Re}\left((1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>0,\ z\in E, \tag{3.28}$$

for a fixed $0\leq\gamma\leq 1$. Although traditionally referred to as γ-starlike in the literature, it in fact generalises both starlike and convex functions depending on the value of γ. Hence, the class $\mathrm{M}_{\gamma,4L}$ may also be viewed as a convex combination of starlike and convex functions.

Setting $\gamma=0$, this class reduces to the well-known class S^{*} of starlike functions that is satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in E, \tag{3.29}$$

while for $\gamma = 1$, it becomes the classical class C of convex functions satisfying

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in E. \tag{3.30}$$

The class $\mathrm{M}_{\gamma,4L}$ is now defined in terms of subordination.

Definition 3.2

A function $f \in S$ is said to belong to the class $\mathrm{M}_{\gamma,4L}$ if and only if

$$(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{5}{6}z+\frac{1}{6}z^5,\ z \in E, \tag{3.31}$$

where $0 \le \gamma \le 1$.

By the definition of subordination, this condition is equivalent to the existence of a Schwarz function $\omega(z)$, analytic in E with $\omega(0)=0$ and $|\omega(z)|<1$, such that

$$(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right) = 1+\frac{5}{6}\omega(z)+\frac{1}{6}\left[\omega(z)\right]^5 = p(z). \tag{3.32}$$

The generalised subclass is examined within the context of epicycloid-type mappings, thus broadening the scope of classical starlike and convex function theory to encompass more intricate geometric domains.

3.7 Analytical Properties of the Defined Class

This section explores various analytical properties associated with the newly defined subclasses of analytic functions. The analysis focuses on deriving coefficient estimates and bounds, as well as establishing several classical inequalities and determinants that play a central role in GFT. These include the Fekete–Szegö inequality, the second Hankel determinant, and Toeplitz determinants.

3.7.1 Coefficient Estimates and Bounds

This section is devoted to determining the coefficient estimates and bounds for functions belonging to the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$ of analytic and univalent functions. The objective is to obtain sharp or best possible estimates for the initial Taylor coefficients a_2, a_3 and a_4 of functions in these subclasses. In addition to estimates, this section also focuses on deriving coefficient bounds, which serve as upper limits for the modulus of the coefficients $|a_2|$, $|a_3|$ and $|a_4|$.

Derivation of these results relies on the subordination principle, which allows the analytic functions $f \in S^*_{T,n-1}(\alpha)$ and $f \in \mathrm{M}_{\gamma,4L}$ to be expressed in terms of a Schwarz function $\omega(z)$, belonging to the class of analytic functions in E with $\omega(0)=0$ and $|\omega(z)|<1$. The auxiliary function $p(z)=\dfrac{1+\omega(z)}{1-\omega(z)} \in P$, the Carathéodory class of functions with positive real part, plays a central role in the derivation of coefficient bounds.

By expanding both sides of the subordination condition and comparing the coefficients of powers of z, the coefficients a_n can be expressed in terms of c_n. Algebraic simplification and manipulation are applied, and the use of known lemmas, such as those by Pommerenke, Duren, and Efraimidis, allows the establishment of the best possible bounds.

The resulting coefficient estimates not only provide deeper insights into the nature of the subclass but also form the basis for further investigations involving coefficient-related problems. These include the Fekete–Szegö functional, the second Hankel determinant, and Toeplitz determinants, which will be addressed in the subsequent subsections.

3.7.2 Fekete–Szegö Inequality

In this section, the Fekete–Szegö problem is addressed for the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$, aiming to determine the sharp upper bounds for the functional $|a_3 - \mu a_2^2|$. The parameter μ is treated as a complex number across both subclasses to ensure mathematical rigor and analytical generality. This unified approach follows the framework established by Ma and Minda (1994), where the Fekete–Szegö functional is examined for the general case $\mu \in \mathbb{C}$ to capture the complete range of extremal behaviors. In the case of the tilted starlike class, this choice aligns with the rotational influence of the tilted factor $e^{i\alpha}$ on the image domain. Similarly, for the convex-combination subclass, adopting a complex μ allows for a comprehensive study of coefficient stability under the influence of the epicycloid domain, regardless of the structural blending of operators $\gamma \in [0,1]$.

The methodology involves expressing the coefficients a_2 and a_3 in terms of the coefficients c_1 and c_2 of the associated Carathéodory function $p(z)$. In particular, Lemma 3.2 is applied to facilitate the estimation process, enabling the derivation of sharp coefficient inequalities. This inequality provides important insights into the coefficient structure and plays a significant role in the theory of univalent functions.

3.7.3 Second Hankel Determinant

The second Hankel determinant, defined by $H_2(2) = a_2 a_4 - a_3^2$ is examined in this subsection. The objective is to establish an upper bound for $|H_2(2)|$ for functions belonging to the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$. This determinant plays a significant role in GFT, as it is closely associated with the regularity, geometric stability, and structural behaviour of analytic and univalent functions.

The previously obtained coefficient estimates for a_2, a_3 and a_4 are utilised to compute $H_2(2)$. These coefficients are substituted into the determinant formula, followed by algebraic manipulations and simplifications. The process is guided by the application of relevant lemmas introduced earlier, which provide essential tools for establishing the sharpness and validity of the resulting bounds.

3.7.4 Toeplitz Determinant

Toeplitz determinant is another important analytic tool used to capture coefficient interactions and structural behavior of univalent functions. This subsection addresses the derivation of upper bounds for the second- and third-order Toeplitz determinants for functions in the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$. The Toeplitz determinants of interest are defined as follows:

$$T_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2, \tag{3.33}$$

$$T_3(1)=\begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}=1\left(1-a_2^2\right)-a_2\left(a_2-a_2a_3\right)+a_3\left(a_2^2-a_3\right), \tag{3.34}$$

$$T_3(2)=\begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_2 & a_2 \end{vmatrix}=a_2\left(a_2^2-a_2a_3\right)-a_3\left(a_2a_3-a_3a_4\right)+a_4\left(a_2a_3-a_2a_4\right). \tag{3.35}$$

The determinants are evaluated by substituting the earlier derived coefficient estimates a_2, a_3 and a_4 into their corresponding determinant expressions. This is followed by algebraic simplification and analysis to obtain sharp or best-possible upper bounds. Alternatively, where applicable, the coefficient bounds derived earlier may also be used in place of the coefficient estimates to provide general upper limits for the Toeplitz determinants. Moreover, the results obtained for the second Hankel determinant, which involves the same coefficients can also be utilised to substitute into the Toeplitz determinant expressions.

The process is supported by the application of the proposed lemmas which provide essential bounds and structural relationships for the coefficients associated with Carathéodory functions. These lemmas play a crucial role in ensuring that the resulting inequalities are mathematically rigorous. Through this approach, the Toeplitz determinants yield meaningful insights into the geometric and analytic structure of the defined subclasses, complementing the results obtained for coefficient estimates, the Fekete–Szegö problem, and the Hankel determinants.

CHAPTER 4
ANALYTICAL RESULTS ON THE SUBCLASSES $S^*_{T,n-1}(\alpha)$ AND $\mathrm{M}_{\gamma,4L}$

4.1 Introduction

This chapter focuses on two important subclasses of analytic functions that are associated with the epicycloid domain. The first subclass is the subclass of tilted starlike functions, $S^*_{T,n-1}(\alpha)$ which includes a rotational parameter that affects the geometric behavior of the functions. The second subclass is the subclass of classical starlike–convex functions, $\mathrm{M}_{\gamma,4L}$ defined using a convex combination of the standard starlike and convex conditions. Both subclasses are studied using subordination to the epicycloid mapping function. The main analytical results presented in this chapter include coefficient estimates and bounds, the Fekete–Szegö inequality, the second Hankel determinant, and the second- and third-order Toeplitz determinants. These results are derived using standard techniques from GFT, such as the subordination principle and known bounds for functions with positive real part.

4.2 Tilted Starlike Subclass Associated with the Epicycloid Domain

This section focuses on the analytical properties of the subclass of tilted starlike functions associated with the epicycloid domain. Functions in this subclass are defined using a subordination condition involving a specific epicycloid mapping function. The aim is to investigate key function-theoretic aspects such as coefficient estimates and bounds, the Fekete–Szegö inequality, the second Hankel determinant, and various Toeplitz determinants.

4.2.1 Coefficient Estimates and Bounds

This section begins with the derivation of coefficient estimates and bounds for functions belonging to the tilted starlike subclass associated with the epicycloid domain.

Theorem 4.2.1.

Let $f \in S^{*}_{T,n-1}(\alpha)$ be given in the form (1.1). If $n \geq 2$ and $|\alpha| < \dfrac{\pi}{2}$, then

$$|a_2| \leq \frac{n\cos\alpha}{(n+1)}, \tag{4.1}$$

$$|a_3| \leq \frac{n\cos\alpha}{2(n+1)}, \tag{4.2}$$

and

$$|a_4| \leq \frac{n\cos\alpha}{6(n+1)}\left(\left|2+\frac{n^2 e^{-i2\alpha}\cos^2\alpha}{(n+1)^2}-\frac{3ne^{-i\alpha}\cos\alpha}{(n+1)}\right|+2\right). \tag{4.3}$$

Proof.

Assumed that $f \in S^{*}_{T,n-1}(\alpha)$. Then, using the concept of subordination, it follows that there exists a Schwarz function $\omega(z)$ that satisfy $\omega(0)=0$ and $|\omega(z)|<1$. Hence, from the Definition 3.1, it leads to

$$\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-i\sin\alpha\right)\frac{1}{\cos\alpha}=1+\frac{n}{n+1}\omega(z)+\frac{1}{n+1}\left[\omega(z)\right]^n. \tag{4.4}$$

Let the function be defined as

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+c_3z^3\cdots. \tag{4.5}$$

It is evident that $p \in P$. It follows that

$$\begin{aligned}\omega(z)&=\frac{p(z)-1}{p(z)+1}=\frac{c_1z+c_2z^2+c_3z^3+c_4z^4\cdots}{2+c_1z+c_2z^2+c_3z^3+c_4z^4\cdots}\\&=\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{8}c_1^3-\frac{1}{2}c_1c_2+\frac{1}{2}c_3\right)z^3\\&+\left(\frac{1}{2}c_4-\frac{1}{2}c_1c_3-\frac{1}{4}c_2^2-\frac{1}{16}c_1^4+\frac{3}{8}c_1^2c_2\right)z^4+\cdots.\end{aligned}\tag{4.6}$$

Now, by using (3.1), the expression is expanded as

$$\begin{aligned}\frac{zf'(z)}{f(z)}&=1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3\\&+\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)z^4+\cdots.\end{aligned}\tag{4.7}$$

Next, expanding the left-hand side of (4.4) yields

$$\begin{aligned}&\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-i\sin\alpha\right)\frac{1}{\cos\alpha}\\&=\begin{pmatrix}e^{i\alpha}+e^{i\alpha}a_2z+e^{i\alpha}\left(2a_3-a_2^2\right)z^2+e^{i\alpha}\left(a_2^3-3a_2a_3+3a_4\right)z^3\\+e^{i\alpha}\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)z^4+\cdots-i\sin\alpha\end{pmatrix}\frac{1}{\cos\alpha}.\end{aligned}\tag{4.8}$$

Since $e^{i\alpha}=\cos\alpha+i\sin\alpha$, thus

$$=\begin{pmatrix}\cos\alpha+i\sin\alpha+e^{i\alpha}a_2z+e^{i\alpha}\left(2a_3-a_2^2\right)z^2+e^{i\alpha}\left(a_2^3-3a_2a_3+3a_4\right)z^3\\+e^{i\alpha}\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)z^4+\cdots-i\sin\alpha\end{pmatrix}\frac{1}{\cos\alpha}$$

$$=\begin{pmatrix}\cos\alpha+i\sin\alpha-i\sin\alpha+e^{i\alpha}a_2z+e^{i\alpha}\left(2a_3-a_2^2\right)z^2\\+e^{i\alpha}\left(a_2^3-3a_2a_3+3a_4\right)z^3\\+e^{i\alpha}\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)z^4+\cdots\end{pmatrix}\frac{1}{\cos\alpha}$$

$$\begin{aligned}&=\frac{\cos\alpha}{\cos\alpha}+\frac{e^{i\alpha}a_2}{\cos\alpha}z+\frac{e^{i\alpha}\left(2a_3-a_2^2\right)}{\cos\alpha}z^2+\frac{e^{i\alpha}\left(a_2^3-3a_2a_3+3a_4\right)}{\cos\alpha}z^3\\&\quad+\frac{e^{i\alpha}\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)}{\cos\alpha}z^4+\cdots.\end{aligned}\tag{4.9}$$

Hence, the left-hand side of equation is

$$\begin{aligned}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}-i\sin\alpha\right)\frac{1}{\cos\alpha}&=1+\frac{e^{i\alpha}a_2}{\cos\alpha}z+\frac{e^{i\alpha}\left(2a_3-a_2^2\right)}{\cos\alpha}z^2\\&\quad+\frac{e^{i\alpha}\left(a_2^3-3a_2a_3+3a_4\right)}{\cos\alpha}z^3\\&\quad+\frac{e^{i\alpha}\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)}{\cos\alpha}z^4+\cdots.\end{aligned}\tag{4.10}$$

Next, substituting the series expansion of (4.6) into the right-hand side of (4.4) yields

$$\begin{aligned}&1+\frac{n}{n+1}\omega(z)+\frac{1}{n+1}\left[\omega(z)\right]^n\\&=1+\frac{n}{2(n+1)}c_1z+\left[\frac{n}{2(n+1)}c_2-\frac{n}{4(n+1)}c_1^2\right]z^2\\&\quad+\left[\frac{n}{2(n+1)}c_3-\frac{n}{2(n+1)}c_1c_2+\frac{n}{8(n+1)}c_1^3\right]z^3+\cdots.\end{aligned}\tag{4.11}$$

From the comparison of (4.10) and (4.11), the value of a_2 is obtained as

$$\frac{e^{i\alpha}a_2}{\cos\alpha} = \frac{n}{2(n+1)}c_1$$

$$a_2 = \frac{n\cos\alpha}{2(n+1)e^{i\alpha}}c_1. \qquad (4.12)$$

Proceeding similarly, the coefficient of a_3 is given by

$$\frac{e^{i\alpha}\left(2a_3 - a_2^2\right)}{\cos\alpha} = \frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2$$

$$2a_3 - a_2^2 = \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2\right)$$

$$2a_3 = \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2\right) + a_2^2. \qquad (4.13)$$

Upon substituting the value of a_2, the resulting expression becomes

$$2a_3 = \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2\right) + \left(\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}c_1\right)^2$$

$$= \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2\right) + \frac{n^2\cos^2\alpha}{4(n+1)^2 e^{i2\alpha}}c_1^2$$

$$= \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_2 - \frac{n}{4(n+1)}c_1^2 + \frac{n^2\cos^2\alpha}{4(n+1)^2 e^{i2\alpha}}c_1^2\right)$$

$$= \frac{n\cos\alpha}{4(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2 - (n+1)c_1^2 + \frac{n\cos\alpha}{e^{i\alpha}}c_1^2\right). \qquad (4.14)$$

Hence the value of a_3 is

$$a_3 = \frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2 - \left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right). \tag{4.15}$$

The coefficients of z^3 is compared between equations (4.10) and (4.11), leading to the expression for a_4 as follows:

$$\frac{e^{i\alpha}\left(a_2^3 - 3a_2a_3 + 3a_4\right)}{\cos\alpha} = \frac{n}{2(n+1)}c_3 - \frac{n}{2(n+1)}c_1c_2 + \frac{n}{8(n+1)}c_1^3$$

$$a_2^3 - 3a_2a_3 + 3a_4 = \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_3 - \frac{n}{2(n+1)}c_1c_2 + \frac{n}{8(n+1)}c_1^3\right)$$

$$3a_4 = \frac{\cos\alpha}{e^{i\alpha}}\left(\frac{n}{2(n+1)}c_3 - \frac{n}{2(n+1)}c_1c_2 + \frac{n}{8(n+1)}c_1^3\right) - a_2^3 + 3a_2a_3. \tag{4.16}$$

Solving the equation yields the values of a_2^3 and $3a_2a_3$, resulting in

$$a_2^3 = \left(\frac{\cos\alpha}{e^{i\alpha}} \cdot \frac{n}{2(n+1)}c_1\right)^3$$

$$= \frac{n^3\cos^3\alpha}{8(n+1)^3 e^{i3\alpha}}c_1^3. \tag{4.17}$$

$$3a_2a_3 = 3\left(\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}c_1\right)\left(\frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2 - \left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)\right)$$

$$= \frac{3n^2\cos^2\alpha}{16(n+1)^3 e^{i2\alpha}}c_1\left(2(n+1)c_2 - \left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)$$

$$= \frac{3n^2\cos^2\alpha}{16(n+1)^3 e^{i2\alpha}}\left(2(n+1)c_1c_2 - \left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^3\right). \tag{4.18}$$

Then, substituting the values of a_2^3 and $3a_2a_3$ back gives

$$a_4 = \frac{\cos\alpha}{3e^{i\alpha}}\left(\frac{n}{2(n+1)}c_3 - \frac{n}{2(n+1)}c_1c_2 + \frac{n}{8(n+1)}c_1^3\right) - \frac{n^3\cos^3\alpha}{24(n+1)^3 e^{i3\alpha}}c_1^3$$
$$+\frac{n^2\cos^2\alpha}{16(n+1)^3 e^{i2\alpha}}\left(2(n+1)c_1c_2 - \left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^3\right)$$

$$= \frac{\cos\alpha}{e^{i\alpha}}\left[\begin{array}{l}\left(\frac{n}{6(n+1)}c_3 - \frac{n}{6(n+1)}c_1c_2 + \frac{n}{24(n+1)}c_1^3\right) - \left(\frac{n^3\cos^3\alpha}{24(n+1)^3 e^{i3\alpha}}c_1^3\right)\\ +\left(\frac{n^2\cos^2\alpha}{16(n+1)^3 e^{i2\alpha}}\left[2(n+1)c_1c_2 - \left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^3\right]\right)\end{array}\right]$$

$$= \frac{\cos\alpha}{e^{i\alpha}}\left[\begin{array}{l}\frac{n}{6(n+1)}c_3 - \frac{n}{6(n+1)}c_1c_2 + \frac{n}{24(n+1)}c_1^3 - \frac{n^3\cos^2\alpha}{24(n+1)^3 e^{i2\alpha}}c_1^3\\ +\frac{n^2(n+1)\cos\alpha}{8(n+1)^3 e^{i\alpha}}c_1c_2 - \frac{n^2\cos\alpha}{16(n+1)^3 e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^3\end{array}\right]$$

$$= \frac{\cos\alpha}{e^{i\alpha}}\left[\begin{array}{l}\left(\frac{n}{24(n+1)} - \frac{n^3\cos^2\alpha}{24(n+1)^3 e^{i2\alpha}} - \frac{n^2\cos\alpha}{16(n+1)^3 e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^3\\ +\left(\frac{n^2\cos\alpha}{8(n+1)^2 e^{i\alpha}} - \frac{n}{6(n+1)}\right)c_1c_2 + \frac{n}{6(n+1)}c_3\end{array}\right]$$

$$= \frac{\cos\alpha}{e^{i\alpha}}\left[\begin{array}{l}\left[\frac{n}{48(n+1)^3}\left(2(n+1)^2 - 2n^2\frac{\cos^2\alpha}{e^{i2\alpha}} - 3n\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)\right]c_1^3\\ -\left[\frac{n}{48(n+1)^2}\left(8(n+1)-\frac{6n\cos\alpha}{e^{i\alpha}}\right)\right]c_1c_2 + \frac{n}{6(n+1)}c_3\end{array}\right]$$

$$= \frac{n\cos\alpha}{48(n+1)^3 e^{i\alpha}}\left[\begin{array}{l}\left(2(n+1)^2 - 2n^2\frac{\cos^2\alpha}{e^{i2\alpha}} - 3n\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^3\\ -\left((n+1)\left(8(n+1)-\frac{6n\cos\alpha}{e^{i\alpha}}\right)\right)c_1c_2 + 8(n+1)^2c_3\end{array}\right]. \tag{4.19}$$

Hence the value of a_4 is

$$a_4=\frac{n\cos\alpha}{48(n+1)^3 e^{i\alpha}}\left[\begin{array}{l}\left(2(n+1)^2-2n^2\dfrac{\cos^2\alpha}{e^{i2\alpha}}-3n\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^3\\ -\left((n+1)\left(8(n+1)-\dfrac{6n\cos\alpha}{e^{i\alpha}}\right)\right)c_1c_2+8\left(n^2+2n+1\right)c_3\end{array}\right]. \tag{4.20}$$

Next, by using Lemma 3.1, the bound of a_2 is calculated as

$$|a_2|\le\left|\frac{\cos\alpha}{e^{i\alpha}}\right|\cdot\frac{n}{2(n+1)}|c_1|. \tag{4.21}$$

Since $\left|e^{i\alpha}\right|=1$, the bound of $|a_2|$ is

$$|a_2|\le\frac{n\cos\alpha}{2(n+1)}(2)=\frac{n\cos\alpha}{(n+1)}. \tag{4.22}$$

Finding the bound of $|a_3|$, involves simplifying (4.15), which gives

$$\begin{aligned}a_3&=\frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2-\left(n+1-ne^{-i\alpha}\cos\alpha\right)c_1^2\right)\\&=\frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left[2(n+1)\left(c_2-\frac{1}{2(n+1)}\left(n+1-ne^{-i\alpha}\cos\alpha\right)c_1^2\right)\right]\\&=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[\left(c_2-\frac{1}{2(n+1)}\left(n+1-ne^{-i\alpha}\cos\alpha\right)c_1^2\right)\right]\\&=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[\left(c_2-\left(\frac{1}{2}-\frac{ne^{-i\alpha}\cos\alpha}{2(n+1)}\right)c_1^2\right)\right].\end{aligned} \tag{4.23}$$

Taking the modulus on both sides gives

$$|a_3| = \left|\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\right|\left|c_2 - \left(\frac{1}{2} - \frac{ne^{-i\alpha}\cos\alpha}{2(n+1)}\right)c_1^2\right|. \tag{4.24}$$

Applying Lemma 3.2 and assuming $\mu = \frac{1}{2} - \frac{ne^{-i\alpha}\cos\alpha}{2(n+1)}$, the calculation yields

$$\begin{aligned}
|a_3| &\le \frac{n}{4(n+1)}\left|\frac{\cos\alpha}{e^{i\alpha}}\right| 2\max\left\{1, \left|2\left(\frac{1}{2} - \frac{ne^{-i\alpha}\cos\alpha}{2(n+1)}\right) - 1\right|\right\} \\
&= \frac{n}{2(n+1)}\left|\frac{\cos\alpha}{e^{i\alpha}}\right| \max\left\{1, \left|\left(1 - \frac{ne^{-i\alpha}\cos\alpha}{(n+1)}\right) - 1\right|\right\} \\
&= \frac{n}{2(n+1)}\left|\frac{\cos\alpha}{e^{i\alpha}}\right| \max\left\{1, \left|\frac{ne^{-i\alpha}\cos\alpha}{(n+1)}\right|\right\}.
\end{aligned} \tag{4.25}$$

Since $\left|e^{i\alpha}\right| = 1$, hence

$$|a_3| \le \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \frac{n\cos\alpha}{(n+1)}\right\}. \tag{4.26}$$

Under the assumptions of $|\alpha| < \frac{\pi}{2}$ and $n \ge 2$, $\max\left\{1, \frac{n\cos\alpha}{(n+1)}\right\} = 1$. Thus, the bound of $|a_3|$ can be written as

$$|a_3| \le \frac{n\cos\alpha}{2(n+1)}. \tag{4.27}$$

Lastly, the bound of $|a_4|$ is determined by performing algebraic rearrangements of equation (4.20) in accordance with the structure and conditions outlined in Lemmas 3.1 and 3.2.

$$a_4 = \frac{n\cos\alpha}{48(n+1)^3 e^{i\alpha}} \left[\begin{array}{l} \left(2(n+1)^2 - 2n^2 \frac{\cos^2\alpha}{e^{i2\alpha}} - 3n\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right) c_1^3 \\ -\left((n+1)\left(8(n+1) - \frac{6n\cos\alpha}{e^{i\alpha}}\right)\right) c_1c_2 + 8(n+1)^2 c_3 \end{array} \right]$$

$$= \frac{n\cos\alpha}{6(n+1)e^{i\alpha}} \left[\begin{array}{l} \frac{1}{8(n+1)^2} \left(\begin{array}{l} 2(n+1)^2 - 2n^2 \frac{\cos^2\alpha}{e^{i2\alpha}} \\ -3n(n+1)\frac{\cos\alpha}{e^{i\alpha}} + 3n^2\frac{\cos^2\alpha}{e^{i2\alpha}} \end{array} \right) c_1^3 \\ -\frac{1}{8(n+1)^2}\left(8(n+1)^2 - 6n(n+1)\frac{\cos\alpha}{e^{i\alpha}}\right) c_1c_2 + c_3 \end{array} \right]$$

$$= \frac{n\cos\alpha}{6(n+1)e^{i\alpha}} \left[\begin{array}{l} \frac{1}{8(n+1)^2}\left(2(n+1)^2 + n^2\frac{\cos^2\alpha}{e^{i2\alpha}} - 3n(n+1)\frac{\cos\alpha}{e^{i\alpha}}\right) c_1^3 \\ -\left(1 - \frac{3n}{4(n+1)}\frac{\cos\alpha}{e^{i\alpha}}\right) c_1c_2 + c_3 \end{array} \right]$$

$$= \frac{ne^{-i\alpha}\cos\alpha}{6(n+1)} \left[\begin{array}{l} \left(\frac{1}{4} + \frac{n^2 e^{-i2\alpha}\cos^2\alpha}{8(n+1)^2} - \frac{3ne^{-i\alpha}\cos\alpha}{8(n+1)}\right) c_1^3 \\ +\left[c_3 - \left(1 - \frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right) c_1c_2 \right] \end{array} \right]. \tag{4.28}$$

The modulus of both sides results in

$$|a_4| \leq \left| \frac{ne^{-i\alpha}\cos\alpha}{6(n+1)} \left[\begin{array}{l} \left(\frac{1}{4} + \frac{n^2 e^{-i2\alpha}\cos^2\alpha}{8(n+1)^2} - \frac{3ne^{-i\alpha}\cos\alpha}{8(n+1)}\right) c_1^3 \\ +\left[c_3 - \left(1 - \frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right) c_1c_2 \right] \end{array} \right] \right|. \tag{4.29}$$

Applying the triangle inequality and noting that $\left|e^{-i\alpha}\right|=1$, the value is computed as

$$
\begin{aligned}
|a_4| &\le \left|\frac{ne^{-i\alpha}\cos\alpha}{6(n+1)}\right|\left|\frac{1}{4}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{8(n+1)^2}-\frac{3ne^{-i\alpha}\cos\alpha}{8(n+1)}\right||c_1|^3 \\
&\quad+\left|\frac{ne^{-i\alpha}\cos\alpha}{6(n+1)}\right|\left|c_3-\left(1-\frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right)c_1c_2\right| \\
&=\frac{n\cos\alpha}{6(n+1)}\cdot\left|\frac{1}{4}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{8(n+1)^2}-\frac{3ne^{-i\alpha}\cos\alpha}{8(n+1)}\right|8 \\
&\quad+\frac{n\cos\alpha}{6(n+1)}\left|c_3-\left(1-\frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right)c_1c_2\right| \\
&=\frac{n\cos\alpha}{6(n+1)}\left|2+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{(n+1)^2}-\frac{3ne^{-i\alpha}\cos\alpha}{(n+1)}\right| \\
&\quad+\frac{n\cos\alpha}{6(n+1)}\left|c_3-\left(1-\frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right)c_1c_2\right|.
\end{aligned}
\tag{4.30}
$$

Using Lemma 3.2 with the choice of $\mu=1-\dfrac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}$ yields

$$
\begin{aligned}
\left|c_3-\left(1-\frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right)c_1c_2\right| &\le 2\max\left\{1,\left|2\left(1-\frac{3ne^{-i\alpha}\cos\alpha}{4(n+1)}\right)-1\right|\right\} \\
&=2\max\left\{1,\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}\right|\right\}.
\end{aligned}
\tag{4.31}
$$

Under the assumptions of $|\alpha|<\dfrac{\pi}{2}$ and $n\ge 2$, $\max\left\{1,\left|1-\dfrac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}\right|\right\}=1$. Hence, the bound of $|a_4|$ can be written as

$$|a_4| \le \frac{n\cos\alpha}{6(n+1)}\left|2+\frac{n^2 e^{-i2\alpha}\cos^2\alpha}{(n+1)^2}-\frac{3ne^{-i\alpha}\cos\alpha}{(n+1)}\right|+\frac{n\cos\alpha}{6(n+1)}(2)$$

$$=\frac{n\cos\alpha}{6(n+1)}\left[\left|2+\frac{n^2 e^{-i2\alpha}\cos^2\alpha}{(n+1)^2}-\frac{3ne^{-i\alpha}\cos\alpha}{(n+1)}\right|+2\right]. \tag{4.32}$$

Proof of Theorem 4.2.1 completes.

Setting $\alpha = 0$, yields the subsequent corollary:

Corollary 4.2.1.

Let $f \in S^*_{n-1,L}$. *Then,*

$$|a_2| \le \frac{n\cos(0)}{(n+1)} = \frac{n}{(n+1)}, \tag{4.33}$$

and

$$|a_3| \le \frac{n\cos(0)}{2(n+1)} = \frac{n}{2(n+1)}. \tag{4.34}$$

The inequality in (4.33) and (4.34) of Corollary 4.2.1 aligns to the sharp coefficient bounds obtained by Gandhi et al. (2022) for the class $S^*_{n-1,\mathrm{L}}$.

4.2.2 Fekete–Szegö Inequality

The Fekete–Szegö inequality is examined in light of the previously established coefficient estimates and bounds, which provides further insight into the functional behavior and geometric characteristics of the defined subclass.

Theorem 4.2.2

If $f \in S^*_{T,n-1}(\alpha)$ *be given in the form (1.1), then for any* $\lambda \in \mathbb{C}$, $n \ge 2$ *and* $|\alpha| < \frac{\pi}{2}$,

$$\left|a_3-\lambda a_2^2\right| \le \frac{n\cos\alpha}{2(n+1)}\max\left\{1,\left|\frac{(2\lambda-1)n\cos\alpha}{(n+1)}\right|\right\}. \tag{4.35}$$

Proof.

Assumed that $f \in S^*_{T,n-1}(\alpha)$. Substituting the values from equations (4.12) and (4.15) into the expression $a_3-\lambda a_2^2$ gives

$$\begin{aligned}
a_3-\lambda a_2^2 &= \frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)-\lambda\left(\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}c_1\right)^2\\
&= \frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)-\lambda\left(\frac{n^2\cos^2\alpha}{4(n+1)^2 e^{i2\alpha}}c_1^2\right)\\
&= \frac{n\cos\alpha}{8(n+1)^2 e^{i\alpha}}\left(2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)-\frac{\lambda n^2\cos^2\alpha}{4(n+1)^2 e^{i2\alpha}}c_1^2\\
&= \frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left(c_2-\frac{1}{2(n+1)}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2-\frac{\lambda n\cos\alpha}{(n+1)e^{i\alpha}}c_1^2\right)\\
&= \frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left(c_2-\left(\frac{1}{2}-\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}\right)c_1^2-\frac{\lambda n\cos\alpha}{(n+1)e^{i\alpha}}c_1^2\right)\\
&= \frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left(c_2-\left(\frac{1}{2}-\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}-\frac{\lambda n\cos\alpha}{(n+1)e^{i\alpha}}\right)c_1^2\right)\\
&= \frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left(c_2-\left(\frac{e^{i\alpha}(n+1)-n\cos\alpha+2\lambda n\cos\alpha}{2e^{i\alpha}(n+1)}\right)c_1^2\right)\\
&= \frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left(c_2-\frac{e^{i\alpha}(n+1)+n\cos\alpha(2\lambda-1)}{2e^{i\alpha}(n+1)}c_1^2\right).
\end{aligned} \tag{4.36}$$

Taking the modulus on both sides of the equation yields the following inequality

$$\left|a_3-\lambda a_2^2\right|\le\left|\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\right|\left|c_2-\frac{e^{i\alpha}(n+1)+n\cos\alpha(2\lambda-1)}{2e^{i\alpha}(n+1)}c_1^2\right|. \tag{4.37}$$

Subsequently, by applying Lemma 3.2 with $\mu=\frac{e^{i\alpha}(n+1)+n\cos\alpha(2\lambda-1)}{2e^{i\alpha}(n+1)}$ evaluates to

$$\begin{aligned}\left|a_3-\lambda a_2^2\right|&\le 2\left|\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\right|\max\left\{1,\left|2\left(\frac{e^{i\alpha}(n+1)+n\cos\alpha(2\lambda-1)}{2e^{i\alpha}(n+1)}\right)-1\right|\right\}\\&=\left|\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}\right|\max\left\{1,\left|\frac{e^{i\alpha}(n+1)+n\cos\alpha(2\lambda-1)}{e^{i\alpha}(n+1)}-1\right|\right\}\\&=\left|\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}\right|\max\left\{1,\left|\frac{e^{i\alpha}(n+1)+n\cos\alpha(2\lambda-1)-e^{i\alpha}(n+1)}{e^{i\alpha}(n+1)}\right|\right\}.\end{aligned} \tag{4.38}$$

Since $\left|e^{-i\alpha}\right|=1$, the Fekete–Szegö inequality is obtained as follows:

$$\left|a_3-\lambda a_2^2\right|\le\frac{n\cos\alpha}{2(n+1)}\max\left\{1,\left|\frac{(2\lambda-1)n\cos\alpha}{(n+1)}\right|\right\}. \tag{4.39}$$

Proof of Theorem 4.2.2 completes.

Setting $\alpha=0$ and $\lambda=1$, yields the subsequent corollary:

Corollary 4.2.2.

Let $f\in S^*_{n-1,L}$. *Then,*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{n\cos 0}{2(n+1)} \max\left\{1, \left|\frac{(2(1)-1)n\cos 0}{(n+1)}\right|\right\}$$

$$= \frac{n}{2(n+1)} \max\left\{1, \frac{n}{(n+1)}\right\}.$$

Since the maximum of $\left\{1, \frac{n}{(n+1)}\right\}$ *is 1, then*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{n}{2(n+1)}. \tag{4.40}$$

The inequality in (4.40) of Corollary 4.2.2 aligns to the sharp results obtained by Shi et al. (2022) for the class $S_{n-1,\mathrm{L}}^*$.

4.2.3 Second Hankel Determinant

Having established the Fekete–Szegö functional, the second Hankel determinant for the class $S_{T,n-1}^*(\alpha)$ is now investigated.

Theorem 4.2.3.

If $f \in S_{T,n-1}^*(\alpha)$ *be given in the form (1.1), then for any* $n \ge 2$ *and* $|\alpha| < \frac{\pi}{2}$,

$$\left|a_2 a_4 - a_3^2\right| \le \frac{n^2 \cos^2 \alpha}{4(n+1)^2}. \tag{4.41}$$

Proof.

Assumed that $f \in S_{T,n-1}^*(\alpha)$. Using the values given in (4.12), (4.15), and (4.20), the expression $a_2 a_4 - a_3^2$ is evaluated, begin by evaluating $a_2 a_4$ first:

$$a_2a_4=\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}c_1\left(\frac{n\cos\alpha}{48(n+1)^3e^{i\alpha}}\left[\begin{array}{l}\left(2(n+1)^2-2n^2\frac{\cos^2\alpha}{e^{i2\alpha}}-3n\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^3\\-\left((n+1)\left(8(n+1)^2-\frac{6n(n+1)\cos\alpha}{e^{i\alpha}}\right)\right)c_1c_2+8(n+1)^2c_3\end{array}\right]\right)$$

$$=\frac{\cos\alpha}{e^{i\alpha}}\left[\frac{n}{2(n+1)}c_1\cdot\frac{n}{48(n+1)^3}\left[\begin{array}{l}\left(2(n+1)^2-2n^2\frac{\cos^2\alpha}{e^{i2\alpha}}-3n\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^3\\-\left((n+1)\left(8(n+1)-\frac{6n\cos\alpha}{e^{i\alpha}}\right)\right)c_1c_2+8(n+1)^2c_3\end{array}\right]\right]$$

$$=\frac{\cos\alpha}{e^{i\alpha}}\left[\frac{n}{48(n+1)^3}\left[\begin{array}{l}\left(n(n+1)-\frac{n^3\cos^2\alpha}{(n+1)e^{i2\alpha}}-\frac{3n^2\cos\alpha}{2(n+1)e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^4\\-\left(\frac{n}{2}\right)\left(8(n+1)-\frac{6n\cos\alpha}{e^{i\alpha}}\right)c_1^2c_2+4n(n+1)c_1c_3\end{array}\right]\right]. \tag{4.42}$$

Upon simplification, the expression for a_2a_4 can be written as

$$a_2a_4=\frac{\cos\alpha}{e^{i\alpha}}\left[\frac{n}{48(n+1)^3}\left[\begin{array}{l}\left(n(n+1)-\frac{n^3\cos^2\alpha}{(n+1)e^{i2\alpha}}-\frac{3n^2\cos\alpha}{2(n+1)e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^4\\-\left(4n(n+1)-\frac{3n^2\cos\alpha}{e^{i\alpha}}\right)c_1^2c_2+4n(n+1)c_1c_3\end{array}\right]\right]. \tag{4.43}$$

Next, the expression a_3^2 is evaluated as

$$a_3^2=\left(\frac{n\cos\alpha}{8(n+1)^2e^{i\alpha}}\left[2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right]\right)^2$$

$$=\frac{n^2\cos^2\alpha}{64(n+1)^4e^{i2\alpha}}\left[\begin{array}{l}\left(2(n+1)c_2\right)^2-2\left(2(n+1)c_2\right)\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\\+\left(\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)^2\end{array}\right]. \tag{4.44}$$

After simplifying, a_3^2 is given by

$$a_3^2 = \frac{n^2\cos^2\alpha}{64(n+1)^4 e^{i2\alpha}} \begin{bmatrix} 4(n+1)^2 c_2^2 - 4(n+1)\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2 c_2 + \\ \left((n+1)^2 - \dfrac{2n(n+1)\cos\alpha}{e^{i\alpha}} + \dfrac{n^2\cos^2\alpha}{e^{i2\alpha}}\right)c_1^4 \end{bmatrix}. \tag{4.45}$$

The computed expressions for $a_2 a_4$ and a_3^2 are now substituted into $a_2 a_4 - a_3^2$.

$$a_2a_4 - a_3^2 = \frac{\cos\alpha}{e^{i\alpha}} \left[\frac{n}{48(n+1)^3} \begin{bmatrix} \left(n(n+1) - \dfrac{n^3\cos^2\alpha}{(n+1)e^{i2\alpha}} - \dfrac{3n^2\cos\alpha}{2(n+1)e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^4 \\ -\left(4n(n+1) - \dfrac{3n^2\cos\alpha}{e^{i\alpha}}\right)c_1^2c_2 + 4n(n+1)c_1c_3 \end{bmatrix} \right.$$

$$\left. - \frac{n^2\cos^2\alpha}{64(n+1)^4 e^{i2\alpha}} \begin{bmatrix} 4(n+1)^2 c_2^2 - 4(n+1)\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2 c_2 \\ +(n+1)^2 - \dfrac{2n(n+1)\cos\alpha}{e^{i\alpha}} + \dfrac{n^2\cos^2\alpha}{e^{i2\alpha}} c_1^4 \end{bmatrix} \right]$$

$$= \frac{\cos\alpha}{192(n+1)^4 e^{i\alpha}} \begin{bmatrix} 4n(n+1) \begin{bmatrix} \left(n(n+1) - \dfrac{n^3\cos^2\alpha}{(n+1)e^{i2\alpha}} - \dfrac{3n^2\cos\alpha}{2(n+1)e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^4 \\ -\left(4n(n+1) - \dfrac{3n^2\cos\alpha}{e^{i\alpha}}\right)c_1^2c_2 + 4n(n+1)c_1c_3 \end{bmatrix} \\ -3n^2\dfrac{\cos\alpha}{e^{i\alpha}} \begin{bmatrix} 4(n+1)^2 c_2^2 - 4(n+1)\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2 c_2 \\ +\left((n+1)^2 - \dfrac{2n(n+1)\cos\alpha}{e^{i\alpha}} + \dfrac{n^2\cos^2\alpha}{e^{i2\alpha}}\right) c_1^4 \end{bmatrix} \end{bmatrix}$$

$$= \frac{\cos\alpha}{192(n+1)^4 e^{i\alpha}} \begin{bmatrix} \left(4n^2(n+1)^2 - 4n^4\dfrac{\cos^2\alpha}{e^{i2\alpha}} - 6n^3\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^4 \\ -\left((4n(n+1))^2 - 12n^3(n+1)\dfrac{\cos\alpha}{e^{i\alpha}}\right)c_1^2c_2 \\ +(4n(n+1))^2 c_1c_3 - \left(\dfrac{12n^2(n+1)^2\cos\alpha}{e^{i\alpha}}\right)c_2^2 \\ +\left(\dfrac{12n^2(n+1)\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\right)c_1^2c_2 \\ -\left(\dfrac{3n^2(n+1)^2\cos\alpha}{e^{i\alpha}} - \dfrac{6n^3(n+1)\cos^2\alpha}{e^{i2\alpha}} + \dfrac{3n^4\cos^3\alpha}{e^{i3\alpha}}\right)c_1^4 \end{bmatrix}. \tag{4.46}$$

Following simplification, the resulting expression for $a_2a_4-a_3^2$ is

$$a_2a_4-a_3^2=\frac{\cos\alpha}{192(n+1)^4e^{i\alpha}}\left[\begin{array}{l}\left(\begin{array}{l}4n^2(n+1)^2-4n^4\dfrac{\cos^2\alpha}{e^{i2\alpha}}-6n^3\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)-\\ \dfrac{3n^2(n+1)^2\cos\alpha}{e^{i\alpha}}+\dfrac{6n^3(n+1)\cos^2\alpha}{e^{i2\alpha}}-\dfrac{3n^4\cos^3\alpha}{e^{i3\alpha}}\end{array}\right)c_1^4\\ -\left(\begin{array}{l}\left((4n(n+1))^2-12n^3(n+1)\dfrac{\cos\alpha}{e^{i\alpha}}\right)-\\ \dfrac{12n^2(n+1)\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\end{array}\right)c_1^2c_2\\ +(4n(n+1))^2c_1c_3-\left(\dfrac{12n^2(n+1)^2\cos\alpha}{e^{i\alpha}}\right)c_2^2\end{array}\right]. \tag{4.47}$$

By virtue of Lemma 3.3, the coefficients c_2 and c_3 may be represented in terms of c_1. Upon setting $c_1=c$ with $c\in[0,2]$, the corresponding expressions are obtained as follows:

$$a_2a_4-a_3^2=\frac{\cos\alpha}{192(n+1)^4e^{i\alpha}}\left[\begin{array}{l}\left(\begin{array}{l}4n^2(n+1)^2-4n^4\dfrac{\cos^2\alpha}{e^{i2\alpha}}-6n^3\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)-\\ \dfrac{3n^2(n+1)^2\cos\alpha}{e^{i\alpha}}+\dfrac{6n^3(n+1)\cos^2\alpha}{e^{i2\alpha}}-\dfrac{3n^4\cos^3\alpha}{e^{i3\alpha}}\end{array}\right)c^4\\ -\left(\begin{array}{l}\left((4n(n+1))^2-12n^3(n+1)\dfrac{\cos\alpha}{e^{i\alpha}}\right)-\\ \dfrac{12n^2(n+1)\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)\end{array}\right)c^2\dfrac{1}{2}\left(c^2+(4-c^2)x\right)\\ +(4n(n+1))^2c\left(\dfrac{1}{4}\left(c^3+2cx(4-c^2)-x^2c(4-c^2)+2(1-|x|^2)(4-c^2)\delta\right)\right)\\ -\left(\dfrac{12n^2(n+1)^2\cos\alpha}{e^{i\alpha}}\right)\left(\dfrac{1}{2}\left(c^2+(4-c^2)x\right)\right)^2\end{array}\right]. \tag{4.48}$$

Simplifying the expression within the brackets gives

$$\left(\begin{array}{l} 4n^2(n+1)^2-4n^4\dfrac{\cos^2\alpha}{e^{i2\alpha}}-6n^3\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)- \\ \dfrac{3n^2(n+1)^2\cos\alpha}{e^{i\alpha}}+\dfrac{6n^3(n+1)\cos^2\alpha}{e^{i2\alpha}}-\dfrac{3n^4\cos^3\alpha}{e^{i3\alpha}} \end{array}\right)c^4$$
$$-\left(\begin{array}{l} \left((4n(n+1))^2-12n^3(n+1)\dfrac{\cos\alpha}{e^{i\alpha}}\right)- \\ \dfrac{12n^2(n+1)\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right) \end{array}\right)c^2\frac{1}{2}\left(c^2+(4-c^2)x\right)$$
$$+(4n(n+1))^2c\left(\frac{1}{4}\left(c^3+2cx(4-c^2)-x^2c(4-c^2)+2\left(1-|x|^2\right)(4-c^2)\delta\right)\right)$$
$$-\left(\frac{12n^2(n+1)^2\cos\alpha}{e^{i\alpha}}\right)\left(\frac{1}{2}\left(c^2+(4-c^2)x\right)\right)^2$$

$$=\left(\begin{array}{l} 4n^2(n+1)^2-4n^4\dfrac{\cos^2\alpha}{e^{i2\alpha}}-6n^3\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)- \\ \dfrac{3n^2(n+1)^2\cos\alpha}{e^{i\alpha}}+\dfrac{6n^3(n+1)\cos^2\alpha}{e^{i2\alpha}}-\dfrac{3n^4\cos^3\alpha}{e^{i3\alpha}} \end{array}\right)c^4$$
$$-\left(\frac{1}{2}\left(\begin{array}{l} \left(16n^2(n+1)^2-12n^3(n+1)\dfrac{\cos\alpha}{e^{i\alpha}}\right)- \\ \dfrac{12n^2(n+1)\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right) \end{array}\right)\left(c^4+(4-c^2)c^2x\right)\right)$$
$$+\left(4n^2(n+1)^2\left(c^4+2c^4x(4-c^2)-x^2c^4(4-c^2)+2\left(1-|x|^2\right)(4-c^2)c\delta\right)\right)$$
$$-\left(\frac{1}{4}\left(\frac{12n^2(n+1)^2\cos\alpha}{e^{i\alpha}}\right)\left(c^2+(4-c^2)x\right)^2\right)$$

$$=\left(\begin{array}{l} 4n^4+8n^3+4n^2-4n^4\dfrac{\cos^2\alpha}{e^{i2\alpha}}-6n^3\dfrac{\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right)- \\ \dfrac{\cos\alpha}{e^{i\alpha}}\left(3n^4+6n^3+3n^2\right)+\dfrac{\cos^2\alpha}{e^{i2\alpha}}\left(6n^4+6n^3\right)-\dfrac{3n^4\cos^3\alpha}{e^{i3\alpha}} \end{array}\right)c^4$$
$$-\left(\left(\begin{array}{l} \left(8n^4+16n^3+8n^2-\left(6n^4+6n^3\right)\dfrac{\cos\alpha}{e^{i\alpha}}\right)- \\ \dfrac{\left(6n^3+6n^2\right)\cos\alpha}{e^{i\alpha}}\left(n+1-\dfrac{n\cos\alpha}{e^{i\alpha}}\right) \end{array}\right)\left(c^4+(4-c^2)c^2x\right)\right)$$
$$+\left(4n^4+8n^3+4n^2\left(c^4+2c^2x(4-c^2)-x^2c^2(4-c^2)+2\left(1-|x|^2\right)(4-c^2)c\delta\right)\right)$$
$$-\left(\left(\frac{3n^4+6n^3+3n^2\cos\alpha}{e^{i\alpha}}\right)\left(c^2+(4-c^2)x\right)^2\right).$$

Setting $A=\frac{\cos\alpha}{e^{i\alpha}}$,

$$\begin{pmatrix} 4n^4+8n^3+4n^2-4n^4\frac{\cos^2\alpha}{e^{i2\alpha}}-6n^3\frac{\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)- \\ \frac{\cos\alpha}{e^{i\alpha}}\left(3n^4+6n^3+3n^2\right)+\frac{\cos^2\alpha}{e^{i2\alpha}}\left(6n^4+6n^3\right)-\frac{3n^4\cos^3\alpha}{e^{i3\alpha}} \end{pmatrix}c^4$$
$$-\left(\left(\begin{pmatrix} 8n^4+16n^3+8n^2-\left(6n^4+6n^3\right)\frac{\cos\alpha}{e^{i\alpha}}\end{pmatrix}- \\ \frac{\left(6n^3+6n^2\right)\cos\alpha}{e^{i\alpha}}\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)\right)\left(c^4+\left(4-c^2\right)c^2x\right)\right)$$
$$+\left(4n^4+8n^3+4n^2\left(c^4+2c^2x\left(4-c^2\right)-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\right)$$
$$-\left(\left(\frac{3n^4+6n^3+3n^2\cos\alpha}{e^{i\alpha}}\right)\left(c^2+\left(4-c^2\right)x\right)^2\right)$$

$$\begin{aligned} &=\begin{pmatrix} 4n^4+8n^3+4n^2-4n^4A^2-6n^3A\left(n+1-nA\right)- \\ A\left(3n^4+6n^3+3n^2\right)+A^2\left(6n^4+6n^3\right)-3n^4A^3 \end{pmatrix}c^4 \\ &-\left(\begin{pmatrix} \left(8n^4+16n^3+8n^2-\left(6n^4+6n^3\right)A\right)- \\ \left(6n^3+6n^2\right)A\left(n+1-nA\right) \end{pmatrix}\left(c^4+\left(4-c^2\right)c^2x\right)\right) \\ &+\left(4n^4+8n^3+4n^2\left(c^4+2c^2x\left(4-c^2\right)-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\right) \\ &-\left(\left(3n^4+6n^3+3n^2\right)A\left(c^2+\left(4-c^2\right)x\right)^2\right). \end{aligned} \tag{4.49}$$

Expansion of equation (4.49) results in

$$\begin{pmatrix} 4n^4+8n^3+4n^2-4n^4A^2-6n^3A\left(n+1-nA\right)- \\ A\left(3n^4+6n^3+3n^2\right)+A^2\left(6n^4+6n^3\right)-3n^4A^3 \end{pmatrix}c^4$$

$$-\left(\begin{pmatrix} \left(8n^4+16n^3+8n^2-\left(6n^4+6n^3\right)A\right)- \\ \left(6n^3+6n^2\right)A\left(n+1-nA\right) \end{pmatrix}\left(c^4+\left(4-c^2\right)c^2x\right)\right)$$

$$+\left(4n^4+8n^3+4n^2\left(c^4+2c^2x\left(4-c^2\right)-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\right)$$

$$-\left(\left(3n^4+6n^3+3n^2\right)A\left(c^2+\left(4-c^2\right)x\right)^2\right)$$

$$=\begin{pmatrix} 4n^4+8n^3+4n^2-4n^4A^2-6n^4A-6n^3A+6n^4A^2- \\ 3n^4A-6n^3A-3n^2A+6n^4A^2+6n^3A^2-3n^4A^3 \end{pmatrix}c^4$$

$$-\left(\begin{pmatrix} \left(8n^4+16n^3+8n^2-6n^4A-6n^3A\right)- \\ \left(\left(6n^3A+6n^2A\right)\left(n+1-nA\right)\right) \end{pmatrix}\left(c^4+\left(4-c^2\right)c^2x\right)\right)$$

$$+\left(4n^4+8n^3+4n^2\left(c^4+2c^2x\left(4-c^2\right)-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\right)$$

$$-\left(\left(3n^4A+6n^3A+3n^2A\right)\left(c^4+2c^2\left(4-c^2\right)x+\left(4-c^2\right)^2x^2\right)\right)$$

$$\begin{aligned}
&=\begin{pmatrix} 4n^4+8n^3+4n^2-4n^4A^2-6n^4A-6n^3A+6n^4A^2 \\ -3n^4A-6n^3A-3n^2A+6n^4A^2+6n^3A^2-3n^4A^3 \end{pmatrix}c^4 \\
&-\begin{pmatrix} 8n^4+16n^3+8n^2-6n^4A-6n^3A-6n^4A-6n^3A \\ +6n^4A^2-6n^3A-6n^2A+6n^3A^2 \end{pmatrix}\left(c^4+\left(4-c^2\right)c^2x\right) \\
&+\left(\left(4n^4+8n^3+4n^2\right)\left(c^4+2\left(4-c^2\right)c^2x-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\right) \\
&-\left(3n^4A+6n^3A+3n^2A\right)\left(c^4+2\left(4-c^2\right)c^2x+\left(4-c^2\right)^2x^2\right).
\end{aligned} \tag{4.50}$$

Grouping the equation in terms of the variables gives

$$
\begin{aligned}
&\begin{pmatrix} 4n^4+8n^3+4n^2-4n^4A^2-6n^4A-6n^3A+6n^4A^2 \\ -3n^4A-6n^3A-3n^2A+6n^4A^2+6n^3A^2-3n^4A^3 \end{pmatrix} c^4 \\
&-\begin{pmatrix} 8n^4+16n^3+8n^2-6n^4A-6n^3A-6n^4A-6n^3A \\ +6n^4A^2-6n^3A-6n^2A+6n^3A^2 \end{pmatrix}\left(c^4+\left(4-c^2\right)c^2x\right) \\
&+\left(\left(4n^4+8n^3+4n^2\right)\left(c^4+2\left(4-c^2\right)c^2x-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\right) \\
&-\left(3n^4A+6n^3A+3n^2A\right)\left(c^4+2\left(4-c^2\right)c^2x+\left(4-c^2\right)^2x^2\right).
\end{aligned}
$$

$$
\begin{aligned}
&=\left(4-4A^2-6A+6A^2-3A+6A^2-3A^3-8+6A+6A-6A^2+4-3A\right)n^4\mathrm{c}^4 \\
&+\left(8-6A-6A+6A^2-16+6A+6A+6A-6A^2+8-6A\right)n^3\mathrm{c}^4 \\
&+\left(4-3A-8+6A+4-3A\right)n^2\mathrm{c}^4 \\
&+\begin{pmatrix} -\begin{pmatrix} 8n^4+16n^3+8n^2-6n^4A-6n^3A-6n^4A-6n^3A+6n^4A^2 \\ -6n^3A-6n^2A+6n^3A^2 \end{pmatrix} \\ +2\left(4n^4+8n^3+4n^2\right)-2\left(3n^4A+6n^3A+3n^2A\right) \end{pmatrix}\left(4-c^2\right)c^2x \\
&-\left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right)+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-|x|^2\right)c\delta \\
&-\left(3n^4A+6n^3A+3n^2A\right)\left(4-c^2\right)^2x^2
\end{aligned}
$$

$$
\begin{aligned}
&=\left(2A^2-3A^3\right)n^4\mathrm{c}^4 \\
&-\begin{pmatrix} -8n^4+8n^4-16n^3+16n^3-8n^2+8n^2+6n^4A+6n^4A-6n^4A^2 \\ -6n^4A+6n^3A+6n^3A+6n^3A-6n^3A^2-12n^3A+6n^2A-6n^2A \end{pmatrix}\left(4-c^2\right)c^2x \\
&-\left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right)+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-|x|^2\right)c\delta \\
&-A\left(3n^4+6n^3+3n^2\right)x^2\left(4-c^2\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&=\left(2A^2-3A^3\right)n^4\mathrm{c}^4 \\
&+6A\left(n^4-n^4A+n^3-n^3A\right)\left(4-c^2\right)c^2x \\
&-\left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right)+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-|x|^2\right)c\delta \\
&-A\left(3n^4+6n^3+3n^2\right)x^2\left(4-c^2\right)^2.
\end{aligned}
\tag{4.51}
$$

After simplifying the expression and grouping terms by variable, the substitution of $A=\frac{\cos\alpha}{e^{i\alpha}}$ is reverted to obtain the final form as

$$a_2a_4-a_3^2=\frac{\cos\alpha}{192(n+1)^4e^{i\alpha}}\left[\begin{array}{l}-\frac{\cos\alpha}{e^{i\alpha}}\left(3n^4+6n^3+3n^2\right)x^2\left(4-c^2\right)^2\\+\left(2\frac{\cos^2\alpha}{e^{i2\alpha}}-3\frac{\cos^3\alpha}{e^{i3\alpha}}\right)n^4\mathrm{c}^4\\+6\frac{\cos\alpha}{e^{i\alpha}}\left(n^4-n^4\frac{\cos\alpha}{e^{i\alpha}}+n^3-n^3\frac{\cos\alpha}{e^{i\alpha}}\right)\left(4-c^2\right)c^2x\\+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-|x|^2\right)c\delta\\-\left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right)\end{array}\right]. \qquad (4.52)$$

Then, equation (4.52) is expanded and modulus both sides, yielding

$$\left|a_2a_4-a_3^2\right|=\left|\frac{\cos\alpha}{192(n+1)^4e^{i\alpha}}\left[\begin{array}{l}-\frac{\cos\alpha}{e^{i\alpha}}\left(3n^4+6n^3+3n^2\right)x^2\left(4-c^2\right)^2\\+2\frac{\cos^2\alpha}{e^{i2\alpha}}n^4\mathrm{c}^4-3\frac{\cos^3\alpha}{e^{i3\alpha}}n^4\mathrm{c}^4\\+6\frac{\cos\alpha}{e^{i\alpha}}n^4\left(4-c^2\right)c^2x-6\frac{\cos^2\alpha}{e^{i2\alpha}}n^4\left(4-c^2\right)c^2x\\+6\frac{\cos\alpha}{e^{i\alpha}}n^3\left(4-c^2\right)c^2x-6\frac{\cos^2\alpha}{e^{i2\alpha}}n^3\left(4-c^2\right)c^2x\\+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-|x|^2\right)c\delta-\left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right)\end{array}\right]\right|. \qquad (4.53)$$

Subsequently, the application of the triangle inequality yields

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{192(n+1)^4}\left|\frac{\cos\alpha}{e^{i\alpha}}\right|\left[\begin{array}{l}\left|-\frac{\cos\alpha}{e^{i\alpha}}\left(3n^4+6n^3+3n^2\right)\right|\left|x\right|^2\left|\left(4-c^2\right)^2\right|+\left|2\frac{\cos^2\alpha}{e^{i2\alpha}}n^4\mathrm{c}^4\right|\\+\left|-3\frac{\cos^3\alpha}{e^{i3\alpha}}n^4\mathrm{c}^4\right|+\left|6\frac{\cos\alpha}{e^{i\alpha}}n^4\left(4-c^2\right)c^2x\right|\\+\left|-6\frac{\cos^2\alpha}{e^{i2\alpha}}n^4\left(4-c^2\right)c^2x\right|+\left|6\frac{\cos\alpha}{e^{i\alpha}}n^3\left(4-c^2\right)c^2x\right|\\+\left|-6\frac{\cos^2\alpha}{e^{i2\alpha}}n^3\left(4-c^2\right)c^2x\right|\\+\left|\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-\left|x\right|^2\right)c\delta\right|\\+\left|-\left(4n^4+8n^3+4n^2\right)x^2c^2\left(4-c^2\right)\right|\end{array}\right]. \quad (4.54)$$

Given that $\left|e^{i\alpha}\right|=\left|e^{i2\alpha}\right|=\left|e^{i3\alpha}\right|=1$, the expression simplifies to

$$\left|a_2a_4-a_3^2\right|\le\frac{\cos\alpha}{192(n+1)^4}\left[\begin{array}{l}\cos\alpha\left(3n^4+6n^3+3n^2\right)\left|x\right|^2\left(4-c^2\right)^2+2n^4\mathrm{c}^4\cos^2\alpha\\+3n^4\mathrm{c}^4\cos^3\alpha+6n^4\left(4-c^2\right)c^2\left|x\right|\cos\alpha\\+6n^4\left(4-c^2\right)c^2\left|x\right|\cos^2\alpha+6n^3\left(4-c^2\right)c^2\left|x\right|\cos\alpha\\+6n^3\left(4-c^2\right)c^2\left|x\right|\cos^2\alpha\\+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-\left|x\right|^2\right)c\delta\\+\left(4n^4+8n^3+4n^2\right)\left|x\right|^2c^2\left(4-c^2\right)\end{array}\right]. \quad (4.55)$$

Letting $\left|x\right|=y$ where $y\le1$, and assuming $\left|\delta\right|\le1$, leading to

$$\left|a_2a_4-a_3^2\right|\le\frac{\cos\alpha}{192(n+1)^4}\left\{\begin{array}{l}\cos\alpha\left(3n^4+6n^3+3n^2\right)y^2\left(4-c^2\right)^2+2n^4\mathrm{c}^4\cos^2\alpha\\+3n^4\mathrm{c}^4\cos^3\alpha+6n^4\left(4-c^2\right)c^2y\cos\alpha\\+6n^4\left(4-c^2\right)c^2y\cos^2\alpha+6n^3\left(4-c^2\right)c^2y\cos\alpha\\+6n^3\left(4-c^2\right)c^2y\cos^2\alpha\\+\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)\left(1-y^2\right)c\\+\left(4n^4+8n^3+4n^2\right)y^2c^2\left(4-c^2\right)\end{array}\right\}:=\mathrm{G}(c,y). \quad (4.56)$$

Next, the upper bound of (4.56) is assumed to occur at $c \times y = [0,2] \times [0,1]$. Differentiating equation (4.56) with respect to y results in

$$\frac{\partial G}{\partial y} = \frac{\cos\alpha}{192(n+1)^4} \left\{ \begin{array}{l} 2\cos\alpha\left(3n^4 + 6n^3 + 3n^2\right)\left(4-c^2\right)^2 y + 6n^4\left(4-c^2\right)c^2\cos\alpha \\ +6n^4\left(4-c^2\right)c^2\cos^2\alpha + 6n^3\left(4-c^2\right)c^2\cos\alpha \\ +6n^3\left(4-c^2\right)c^2\cos^2\alpha - 2\left(8n^4+16n^3+8n^2\right)\left(4-c^2\right)cy \\ +2\left(4n^4+8n^3+4n^2\right)\left(4-c^2\right)c^2 y \end{array} \right\}. \tag{4.57}$$

Since $\cos\alpha > 0$ and $(n+1)^4 > 0$, it follows that $\dfrac{\cos\alpha}{192(n+1)^4} > 0$.

Next, observe that each term in the expression contains the factor $\left(4-c^2\right)$. Factoring this out, yields

$$\frac{\partial G}{\partial y} = \frac{\cos\alpha}{192(n+1)^4}\left(4-c^2\right)H(c,y), \tag{4.58}$$

where $H(c,y)$ denotes the remaining expression.

Now, for $0 \le c \le 2$, it follows that $4-c^2 \ge 0$, and for $0 \le y \le 1$, we have $y \ge 0$. Furthermore, since $n \ge 2$, all powers of n are positive by assumption.

By inspection, each term in $H(c,y)$ consists of products of nonnegative quantities such as $\left(4-c^2\right)$, c, c^2, y, and positive constants involving n and $\cos\alpha$. Hence, $H(c,y) \ge 0$.

Given $0 \le y \le 1$ and for any fixed c with $0 \le c \le 2$, it follows that $\dfrac{\partial G}{\partial y} > 0$. Hence, $G(c,y)$ is strictly increasing in y, and consequently, $G(c,y) \le G(c,1) = G(c)$. Therefore, it follows that

$$G(c)=\frac{\cos\alpha}{192(n+1)^4}\left\{\begin{array}{l}\cos\alpha\left(3n^4+6n^3+3n^2\right)\left(4-c^2\right)^2+2n^4c^4\cos^2\alpha\\+3n^4c^4\cos^3\alpha+6n^4\left(4-c^2\right)c^2\cos\alpha\\+6n^4\left(4-c^2\right)c^2\cos^2\alpha+6n^3\left(4-c^2\right)c^2\cos\alpha\\+6n^3\left(4-c^2\right)c^2\cos^2\alpha+\left(4n^4+8n^3+4n^2\right)c^2\left(4-c^2\right)\end{array}\right\}. \tag{4.59}$$

Then, differentiating (4.59) results in

$$G'(c)=\frac{\cos\alpha}{192(n+1)^4}\left\{\begin{array}{l}\cos\alpha\left(3n^4+6n^3+3n^2\right)2\left(4-c^2\right)(-2c)+8n^4c^3\cos^2\alpha\\+12n^4c^3\cos^3\alpha+6n^4\left(\left(4-c^2\right)2c+c^2(-2c)\right)\cos\alpha\\+6n^4\left(\left(4-c^2\right)2c+c^2(-2c)\right)\cos^2\alpha\\+6n^3\left(\left(4-c^2\right)2c+c^2(-2c)\right)\cos\alpha\\+6n^3\left(\left(4-c^2\right)2c+c^2(-2c)\right)\cos^2\alpha\\+\left(4n^4+8n^3+4n^2\right)\left(\left(4-c^2\right)2c+c^2(-2c)\right)\end{array}\right\}$$

$$=\frac{\cos\alpha}{192(n+1)^4}\left\{\begin{array}{l}4\left(3n^4+6n^3+3n^2\right)\left(c^3-4c\right)\cos\alpha+4\left(2n^4c^3\cos^2\alpha+3n^4c^3\cos^3\alpha\right)\\+\left(\left(\begin{array}{l}6n^4\cos\alpha+6n^4\cos^2\alpha\\+6n^3\cos\alpha+6n^3\cos^2\alpha\end{array}\right)+\left(4n^4+8n^3+4n^2\right)\right)\left(8c-4c^3\right)\end{array}\right\}$$

$$=\frac{\cos\alpha}{192(n+1)^4}\left\{\begin{array}{l}4\left(3n^4+6n^3+3n^2\right)\left(c^3-4c\right)\cos\alpha+4\left(2n^4c^3\cos^2\alpha+3n^4c^3\cos^3\alpha\right)\\+4\left(\begin{array}{l}6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\+\left(4n^4+8n^3+4n^2\right)\end{array}\right)\left(2c-c^3\right)\end{array}\right\}$$

$$=\frac{\cos\alpha}{48(n+1)^4}\left\{\begin{array}{l}\left(3n^4+6n^3+3n^2\right)\left(c^3-4c\right)\cos\alpha+\left(2n^4c^3\cos^2\alpha+3n^4c^3\cos^3\alpha\right)\\+\left(\begin{array}{l}6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\+\left(4n^4+8n^3+4n^2\right)\end{array}\right)\left(2c-c^3\right)\end{array}\right\}. \tag{4.60}$$

The solution for c is obtained by setting $G'(c)=0$. Simplifying the contents of the brackets and grouping the terms with respect to each variable result in

$$\left(3n^4+6n^3+3n^2\right)\left(c^3-4c\right)\cos\alpha+\left(2n^4\mathrm{c}^3\cos^2\alpha+3n^4\mathrm{c}^3\cos^3\alpha\right)+$$
$$\left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)\left(2c-c^3\right)$$

$$=\left(3n^4+6n^3+3n^2\right)c^3\cos\alpha-4\left(3n^4+6n^3+3n^2\right)c\cos\alpha+2n^4\mathrm{c}^3\cos^2\alpha+$$
$$3n^4\mathrm{c}^3\cos^3\alpha+2\left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)c$$
$$-\left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)c^3$$

$$\left(3n^4+6n^3+3n^2\right)c^3\cos\alpha-4\left(3n^4+6n^3+3n^2\right)c\cos\alpha+2n^4\mathrm{c}^3\cos^2\alpha$$
$$+3n^4\mathrm{c}^3\cos^3\alpha+2\left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)c$$
$$-\left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)c^3$$

$$=\begin{pmatrix}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha-\\ \left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)\end{pmatrix}c^3$$
$$+\begin{pmatrix}2\left(6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+\left(4n^4+8n^3+4n^2\right)\right)\\ -4\left(3n^4+6n^3+3n^2\right)\cos\alpha\end{pmatrix}c$$

$$=\begin{pmatrix}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha-\\ 6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right)\end{pmatrix}c^3$$
$$+\begin{pmatrix}12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)+2\left(4n^4+8n^3+4n^2\right)\\ -4\left(3n^4+6n^3+3n^2\right)\cos\alpha\end{pmatrix}c. \tag{4.61}$$

As a result, the following expression is obtained:

$$G'(c)=\frac{\cos\alpha}{48(n+1)^4}\left\{\begin{array}{l}\begin{pmatrix}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha-\\ 6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ -\left(4n^4+8n^3+4n^2\right)\end{pmatrix}c^3\\ +\begin{pmatrix}12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ +2\left(4n^4+8n^3+4n^2\right)-4\left(3n^4+6n^3+3n^2\right)\cos\alpha\end{pmatrix}c\end{array}\right\}=0. \tag{4.62}$$

The expression is solved to get the solutions of c.

$$\frac{\cos\alpha}{48(n+1)^4}\left\{\begin{array}{l}\left(\begin{array}{l}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha\\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ -\left(4n^4+8n^3+4n^2\right)\end{array}\right)c^3\\ +\left(\begin{array}{l}12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ +2\left(4n^4+8n^3+4n^2\right)-4\left(3n^4+6n^3+3n^2\right)\cos\alpha\end{array}\right)c\end{array}\right\}=0$$

$$\begin{array}{l}\left(\begin{array}{l}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha\\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ -\left(4n^4+8n^3+4n^2\right)\end{array}\right)c^3\\ +\left(\begin{array}{l}12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ +2\left(4n^4+8n^3+4n^2\right)-4\left(3n^4+6n^3+3n^2\right)\cos\alpha\end{array}\right)c=0\end{array}$$

$$c\left[\begin{array}{l}\left(\begin{array}{l}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha\\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ -\left(4n^4+8n^3+4n^2\right)\end{array}\right)c^2\\ +\left(\begin{array}{l}12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\\ +2\left(4n^4+8n^3+4n^2\right)-4\left(3n^4+6n^3+3n^2\right)\cos\alpha\end{array}\right)\end{array}\right]=0. \tag{4.63}$$

Upon simplifying equation (4.63), the results are

$$c=0$$

and

$$\begin{pmatrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha \\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right) \end{pmatrix}c^2+$$
$$\begin{pmatrix} 12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right) \\ +2\left(4n^4+8n^3+4n^2\right)-4\left(3n^4+6n^3+3n^2\right)\cos\alpha \end{pmatrix}=0$$

$$\begin{pmatrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha \\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right) \end{pmatrix}c^2$$
$$=-\begin{pmatrix} 12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right) \\ +2\left(4n^4+8n^3+4n^2\right)-4\left(3n^4+6n^3+3n^2\right)\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha \\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right) \end{pmatrix}c^2$$
$$=\begin{pmatrix} -12\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right) \\ -2\left(4n^4+8n^3+4n^2\right)+4\left(3n^4+6n^3+3n^2\right)\cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha \\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right) \end{pmatrix}c^2$$
$$=4\begin{bmatrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha-2\left(n^4+2n^3+n^2\right) \\ -3\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right) \end{bmatrix}$$

$$c^2=\frac{4\begin{bmatrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha-2\left(n^4+2n^3+n^2\right) \\ -3\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right) \end{bmatrix}}{\begin{matrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha \\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right) \end{matrix}}$$

$$c=\pm\sqrt{4}\sqrt{\frac{\begin{matrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha-2\left(n^4+2n^3+n^2\right) \\ -3\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right) \end{matrix}}{\begin{matrix} \left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha \\ -6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right) \end{matrix}}}.$$

Hence, the solutions of c are

$$c=0,\ c=\pm 2\sqrt{\frac{\begin{array}{c}\left(3n^4+6n^3+3n^2\right)\cos\alpha-2\left(n^4+2n^3+n^2\right)-\\ 3\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)\end{array}}{\begin{array}{l}\left(3n^4+6n^3+3n^2\right)\cos\alpha+2n^4\cos^2\alpha+3n^4\cos^3\alpha-\\ 6\left(n^4\cos\alpha+n^4\cos^2\alpha+n^3\cos\alpha+n^3\cos^2\alpha\right)-\left(4n^4+8n^3+4n^2\right)\end{array}}}. \tag{4.64}$$

Under the assumption $c\in[0,2]$, the maximum of $G(c)$ is attained at $c=0$. Accordingly, from equation (4.56), the upper bound of $G(c,y)$ corresponds to $c=0$ and $y=1$, resulting in

$$\left|a_2a_4-a_3^2\right|\leq\frac{\cos\alpha}{192(n+1)^4}\left\{\begin{array}{l}\cos\alpha\left(3n^4+6n^3+3n^2\right)1^2\left(4-0^2\right)^2+2n^4(0)^4\cos^2\alpha\\ +3n^4(0)^4\cos^3\alpha+6n^4\left(4-0^2\right)(0)^2(1)\cos\alpha\\ +6n^4\left(4-c^2\right)(0)^2(1)\cos^2\alpha\\ +6n^3\left(4-0^2\right)0^2 1\cos\alpha+6n^3\left(4-c^2\right)(0)^2(1)\cos^2\alpha\\ +\left(8n^4+16n^3+8n^2\right)\left(4-0^2\right)\left(1-1^2\right)(0)\\ +\left(4n^4+8n^3+4n^2\right)(1)^2(0)^2\left(4-0^2\right)\end{array}\right\}$$

$$=\frac{\cos\alpha}{192(n+1)^4}\left(\cos\alpha\left(3n^4+6n^3+3n^2\right)\left(4-0^2\right)^2\right)$$

$$=\frac{\left(3n^4+6n^3+3n^2\right)(16)}{192(n+1)^4}\cos^2\alpha$$

$$=\frac{48\left(n^4+2n^3+n^2\right)}{192(n+1)^4}\cos^2\alpha$$

$$=\frac{n^2(n+1)^2}{4(n+1)^4}\cos^2\alpha. \tag{4.65}$$

Finally, the upper bound for the second Hankel determinant is given by

$$\left|a_2a_4 - a_3^2\right| \leq \frac{n^2\cos^2\alpha}{4(n+1)^2}. \tag{4.66}$$

Proof of Theorem 4.2.3 completes.

Setting $\alpha = 0$, the subsequent corollary is obtained:

Corollary 4.2.3.

Let $f \in S^*_{n-1,L}$. Then,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{n^2\cos^2(0)}{4(n+1)^2} = \frac{n^2}{4(n+1)^2}. \tag{4.67}$$

The inequality in (4.67) of Corollary 4.2.3 aligns to the sharp bounds obtained by Shi et al. (2022) for the class $S^*_{n-1,\mathrm{L}}$.

4.2.4 Toeplitz Determinant

Having established the inequality for the second Hankel determinant, attention is now turned to the Toeplitz determinant. In this section, three specific cases, namely $T_2(2)$, $T_3(1)$ and $T_3(2)$, are considered and their respective bounds are determined for functions belonging to the class $S^*_{T,n-1}(\alpha)$.

Theorem 4.2.4.

Let $f \in S^*_{T,n-1}(\alpha)$ be the expression given in the form (1.1). If $n \geq 2$ and $|\alpha| < \frac{\pi}{2}$, then

$$\left|T_2(2)\right| \leq \frac{5n^2\cos^2\alpha}{4(n+1)^2}. \tag{4.68}$$

Proof.

Assumed that $f \in S^{*}_{T,n-1}(\alpha)$. Then, the expression for the Toeplitz determinant $T_2(2)$ is given by

$$|T_2(2)| = |a_2^2 - a_3^2|. \tag{4.69}$$

Applying the triangle inequality gives

$$|T_2(2)| \le |a_2^2| + |a_3^2|. \tag{4.70}$$

Squaring $|a_2|$ and $|a_3|$ from equations (4.22) and (4.27) gives the following expressions:

$$|a_2^2| \le \left(\frac{n\cos\alpha}{(n+1)}\right)^2$$

$$= \frac{n^2\cos^2\alpha}{(n+1)^2}, \tag{4.71}$$

and

$$|a_3^2| \le \left(\frac{n\cos\alpha}{2(n+1)}\right)^2$$

$$= \frac{n^2\cos^2\alpha}{4(n+1)^2}. \tag{4.72}$$

Substituting (4.71) and (4.72) into (4.70) yields the desired results

$$\left|T_2(2)\right| \le \frac{n^2\cos^2\alpha}{(n+1)^2}+\frac{n^2\cos^2\alpha}{4(n+1)^2}$$

$$=\frac{5n^2\cos^2\alpha}{4(n+1)^2}. \tag{4.73}$$

Proof of Theorem 4.2.4 completes.

Setting $\alpha=0$, the subsequent corollary is obtained:

Corollary 4.2.4.

Let $f \in S^*_{T,n-1}(0)$. *Then,*

$$\left|T_2(2)\right| \le \frac{5n^2}{4(n+1)^2}. \tag{4.74}$$

Theorem 4.2.5.

Let $f \in S^*_{T,n-1}(\alpha)$ *be the expression given in the form (1.1). If* $n \ge 2$ *and* $|\alpha| < \frac{\pi}{2}$ *, then*

$$\left|T_3(1)\right| \le 1+\frac{n^2\cos^2\alpha}{(n+1)^2}\left[2+\frac{1}{4}\max\left\{1,\left|\frac{3n\cos\alpha}{(n+1)}\right|\right\}\right]. \tag{4.75}$$

Proof.

Assumed that $f \in S^*_{T,n-1}(\alpha)$. Then, by expanding the Toeplitz determinant $T_3(1)$, we obtain

$$\left|T_3(1)\right| \le \left|1-2a_2^2+2a_2^2a_3-a_3^2\right|$$

$$=1+2\left|a_2^2\right|+\left|a_3\right|\left|a_3-2a_2^2\right|. \tag{4.76}$$

Evaluation of $a_3-2a_2^2$ using equations (4.12) and (4.15) gives

$$a_3-2a_2^2=\frac{n\cos\alpha}{8(n+1)^2e^{i\alpha}}\left(2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)-2\left(\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}c_1\right)^2$$

$$=\frac{n\cos\alpha}{8(n+1)^2e^{i\alpha}}\left(2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2\right)-\left(\frac{n^2\cos^2\alpha}{2(n+1)^2e^{i2\alpha}}c_1^2\right)$$

$$=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[\frac{1}{2(n+1)}\left(2(n+1)c_2-\left(n+1-\frac{n\cos\alpha}{e^{i\alpha}}\right)c_1^2-\frac{2n\cos\alpha}{e^{i\alpha}}c_1^2\right)\right]$$

$$=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[c_2-\left(\frac{1}{2}-\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}\right)c_1^2-\frac{2n\cos\alpha}{(n+1)e^{i\alpha}}c_1^2\right]$$

$$=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[c_2-\left(\frac{1}{2}-\frac{n\cos\alpha}{2(n+1)e^{i\alpha}}+\frac{2n\cos\alpha}{(n+1)e^{i\alpha}}\right)c_1^2\right]$$

$$=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[c_2-\left(\frac{(n+1)e^{i\alpha}-n\cos\alpha+4n\cos\alpha}{2(n+1)e^{i\alpha}}\right)c_1^2\right]$$

$$=\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[c_2-\left(\frac{(n+1)e^{i\alpha}+3n\cos\alpha}{2(n+1)e^{i\alpha}}\right)c_1^2\right]. \tag{4.77}$$

Taking modulus of both sides yields

$$\left|a_3-2a_2^2\right|\le\left|\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\left[c_2-\left(\frac{(n+1)e^{i\alpha}+3n\cos\alpha}{2(n+1)e^{i\alpha}}\right)c_1^2\right]\right|. \tag{4.78}$$

Applying Lemma 3.2 with $\mu=\dfrac{(n+1)e^{i\alpha}+3n\cos\alpha}{2(n+1)e^{i\alpha}}$ with $\left|e^{i\alpha}\right|=1$ results in

$$\left|a_3 - 2a_2^2\right| \le 2\left|\frac{n\cos\alpha}{4(n+1)e^{i\alpha}}\right| \max\left\{1, \left|2\left(\frac{(n+1)e^{i\alpha} + 3n\cos\alpha}{2(n+1)e^{i\alpha}}\right) - 1\right|\right\}$$

$$= \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \left|2\left(\frac{(n+1)e^{i\alpha} + 3n\cos\alpha}{2(n+1)e^{i\alpha}}\right) - 1\right|\right\}$$

$$= \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \left|\left(\frac{(n+1)e^{i\alpha} + 3n\cos\alpha}{(n+1)e^{i\alpha}}\right) - 1\right|\right\}$$

$$= \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \left|\frac{(n+1)e^{i\alpha} + 3n\cos\alpha - (n+1)e^{i\alpha}}{(n+1)e^{i\alpha}}\right|\right\}$$

$$= \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \frac{3n\cos\alpha}{(n+1)e^{i\alpha}}\right\}. \tag{4.79}$$

Noting once more that $\left|e^{i\alpha}\right| = 1$, it follows that

$$\left|a_3 - 2a_2^2\right| \le \frac{n\cos\alpha}{2(n+1)} \max\left\{1, \frac{3n\cos\alpha}{(n+1)}\right\}. \tag{4.80}$$

Next, by substituting equations from (4.27), (4.71), and (4.80) into (4.76) gives

$$\left|T_3(1)\right| \le 1 + 2\left|a_2^2\right| + \left|a_3\right|\left|a_3 - 2a_2^2\right|$$

$$= 1 + 2\left(\frac{n^2\cos^2\alpha}{(n+1)^2}\right) + \frac{n\cos\alpha}{2(n+1)}\left(\frac{n\cos\alpha}{2(n+1)} \max\left\{1, \frac{3n\cos\alpha}{(n+1)}\right\}\right). \tag{4.81}$$

Proceeding with the simplification, it follows that

$$\left|T_3(1)\right| \le 1 + \frac{2n^2\cos^2\alpha}{(n+1)^2} + \frac{n^2\cos^2\alpha}{4(n+1)^2}\max\left\{1, \frac{3n\cos\alpha}{(n+1)}\right\}$$

$$= 1 + \frac{n^2\cos^2\alpha}{(n+1)^2}\left[2 + \frac{1}{4}\max\left\{1, \frac{3n\cos\alpha}{(n+1)}\right\}\right]. \tag{4.82}$$

Proof of Theorem 4.2.5 completes.

Setting $\alpha = 0$, the subsequent corollary is obtained:

Corollary 4.2.5.

Let $f \in S^*_{T,n-1}(0)$. Then,

$$\left|T_3(1)\right| \le 1 + \frac{n^2}{(n+1)^2}\left[2 + \frac{1}{4}\max\left\{1, \frac{3n}{(n+1)}\right\}\right]. \tag{4.83}$$

Theorem 4.2.6.

Let $f \in S^*_{T,n-1}(\alpha)$ be the expression given in the form (1.1). If $n \ge 2$ and $|\alpha| < \frac{\pi}{2}$, then

$$\left|T_3(2)\right| \le \frac{3n^3\cos^3\alpha}{2(n+1)^3}\left(\frac{1}{3}\rho + \frac{4}{3}\right), \tag{4.84}$$

where $\rho = \left|1 - \frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)} + \frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|.$

Proof.

Assumed that $f \in S^*_{T,n-1}(\alpha)$. Then, the expression of Toeplitz determinant $T_3(2)$ can be written as

$$\begin{aligned}\left|T_3(2)\right| &\le \left|\left(a_2-a_4\right)\left(a_2^2-2a_3^2+a_2a_4\right)\right| \\ &= \left|a_2-a_4\right|\left|a_2^2-2a_3^2+a_2a_4\right| \\ &= \left|a_2-a_4\right|\left|a_2^2-a_3^2-a_3^2+a_2a_4\right| \\ &= \left|a_2-a_4\right|\left(\left|a_2^2-a_3^2\right|+\left|a_2a_4-a_3^2\right|\right). \end{aligned} \tag{4.85}$$

Firstly, the expression $\left|a_2-a_4\right|$ is solved. By applying the triangle inequality, along with the substitutions from equations (4.22) and (4.32) gives

$$\begin{aligned}\left|a_2-a_4\right| &\le \left|a_2\right|+\left|a_4\right| \\ &= \frac{n\cos\alpha}{(n+1)}+\frac{n\cos\alpha}{3(n+1)}\left[\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+1\right]. \end{aligned} \tag{4.86}$$

Upon simplification, the expression becomes

$$\left|a_2\right|+\left|a_4\right| \le \frac{n\cos\alpha}{(n+1)}\left(1+\frac{1}{3}\left[\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+1\right]\right). \tag{4.87}$$

Next, by substituting the results from equations (4.66), (4.73), and (4.87) into (4.85) gives

$$\begin{aligned}\left|T_3(2)\right| &\le \left|a_2-a_4\right|\left(\left|a_2^2-a_3^2\right|+\left|a_2a_4-a_3^2\right|\right) \\ &= \frac{n\cos\alpha}{(n+1)}\left(1+\frac{1}{3}\left[\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+1\right]\right)\left(\frac{5n^2\cos^2\alpha}{4(n+1)^2}+\frac{n^2\cos^2\alpha}{4(n+1)^2}\right). \end{aligned} \tag{4.88}$$

Simplifying the expression gives

$$\left|T_3(2)\right| \leq \frac{3n^2\cos^2\alpha}{2(n+1)^2}\left(\frac{n\cos\alpha}{(n+1)}\left(1+\frac{1}{3}\left[\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+1\right]\right)\right)$$

$$=\frac{3n^2\cos^2\alpha}{2(n+1)^2}\cdot\frac{n\cos\alpha}{(n+1)}\left(1+\frac{1}{3}\left[\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+1\right]\right)$$

$$=\frac{3n^3\cos^3\alpha}{2(n+1)^3}\left(1+\frac{1}{3}\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+\frac{1}{3}\right)$$

$$=\frac{3n^3\cos^3\alpha}{2(n+1)^3}\left(\frac{1}{3}\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|+\frac{4}{3}\right). \tag{4.89}$$

The upper bound of $\left|T_3(2)\right|$ is determined as

$$\left|T_3(2)\right| \leq \frac{3n^3\cos^3\alpha}{2(n+1)^3}\left(\frac{1}{3}\rho+\frac{4}{3}\right), \tag{4.90}$$

where $\rho=\left|1-\frac{3ne^{-i\alpha}\cos\alpha}{2(n+1)}+\frac{n^2e^{-i2\alpha}\cos^2\alpha}{2(n+1)^2}\right|$.

Proof of Theorem 4.2.6 completes.

Setting $\alpha=0$, the subsequent corollary is obtained:

Corollary 4.2.6.

Let $f\in S^*_{T,n-1}(0)$. *Then,*

$$\left|T_3(2)\right| \leq \frac{3n^3}{2(n+1)^3}\left(\frac{1}{3}\rho+\frac{4}{3}\right), \tag{4.91}$$

where $\rho=\left|1-\frac{3n}{2(n+1)}+\frac{n^2}{2(n+1)^2}\right|$.

4.3 Classical Starlike and Convex Functions Associated with the Epicycloid Domain

This section investigates the analytical properties of convex combinations of starlike and convex functions associated with the epicycloid domain. These functions are characterised via a subordination condition involving a specific epicycloid mapping function. The objective is to examine essential aspects of geometric function theory, including coefficient estimates, the Fekete–Szegö inequality, the second Hankel determinant, and various Toeplitz determinants.

4.3.1 Coefficient Estimates and Bounds

This section begins with the derivation of coefficient estimates for functions belonging to the classical subclasses of starlike and convex functions associated with the epicycloid domain.

Theorem 4.3.1.

Let $f \in \mathrm{M}_{\gamma,4L}$ be given in the form (1.1). If $0 \le \gamma \le 1$, then

$$|a_2| \le \frac{5}{6(1+\gamma)}, \tag{4.92}$$

$$|a_3| \le \frac{5}{12(1+2\gamma)}, \tag{4.93}$$

and

$$|a_4| \le \frac{5}{18(1+3\gamma)}. \tag{4.94}$$

Proof.

Assumed that $f \in \mathrm{M}_{\gamma,4L}$. Then, using the concept of subordination, it follows that there exists a Schwarz function $\omega(z)$ that satisfy $\omega(0)=0$ and $|\omega(z)|<1$. Hence, from the Definition 3.2, leading to

$$(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)=1+\frac{5}{6}\omega(z)+\frac{1}{6}\left[\omega(z)\right]^{5}. \tag{4.95}$$

Let the function be defined as in equation (4.5), and assume it satisfies the condition given in equation (4.6).

Now, by applying equation (3.1), the convex part of the expression is expanded and obtain the following series representation

$$\begin{aligned}1+\frac{zf''(z)}{f'(z)}&=1+2a_2z+(6a_3-4a_2^2)z^2+\left(12a_4-18a_2a_3+8a_3^2\right)z^3\\&\quad+\left(-16a_2^4+48a_2^2a_3-32a_2a_4-18a_3^2+20a_5\right)\cdots.\end{aligned} \tag{4.96}$$

Using the expressions from equations (4.7) and (4.96), the left-hand side of equation (4.95) is expanded as follows:

$$\begin{aligned}&(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\\&=(1-\gamma)\left(1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3+\cdots\right)\\&\quad+\gamma\left(1+2a_2z+(6a_3-4a_2^2)z^2+\left(12a_4-18a_2a_3+8a_3^2\right)z^3+\cdots\right)\\&=1-\gamma+\gamma+\left((1-\gamma)a_2+2a_2\gamma\right)z+\left((1-\gamma)\left(2a_3-a_2^2\right)+\left(6a_3-4a_2^2\right)\gamma\right)z^2\\&\quad+\left((1-\gamma)\left(a_2^3-3a_2a_3+3a_4\right)+\left(12a_4-18a_2a_3+8a_3^2\right)\gamma\right)z^3+\cdots\\&=1+\left(a_2-a_2\gamma+2a_2\gamma\right)z+\left(2a_3-a_2^2-2a_3\gamma+a_2^2\gamma+6a_3\gamma-4a_2^2\gamma\right)z^2\\&\quad+\left(a_2^3-3a_2a_3+3a_4-a_2^3\gamma+3a_2a_3\gamma-3a_4\gamma+12a_4\gamma-18a_2a_3\gamma+8a_3^2\gamma\right)z^3+\cdots.\end{aligned} \tag{4.97}$$

Thus, the left-hand side of the equation is expressed as

$$(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)=1+\left(a_2+a_2\gamma\right)z+\left(2a_3-a_2^2+4a_3\gamma-3a_2^2\gamma\right)z^2 \\ +\left(a_2^3-3a_2a_3+3a_4+7a_3^2\gamma-15a_2a_3\gamma+9a_4\gamma\right)z^3+\cdots. \tag{4.98}$$

Next, for the right-hand side of equation (4.95), the series expansion from equation (4.6) is substituted into equation (4.95), resulting in

$$1+\frac{5}{6}\omega(z)+\frac{1}{6}\left[\omega(z)\right]^5=1+\frac{5}{12}c_1z+\left(\frac{5}{12}c_2-\frac{5}{24}c_1^2\right)z^2 \\ +\left(\frac{5}{12}c_3-\frac{5}{12}c_1c_2+\frac{5}{48}c_1^3\right)z^3+\cdots. \tag{4.99}$$

A term-by-term comparison of (4.98) and (4.99) yields the value of a_2 as

$$a_2+a_2\gamma=\frac{5}{12}c_1$$

$$a_2\left(1+\gamma\right)=\frac{5}{12}c_1$$

$$a_2=\frac{5}{12\left(1+\gamma\right)}c_1. \tag{4.100}$$

Next, the value of a_3 is derived as follows:

$$2a_3-a_2^2+4a_3\gamma-3a_2^2\gamma=\frac{5}{12}c_2-\frac{5}{24}c_1^2$$

$$2a_3\left(1+2\gamma\right)-a_2^2\left(1+3\gamma\right)=\frac{5}{12}c_2-\frac{5}{24}c_1^2. \tag{4.101}$$

Substituting the value of a_2 from (4.100) yields

$$2a_3(1+2\gamma)=\frac{5}{12}c_2-\frac{5}{24}c_1^2+\left(\frac{5}{12(1+\gamma)}c_1\right)^2(1+3\gamma)$$

$$=\frac{5}{12}c_2-\frac{5}{24}c_1^2+\frac{25(1+3\gamma)}{144(1+\gamma)^2}c_1^2. \qquad (4.102)$$

The equation simplifies and rearranges to

$$a_3=\frac{5}{24(1+2\gamma)}c_2-\frac{5}{48(1+2\gamma)}c_1^2+\frac{25(1+3\gamma)}{288(1+\gamma)^2(1+2\gamma)}c_1^2$$

$$=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2-\frac{6}{(1+2\gamma)}c_1^2+\frac{5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}c_1^2\right)$$

$$=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2-\left(\frac{6}{(1+2\gamma)}-\frac{5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right). \qquad (4.103)$$

Hence, the value of a_3 is

$$a_3=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right). \qquad (4.104)$$

Proceeding to the next coefficient, the coefficient of a_4 is determined by comparing the coefficients of z^3 as

$$a_2^3-3a_2a_3+3a_4+7a_3^2\gamma-15a_2a_3\gamma+9a_4\gamma=\frac{5}{12}c_3-\frac{5}{12}c_1c_2+\frac{5}{48}c_1^3$$

$$(1+7\gamma)a_2^3-(3+15\gamma)a_2a_3+3(1+3\gamma)a_4=\frac{5}{12}c_3-\frac{5}{12}c_1c_2+\frac{5}{48}c_1^3$$

$$3(1+3\gamma)a_4 = \frac{5}{12}c_3 - \frac{5}{12}c_1c_2 + \frac{5}{48}c_1^3 - (1+7\gamma)a_2^3 + (3+15\gamma)a_2a_3. \tag{4.105}$$

The computation of a_4 begins with the evaluation of a_2^3 and a_2a_3 as

$$a_2^3 = \left(\frac{5}{12(1+\gamma)}c_1\right)^3$$

$$= \frac{125}{1728(1+\gamma)^3}c_1^3. \tag{4.106}$$

$$a_2a_3 = \frac{5}{12(1+\gamma)}c_1\left(\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2 - \left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)\right)$$

$$= \frac{25}{3456(1+\gamma)}c_1\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_2 - c_1^2\right)\right)$$

$$= \frac{25}{3456(1+\gamma)}\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_1c_2 - c_1^3\right)\right)$$

$$= \frac{150(1+\gamma)^2-125(1+3\gamma)}{3456(1+\gamma)^3(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_1c_2 - c_1^3\right). \tag{4.107}$$

Then, substituting the values of a_2^3 and $3a_2a_3$ into equation (4.105), the expression can be simplified and rearranged to obtain

$$3(1+3\gamma)a_4=\frac{5}{12}c_3-\frac{5}{12}c_1c_2+\frac{5}{48}c_1^3-\frac{125(1+7\gamma)}{1728(1+\gamma)^3}c_1^3+$$

$$\frac{3+15\gamma\left(150(1+\gamma)^2-125(1+3\gamma)\right)}{3456(1+\gamma)^3(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_1c_2-c_1^3\right)$$

$$3a_4=\frac{1}{1+3\gamma}\left(\begin{array}{l}\frac{5}{12}c_3-\frac{5}{12}c_1c_2+\frac{5}{48}c_1^3-\frac{125(1+7\gamma)}{1728(1+\gamma)^3}c_1^3+\\ \frac{3+15\gamma\left(150(1+\gamma)^2-125(1+3\gamma)\right)}{3456(1+\gamma)^3(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_1c_2-c_1^3\right)\end{array}\right)$$

$$a_4=\frac{5}{3456(1+3\gamma)}\left(\begin{array}{l}288c_3-288c_1c_2+72c_1^3-\frac{50(1+7\gamma)}{(1+\gamma)^3}c_1^3+\\ \frac{3+15\gamma\left(30(1+\gamma)^2-25(1+3\gamma)\right)}{(1+\gamma)^3(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_1c_2-c_1^3\right)\end{array}\right)$$

$$=\frac{5}{10368(1+3\gamma)}\left(\begin{array}{l}288c_3-288c_1c_2+72c_1^3-\frac{50(1+7\gamma)}{(1+\gamma)^3}c_1^3+\\ \frac{3+15\gamma\left(30(1+\gamma)^2-25(1+3\gamma)\right)}{(1+\gamma)^3(1+2\gamma)}\left(\frac{12(1+\gamma)^2}{6(1+\gamma)^2-5(1+3\gamma)}c_1c_2\right)-\\ \frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{(1+\gamma)^3(1+2\gamma)}c_1^3\end{array}\right)$$

$$=\frac{5}{10368(1+3\gamma)}\left(\begin{array}{l}288c_3-288c_1c_2+72c_1^3-\frac{50(1+7\gamma)}{(1+\gamma)^3}c_1^3\\ +\frac{360(1+\gamma)^2(3+15\gamma)-300(1+3\gamma)(3+15\gamma)}{(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}c_1c_2\\ -\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{(1+\gamma)^3(1+2\gamma)}c_1^3\end{array}\right). \tag{4.108}$$

Hence, the value of a_4 is

$$a_4=\frac{5}{10368(1+3\gamma)}\left[\begin{array}{l}\left(72-\frac{50(1+7\gamma)}{(1+\gamma)^3}-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{(1+\gamma)^3(1+2\gamma)}\right)c_1^3-\\ \left(288-\frac{360(1+\gamma)^2(3+15\gamma)-300(1+3\gamma)(3+15\gamma)}{(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1c_2+288c_3\end{array}\right]. \tag{4.109}$$

Having determined the expressions for a_2, a_3 and a_4, the bound of $|a_2|$ is now estimated by using Lemma 3.1 and get

$$|a_2|\le\frac{5}{12(1+\gamma)}|c_1|=\frac{5}{12(1+\gamma)}(2)$$

$$=\frac{5}{6(1+\gamma)}. \tag{4.110}$$

The bound of a_3 is determined by simplifying equation (4.104), which yields

$$a_3=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)$$

$$=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}\left(c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)\left(\frac{(1+2\gamma)}{12}\right)c_1^2\right)\right)$$

$$=\frac{5}{24(1+2\gamma)}\left(c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}\right)c_1^2\right). \tag{4.111}$$

By applying the modulus to both sides results in

$$|a_3|=\frac{5}{24(1+2\gamma)}\left|c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}\right)c_1^2\right|. \tag{4.112}$$

Applying Lemma 3.2 and assuming $\mu=\dfrac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}$, the calculation gives

$$|a_3|\le\frac{5}{24(1+2\gamma)}\cdot 2\max\left\{1,\left|2\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}-1\right)\right|\right\}$$

$$=\frac{5}{12(1+2\gamma)}\max\left\{1,\left|\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{6(1+\gamma)^2}\right)-1\right|\right\}$$

$$=\frac{5}{12(1+2\gamma)}\max\left\{1,\left|\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{6(1+\gamma)^2}\right)-1\right|\right\}$$

$$=\frac{5}{12(1+2\gamma)}\max\left\{1,\frac{5(1+3\gamma)}{6(1+\gamma)^2}\right\}. \tag{4.113}$$

With the assumptions of $0\le\gamma\le 1$, $\max\left\{1,\dfrac{5(1+3\gamma)}{6(1+\gamma)^2}\right\}=1$. Thus, the bound of $|a_3|$ can be written as

$$|a_3|\le\frac{5}{12(1+2\gamma)}. \tag{4.114}$$

Lastly, the bound of $|a_4|$ is obtained by algebraically manipulating equation (4.109) following the conditions given in Lemmas 3.1 and 3.2.

$$a_4 = \frac{5}{10368(1+3\gamma)}\left[\begin{array}{l}\left(72-\frac{50(1+7\gamma)}{(1+\gamma)^3}-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{(1+\gamma)^3(1+2\gamma)}\right)c_1^3 \\ -\left(288-\frac{360(1+\gamma)^2(3+15\gamma)-300(1+3\gamma)(3+15\gamma)}{(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1c_2+288c_3\end{array}\right]$$

$$= \frac{5}{36(1+3\gamma)}\left[\begin{array}{l}\frac{1}{288}\left(72-\frac{50(1+7\gamma)}{(1+\gamma)^3}-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{(1+\gamma)^3(1+2\gamma)}\right)c_1^3 \\ -\frac{1}{288}\left(288-\frac{360(1+\gamma)^2(3+15\gamma)-300(1+3\gamma)(3+15\gamma)}{(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1c_2+c_3\end{array}\right]$$

$$= \frac{5}{36(1+3\gamma)}\left[\begin{array}{l}c_3-\left(1-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{24(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1c_2+ \\ \left(\frac{1}{4}-\frac{25(1+7\gamma)}{144(1+\gamma)^3}-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{288(1+\gamma)^3(1+2\gamma)}\right)c_1^3\end{array}\right]. \tag{4.115}$$

Taking modulus on the both sides of equation (4.115) yields

$$|a_4| \le \frac{5}{36(1+3\gamma)}\left|\begin{array}{l}c_3-\left(1-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{24(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1c_2+ \\ \left(\frac{1}{4}-\frac{25(1+7\gamma)}{144(1+\gamma)^3}-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{288(1+\gamma)^3(1+2\gamma)}\right)c_1^3\end{array}\right|. \tag{4.116}$$

It is observed that the expression in (4.116) follows the same structure as in Lemma 3.4, in which

$$\left|c_3-2\beta c_1c_2+\delta^* c_1^3\right|=\left|\begin{array}{l} c_3-\left(1-\dfrac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{24(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1c_2+ \\ \left(\dfrac{1}{4}-\dfrac{25(1+7\gamma)}{144(1+\gamma)^3}-\dfrac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{288(1+\gamma)^3(1+2\gamma)}\right)c_1^3 \end{array}\right|\leq 2. \quad (4.117)$$

To ensure the validity of the result, we verify that the conditions required in Lemma 3.4 hold. In particular,

$$\beta=\frac{1}{2}\left(1-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{24(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right), \quad (4.118)$$

and

$$\delta^*=\frac{1}{4}-\frac{25(1+7\gamma)}{144(1+\gamma)^3}-\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{288(1+\gamma)^3(1+2\gamma)}. \quad (4.119)$$

For the given constraint $0\leq\gamma\leq 1$, the conditions are satisfied such that $0\leq\beta\leq 1$ and $\beta-\delta^*>0$, where

$$\begin{aligned}\beta-\delta^*=\frac{1}{4}-&\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{48(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ &+\frac{25(1+7\gamma)}{144(1+\gamma)^3}+\frac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{288(1+\gamma)^3(1+2\gamma)}.\end{aligned} \quad (4.120)$$

Next, it satisfies $\beta(2\beta-1)<0<\delta^*\leq\beta$, where

$$\beta(2\beta-1)=\left[\begin{array}{c}-\dfrac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{24(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\cdot\\ \left(\dfrac{1}{2}-\dfrac{30(1+\gamma)^2(3+15\gamma)-25(1+3\gamma)(3+15\gamma)}{48(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)\end{array}\right]. \tag{4.121}$$

Thus, substituting (4.117) into $|a_4|$, yields

$$\begin{aligned}|a_4| &\leq \frac{5}{36(1+3\gamma)}(2)\\ &= \frac{5}{18(1+3\gamma)}.\end{aligned} \tag{4.122}$$

Proof of Theorem 4.3.1 completes.

Setting $\gamma=0$, the subsequent corollary is obtained:

Corollary 4.3.1.

Let $f\in S^*_{4,L}$. Then,

$$|a_2|\leq\frac{5}{6}, \tag{4.123}$$

$$|a_3|\leq\frac{5}{12}, \tag{4.124}$$

and

$$|a_4|\leq\frac{5}{18}. \tag{4.125}$$

The inequality of (4.123)-(4.125) from Corollary 4.3.1 aligns to the sharp bounds obtained by Gandhi et al. (2022) for the class of $S^*_{4,\mathrm{L}}$.

4.3.2 Fekete–Szegö Inequality

Having established the coefficient estimates and their corresponding bounds, this section proceeds to investigate the Fekete–Szegö inequality. This inequality serves as a significant analytic tool for exploring the functional characteristics and geometric properties of the proposed subclass.

Theorem 4.3.2.

If $f \in \mathrm{M}_{\gamma,4L}$ be given in the form (1.1), then for any $\lambda \in \mathbb{C}$ and $0 \le \gamma \le 1$,

$$\left|a_3 - \lambda a_2^2\right| \le \frac{5}{12(1+2\gamma)} \max\left\{1, \frac{5(2\lambda-1)+5\gamma(4\lambda-3)}{6(1+\gamma)^2}\right\}. \tag{4.126}$$

Proof.

Assumed that $f \in \mathrm{M}_{\gamma,4L}$. By substituting the expressions of a_2 and a_3 obtained from equations (4.100) and (4.104) respectively into the functional $a_3 - \lambda a_2^2$, the following expression is obtained

$$a_3-\lambda a_2^2=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)-\lambda\left(\frac{5}{12(1+\gamma)}c_1\right)^2$$

$$=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)-\frac{25\lambda}{144(1+\gamma)^2}c_1^2$$

$$=\frac{5}{288}\left(\frac{12}{(1+2\gamma)}\left[c_2-\frac{(1+2\gamma)}{12}\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right]\right)-\frac{25\lambda}{144(1+\gamma)^2}c_1^2$$

$$=\frac{5}{24(1+2\gamma)}\left[c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}\right)c_1^2\right]-\frac{25\lambda}{144(1+\gamma)^2}c_1^2$$

$$=\frac{5}{24(1+2\gamma)}\left(\left[c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}\right)c_1^2\right]-\frac{5(1+2\gamma)\lambda}{6(1+\gamma)^2}c_1^2\right)$$

$$=\frac{5}{24(1+2\gamma)}\left(c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{12(1+\gamma)^2}+\frac{5(1+2\gamma)\lambda}{6(1+\gamma)^2}\right)c_1^2\right). \tag{4.127}$$

Consequently, by taking the modulus on both sides, the inequality becomes

$$\left|a_3-\lambda a_2^2\right|\le\frac{5}{24(1+2\gamma)}\left|c_2-\left(\frac{6(1+\gamma)^2-5(1+3\gamma)+10\lambda(1+2\gamma)}{12(1+\gamma)^2}\right)c_1^2\right|. \tag{4.128}$$

Thereafter, by invoking Lemma 3.2 with $\mu=\dfrac{6(1+\gamma)^2-5(1+3\gamma)+10\lambda(1+2\gamma)}{12(1+\gamma)^2}$ gives

$$\left|a_3-\lambda a_2^2\right| \le 2\left|\frac{5}{24(1+2\gamma)}\right| \max\left\{1, \left|2\left(\frac{6(1+\gamma)^2-5(1+3\gamma)+10\lambda(1+2\gamma)}{12(1+\gamma)^2}\right)-1\right|\right\}$$

$$=\frac{5}{12(1+2\gamma)} \max\left\{1, \left|\left(\frac{6(1+\gamma)^2-5(1+3\gamma)+10\lambda(1+2\gamma)}{6(1+\gamma)^2}\right)-1\right|\right\}$$

$$=\frac{5}{12(1+2\gamma)} \max\left\{1, \left|\frac{6(1+\gamma)^2-5(1+3\gamma)+10\lambda(1+2\gamma)-6(1+\gamma)^2}{6(1+\gamma)^2}\right|\right\}$$

$$=\frac{5}{12(1+2\gamma)} \max\left\{1, \left|\frac{10\lambda+20\lambda\gamma-5-15\gamma}{6(1+\gamma)^2}\right|\right\}. \tag{4.129}$$

Therefore, the Fekete–Szegö inequality for the class is established as follows:

$$\left|a_3-\lambda a_2^2\right| \le \frac{5}{12(1+2\gamma)} \max\left\{1, \frac{5(2\lambda-1)+5\gamma(4\lambda-3)}{6(1+\gamma)^2}\right\}. \tag{4.130}$$

Proof of Theorem 4.3.2 completes.

Setting $\lambda=1$ and $\gamma=0$, the subsequent corollary is obtained:

Corollary 4.3.2.

Let $f \in S_{4,L}^{*}$. Then,

$$\left|a_3-\lambda a_2^2\right| \le \frac{5}{12}. \tag{4.131}$$

The inequality of (4.131) from Corollary 4.3.2 aligns to the sharp inequality obtained by Shi et al. (2022) for the class of $S_{4,\mathrm{L}}^{*}$.

4.3.3 Second Hankel Determinant

Following the derivation of the Fekete–Szegö inequality, the next focus is the computation of the second Hankel determinant associated with the subclass $\mathrm{M}_{\gamma,4L}$.

Theorem 4.3.3.

If $f \in \mathrm{M}_{\gamma,4L}$ *be given in the form (1.1), then for* $0 \le \gamma \le 1$,

$$\left|a_2 a_4 - a_3^2\right| \le \frac{25}{144(1+2\gamma)^2}. \tag{4.132}$$

Proof.

Assumed that $f \in \mathrm{M}_{\gamma,4L}$. The coefficient values obtained from equations (4.100), (4.104) and (4.109) are substituted into the corresponding expression for the second Hankel determinant. Evaluation of $a_2 a_4$ first gives

$$a_2 a_4 = \frac{5}{12(1+\gamma)} c_1 \left(\frac{5}{10368(1+3\gamma)} \left[\begin{array}{l} \left(72 - \frac{50(1+7\gamma)}{(1+\gamma)^3} - \frac{30(1+\gamma)^2(3+15\gamma) - 25(1+3\gamma)(3+15\gamma)}{(1+\gamma)^3(1+2\gamma)} \right) c_1^3 \\ - \left(288 - \frac{360(1+\gamma)^2(3+15\gamma) - 300(1+3\gamma)(3+15\gamma)}{(1+\gamma)(1+2\gamma)\left(6(1+\gamma)^2 - 5(1+3\gamma)\right)} \right) c_1 c_2 + 288 c_3 \end{array} \right] \right)$$

$$= \frac{5}{10368(1+3\gamma)} \left(\begin{array}{l} \left(\frac{30}{(1+\gamma)} - \frac{125(1+7\gamma)}{6(1+\gamma)^4} - \frac{150(1+\gamma)^2(3+15\gamma) - 125(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)} \right) c_1^4 \\ - \left(\frac{120}{(1+\gamma)} - \frac{1800(1+\gamma)^2(3+15\gamma) - 1500(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2 - 5(1+3\gamma)\right)} \right) c_1^2 c_2 + \frac{120}{(1+\gamma)} c_1 c_3 \end{array} \right). \tag{4.133}$$

Next, the expression of a_3^2 is evaluated as

$$a_3^2 = \left(\frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)\right)^2$$

$$= \frac{25}{82944}\begin{pmatrix} \left(\frac{12}{(1+2\gamma)}c_2\right)^2 - 2\left(\frac{12}{(1+2\gamma)}c_2\right)\left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2 \\ +\left(\left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)^2 \end{pmatrix}$$

$$= \frac{25}{82944}\begin{pmatrix} \frac{144}{(1+2\gamma)^2}c_2^2 - \left(\frac{144(1+\gamma)^2 - 120(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)^2}\right)c_1^2 c_2 \\ +\left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)^2 c_1^4 \end{pmatrix}. \tag{4.134}$$

Substituting the computed values of $a_2 a_4$ and a_3^2 from (4.133) and (4.134) respectively into the formula of the second Hankel determinant yields

$$a_2a_4-a_3^2$$

$$=\frac{5}{10368(1+3\gamma)}\left(\begin{array}{l}\left(\frac{30}{(1+\gamma)}-\frac{125(1+7\gamma)}{6(1+\gamma)^4}-\frac{150(1+\gamma)^2(3+15\gamma)-125(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\right)c_1^4\\ -\left(\frac{120}{(1+\gamma)}-\frac{1800(1+\gamma)^2(3+15\gamma)-1500(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1^2c_2+\frac{120}{(1+\gamma)}c_1c_3\end{array}\right)$$

$$-\frac{25}{82944}\left(\frac{144}{(1+2\gamma)^2}c_2^2-\left(\frac{144(1+\gamma)^2-120(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)^2}\right)c_1^2c_2+\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)^2c_1^4\right)$$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}120\left(\begin{array}{l}\left(\frac{30}{(1+\gamma)}-\frac{125(1+7\gamma)}{6(1+\gamma)^4}-\frac{150(1+\gamma)^2(3+15\gamma)-125(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\right)c_1^4-\\ \left(\frac{120}{(1+\gamma)}-\frac{1800(1+\gamma)^2(3+15\gamma)-1500(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right)c_1^2c_2+\frac{120}{(1+\gamma)}c_1c_3\end{array}\right)\\ -75(1+3\gamma)\left(\begin{array}{l}\frac{144}{(1+2\gamma)^2}c_2^2-\left(\frac{144(1+\gamma)^2-120(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)^2}\right)c_1^2c_2\\ +\left(\frac{6(1+\gamma)^2-5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)^2c_1^4\end{array}\right)\end{array}\right]$$

$$=\frac{1}{248832(1+3\alpha)}\left[\begin{array}{l}\left(\begin{array}{l}\frac{3600}{(1+\gamma)}-\frac{2500(1+7\gamma)}{(1+\gamma)^4}-\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\ -75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c_1^4\\ -\left(\begin{array}{l}\frac{14400}{(1+\gamma)}-\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ -\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)c_1^2c_2+\\ \frac{14400}{(1+\gamma)}c_1c_3-\frac{10800(1+3\gamma)}{(1+2\gamma)^2}c_2^2\end{array}\right]. \qquad (4.135)$$

By applying Lemma 3.3, the coefficients c_2 and c_3 may be represented in terms of c_1. Upon setting $c_1=c$ with $c\in[0,2]$, these coefficients take the following form

$$a_2a_4-a_3^2$$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\left(\begin{array}{l}\dfrac{3600}{(1+\gamma)}-\dfrac{2500(1+7\gamma)}{(1+\gamma)^4}-\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\-75(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^4\\-\left(\begin{array}{l}\dfrac{14400}{(1+\gamma)}-\dfrac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\\-\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}\right)c^2\left(\dfrac{1}{2}\left(c^2+\left(4-c^2\right)x\right)\right)\\+\dfrac{14400}{(1+\gamma)}c\left(\dfrac{1}{4}\left(c^3+2cx\left(4-c^2\right)-x^2c\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)\delta\right)\right)\\-\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}\left(\dfrac{1}{2}\left(c^2+\left(4-c^2\right)x\right)\right)^2\end{array}\right]$$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\left(\begin{array}{l}\dfrac{3600}{(1+\gamma)}-\dfrac{2500(1+7\gamma)}{(1+\gamma)^4}-\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\-75(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^4\\-\left(\begin{array}{l}\dfrac{7200}{(1+\gamma)}-\dfrac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\\-\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}\right)\left(c^4+\left(4-c^2\right)c^2x\right)\\+\dfrac{3600}{(1+\gamma)}\left(c^4+2\left(4-c^2\right)c^2x-x^2c^2\left(4-c^2\right)+2\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right)\\-\dfrac{2700(1+3\gamma)}{(1+2\gamma)^2}\left(c^4+2\left(4-c^2\right)c^2x+\left(4-c^2\right)^2x^2\right)\end{array}\right]. \tag{4.136}$$

Simplifying the expression within the brackets and expanding the equation in terms of the relevant variables gives

$a_2a_4 - a_3^2$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\left(\begin{array}{l}\frac{3600}{(1+\gamma)}-\frac{2500(1+7\gamma)}{(1+\gamma)^4}-75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\-\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\end{array}\right)c^4\\-\left(\begin{array}{l}\frac{7200}{(1+\gamma)}-\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\\-\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}\right)c^4\\-\left(\begin{array}{l}\frac{7200}{(1+\gamma)}-\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\\-\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}\right)\left(4-c^2\right)c^2x\\+\frac{3600}{(1+\gamma)}c^4+\frac{7200}{(1+\gamma)}\left(4-c^2\right)c^2x-\frac{3600}{(1+\gamma)}x^2c^2\left(4-c^2\right)\\+\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}c^4\\-\frac{5400(1+3\gamma)}{(1+2\gamma)^2}\left(4-c^2\right)c^2x-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\left(4-c^2\right)^2x^2\end{array}\right]$$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\left(\begin{array}{l}\frac{3600}{(1+\gamma)}-\frac{2500(1+7\gamma)}{(1+\gamma)^4}-75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\-\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}-\frac{7200}{(1+\gamma)}\\+\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\\+\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\frac{3600}{(1+\gamma)}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^4\\+\left(\begin{array}{l}-\frac{7200}{(1+\gamma)}+\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\\+\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\frac{7200}{(1+\gamma)}-\frac{5400(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)\left(4-c^2\right)c^2x\\-\frac{3600}{(1+\gamma)}x^2c^2\left(4-c^2\right)+\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta\\-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\left(4-c^2\right)^2x^2\end{array}\right]$$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}x^2\left(4-c^2\right)^2+\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta-\frac{3600}{(1+\gamma)}x^2c^2\left(4-c^2\right)+\\ \left(\begin{array}{l}\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\\ +\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ -\frac{5400(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)\left(4-c^2\right)c^2x+\\ \left(\begin{array}{l}-\frac{2500(1+7\gamma)}{(1+\gamma)^4}-\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\ -75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^4\end{array}\right]$$

$$=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}x^2\left(4-c^2\right)^2+\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta-\frac{3600}{(1+\gamma)}x^2c^2\left(4-c^2\right)+\\ \left(\begin{array}{l}\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}+\\ \frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}-\frac{5400(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)\left(4-c^2\right)c^2x+\\ \left(\begin{array}{l}-\frac{2500(1+7\gamma)}{(1+\gamma)^4}-\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\ -75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^4\end{array}\right].$$

Subsequently, the final form is obtained as

$$a_2a_4-a_3^2=\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}x^2\left(4-c^2\right)^2+\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta-\frac{3600}{(1+\gamma)}x^2c^2\left(4-c^2\right)+\\ \left(\begin{array}{l}\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}+\\ \frac{-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)\left(4-c^2\right)c^2x+\\ \left(\begin{array}{l}-\frac{2500(1+7\gamma)}{(1+\gamma)^4}-\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\ -75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^4\end{array}\right]. \quad (4.137)$$

Taking the modulus on both sides and applying the triangle inequality gives

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\left|-\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\right||x|^2\left|\left(4-c^2\right)^2\right|+\left|\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta\right|+\left|-\frac{3600}{(1+\gamma)}\right||x|^2\left|c^2\left(4-c^2\right)\right|\\ +\left|\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\right|\left|\left(4-c^2\right)c^2\right||x|\\ +\left|\frac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\right|\left|\left(4-c^2\right)c^2\right||x|+\left|\frac{2500(1+7\gamma)}{(1+\gamma)^4}\mathrm{c}^4\right|\\ +\left|\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\mathrm{c}^4\right|\\ +\left|75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\mathrm{c}^4\right|\\ +\left|\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\mathrm{c}^4\right|\\ +\left|\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}c_4\right|+\left|\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\mathrm{c}^4\right|\end{array}\right]$$

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\frac{2700(1+3\gamma)}{(1+2\gamma)^2}|x|^2\left(4-c^2\right)^2+\frac{7200}{(1+\gamma)}\left(1-|x|^2\right)\left(4-c^2\right)c\delta+\frac{3600}{(1+\gamma)}|x|^2c^2\left(4-c^2\right)\\+\left(\begin{array}{l}\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\frac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)|x|\left(4-c^2\right)c^2\\+\left(\begin{array}{l}\frac{2500(1+7\gamma)}{(1+\gamma)^4}+\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\+75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\+\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}+\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^4\end{array}\right].$$

Letting $|x|=y$ where $y\le 1$, and assuming $|\delta|\le 1$, leading to

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\frac{2700(1+3\gamma)}{(1+2\gamma)^2}y^2\left(4-c^2\right)^2+\frac{7200}{(1+\gamma)}\left(1-y^2\right)\left(4-c^2\right)c\\+\frac{3600}{(1+\gamma)}y^2c^2\left(4-c^2\right)\\+\left(\begin{array}{l}\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\frac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)y\left(4-c^2\right)c^2\\+\left(\begin{array}{l}\frac{2500(1+7\gamma)}{(1+\gamma)^4}+\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\+75(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\+\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\frac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}+\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^4\end{array}\right]:=\mathrm{G}(c,y). \quad (4.138)$$

Next, the upper bound of (4.138) is assumed to occur at $c \times y = [0,2] \times [0,1]$. Differentiating equation (4.138) with respect to y gives

$$\frac{\partial G}{\partial y} = \frac{1}{248832(1+3\gamma)} \left\{ \begin{array}{l} \dfrac{5400(1+3\gamma)}{(1+2\gamma)^2} y(4-c^2)^2 + \dfrac{7200}{(1+\gamma)}(-2y)(4-c^2)c + \dfrac{7200}{(1+\gamma)} yc^2(4-c^2) \\ + \left(\begin{array}{l} \dfrac{216000(1+\gamma)^2(3+15\gamma) - 180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2 - 5(1+3\gamma)\right)} \\ + \dfrac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2} \end{array} \right) (4-c^2)c^2 \end{array} \right\}$$

$$= \frac{1}{248832(1+3\gamma)} \left\{ \begin{array}{l} \dfrac{5400(1+3\gamma)}{(1+2\gamma)^2} y(4-c^2)^2 - \dfrac{14400}{(1+\gamma)} y(4-c^2)c + \dfrac{7200}{(1+\gamma)} yc^2(4-c^2) \\ + \left(\begin{array}{l} \dfrac{216000(1+\gamma)^2(3+15\gamma) - 180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2 - 5(1+3\gamma)\right)} \\ + \dfrac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2} \end{array} \right) (4-c^2)c^2 \end{array} \right\}. \quad (4.139)$$

Since $0 \le \gamma \le 1$, it follows that $1+3\gamma > 0$, hence $\dfrac{1}{248832(1+3\gamma)} > 0$.

Next, observe that each term in the expression contains the factor $(4-c)^2$. Factoring this term, we obtain

$$\frac{\partial G}{\partial y} = \frac{1}{248832(1+3\gamma)}(4-c)^2 K(c,y), \quad (4.140)$$

where $K(c,y)$ denotes the remaining expression.

Now, for $0 \le c \le 2$, it follows that $(4-c)^2 \ge 0$, and for $0 \le y \le 1$, we have $y \ge 0$. Morever, since $0 \le \gamma \le 1$, all expressions involving $(1+\gamma)$, $(1+2\gamma)$ and $(1+3\gamma)$ are strictly positive.

Next, considering the terms $y\left(4-c^2\right)^2$, $\left(4-c^2\right)cy$, and $\left(4-c^2\right)c^2y$ inside $K(c,y)$ are nonnegative for $0 \le c \le 2$ and $0 \le y \le 1$.

The remaining term of $\left(4-c^2\right)c^2$ with coefficients depending on γ is also nonnegative since all denominators are positive and the numerators consist of sums of positive quantities for $0 \le \gamma \le 1$. Thus, $K(c,y) > 0$.

Given $0 \le y \le 1$ and for any fixed c with $0 \le c \le 2$, it follows that $\dfrac{\partial G}{\partial y} > 0$. Hence, $G(c,y)$ is increasing in y, and consequently, $G(c,y) \le G(c,1) = G(c)$. Then, we get

$$G(c) = \frac{1}{248832(1+3\gamma)} \left\{ \begin{array}{l} \dfrac{2700(1+3\gamma)}{(1+2\gamma)^2}\left(4-c^2\right)^2 + \dfrac{3600}{(1+\gamma)} c^2\left(4-c^2\right) \\ + \left(\begin{array}{l} \dfrac{216000(1+\gamma)^2(3+15\gamma) - 180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2 - 5(1+3\gamma)\right)} \\ + \dfrac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2} \end{array} \right) \left(4-c^2\right)c^2 \\ + \left(\begin{array}{l} \dfrac{2500(1+7\gamma)}{(1+\gamma)^4} + \dfrac{18000(1+\gamma)^2(3+15\gamma) - 15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)} \\ +75(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma) - 5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)} \right)^2 \\ + \dfrac{216000(1+\gamma)^2(3+15\gamma) - 180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2 - 5(1+3\gamma)\right)} \\ + \dfrac{10800(1+\gamma)^2(1+3\gamma) - 9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2} + \dfrac{2700(1+3\gamma)}{(1+2\gamma)^2} \end{array} \right) \mathrm{c}^4 \end{array} \right\}. \tag{4.141}$$

Next, differentiating equation (4.141) yields

$$G'(c)=\frac{1}{248832(1+3\gamma)}\left\{\begin{array}{l}\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}\left(c^3-4c\right)+\dfrac{3600}{(1+\gamma)}\left(8c-4c^3\right)\\[2ex]+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\[2ex]+\dfrac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)\left(8c-4c^3\right)\\[2ex]+\left(\begin{array}{l}\dfrac{10000(1+7\gamma)}{(1+\gamma)^4}+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}+\\[2ex]300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\[2ex]+\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{6(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}+\\[2ex]\dfrac{21600(1+\gamma)^2(1+3\gamma)-18000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c^3\end{array}\right\}. \tag{4.142}$$

Setting $G'(c)=0$ and solving the resulting equation yields the possible values of c. Simplifying the contents of the bracket gives

$$\frac{10800(1+3\gamma)}{(1+2\gamma)^2}\left(c^3-4c\right)+\frac{3600}{(1+\gamma)}\left(8c-4c^3\right)$$

$$+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\dfrac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)\left(8c-4c^3\right)$$

$$+\left(\begin{array}{l}\dfrac{10000(1+7\gamma)}{(1+\gamma)^4}+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}+\\ 300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{6(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}+\\ \dfrac{21600(1+\gamma)^2(1+3\gamma)-18000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)\mathrm{c}^3$$

$$=\frac{10800(1+3\gamma)}{(1+2\gamma)^2}c^3-\frac{43200(1+3\gamma)}{(1+2\gamma)^2}c+\frac{28800}{(1+\gamma)}c-\frac{14400}{(1+\gamma)}c^3$$

$$+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)c$$

$$-\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{6(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\dfrac{18000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)c^3$$

$$+\left(\begin{array}{l}\dfrac{10000(1+7\gamma)}{(1+\gamma)^4}+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}+\\ 300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{6(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}+\\ \dfrac{21600(1+\gamma)^2(1+3\gamma)-18000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)\mathrm{c}^3$$

$$
= \begin{pmatrix}
\frac{10800(1+3\gamma)}{(1+2\gamma)^2} - \frac{14400}{(1+\gamma)} \\
-\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{6(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)} \\
-\frac{18000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2} + \frac{10000(1+7\gamma)}{(1+\gamma)^4} \\
+\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)} \\
+300(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2 \\
+\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{6(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)} \\
+\frac{21600(1+\gamma)^2(1+3\gamma)-18000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2} + \frac{10800(1+3\gamma)}{(1+2\gamma)^2}
\end{pmatrix} c^3
+ \begin{pmatrix}
\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)} \\
+\frac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\frac{28800}{(1+\gamma)} - \frac{43200(1+3\gamma)}{(1+2\gamma)^2}
\end{pmatrix} c
$$

$$
= \begin{pmatrix}
\frac{21600(1+3\gamma)}{(1+2\gamma)^2} - \frac{14400}{(1+\gamma)} + \frac{10000(1+7\gamma)}{(1+\gamma)^4} \\
+\frac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)} \\
+300(1+3\gamma)\left(\frac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2 \\
+\frac{21600(1+\gamma)^2(1+3\gamma)-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}
\end{pmatrix} c^3
+ \begin{pmatrix}
\frac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)} \\
+\frac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\frac{28800}{(1+\gamma)} - \frac{43200(1+3\gamma)}{(1+2\gamma)^2}
\end{pmatrix} c. \tag{4.143}
$$

As a result, the following expression is obtained

$$G'(c)=\frac{1}{248832(1+3\gamma)}\left\{\begin{array}{l}\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\\+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\\+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^3\\+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\dfrac{28800}{(1+\gamma)}-\dfrac{43200(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c\end{array}\right\}=0. \quad (4.144)$$

The expression is then solved to obtain the solutions for c.

$$\frac{1}{248832(1+3\gamma)}\left\{\begin{array}{l}\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\\+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\\+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^3\\+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\dfrac{28800}{(1+\gamma)}-\dfrac{43200(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c\end{array}\right\}=0$$

$$\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\\+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^3$$

$$+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\dfrac{28800}{(1+\gamma)}-\dfrac{43200(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)c=0$$

$$c\left[\begin{array}{l}\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\\+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^2\\+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\dfrac{28800}{(1+\gamma)}-\dfrac{43200(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)\end{array}\right]=0. \quad (4.145)$$

From equation (4.145), the solutions of c are

$$c=0$$

and

$$\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\\+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^2$$

$$+\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{28800}{(1+\gamma)}-\dfrac{43200(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)=0$$

$$\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^2}{(1+\gamma)^3}\\+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^2$$

$$=-\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\+\dfrac{3600(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{28800}{(1+\gamma)}-\dfrac{43200(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)$$

$$\left(\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\\ +\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\ +300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\end{array}\right)c^2$$

$$=4\left(\begin{array}{l}\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}-\dfrac{900(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}-\dfrac{7200}{(1+\gamma)}\\ -\dfrac{5400(1+\gamma)^2(3+15\gamma)-45000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}\right)$$

$$c^2=\frac{4\left(\begin{array}{l}\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}-\dfrac{900(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}-\dfrac{7200}{(1+\gamma)}\\ -\dfrac{5400(1+\gamma)^2(3+15\gamma)-45000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}\right)}{\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\end{array}}$$

$$c=\pm\sqrt{4}\sqrt{\frac{\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}-\dfrac{900(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}-\dfrac{7200}{(1+\gamma)}-\dfrac{5400(1+\gamma)^2(3+15\gamma)-45000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}}{\begin{array}{l}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}+300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}+\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\end{array}}}.$$

Hence, the solutions for c are

$$c=0,\ c\pm 2\sqrt{\frac{\begin{array}{c}\dfrac{10800(1+3\gamma)}{(1+2\gamma)^2}-\dfrac{900(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}-\dfrac{7200}{(1+\gamma)}\\ -\dfrac{5400(1+\gamma)^2(3+15\gamma)-45000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\end{array}}{\begin{array}{c}\dfrac{43200(1+3\gamma)(1+\gamma)^2-36000(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\\ +300(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{3(1+\gamma)^4(1+2\gamma)}\\ +\dfrac{10000(1+7\gamma)-14400(1+\gamma)^3}{(1+\gamma)^4}\end{array}}}. \tag{4.146}$$

Under the assumption $c\in[0,2]$, the maximum of $G(c)$ is attained at $c=0$. Accordingly, from equation (4.138), the upper bound of $G(c,y)$ corresponds to $c=0$ and $y=1$, resulting in

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{248832(1+3\gamma)}\left[\begin{array}{l}\dfrac{2700(1+3\gamma)}{(1+2\gamma)^2}\left(4-(0)^2\right)^2+\dfrac{7200}{(1+\gamma)}\left(1-(1)^2\right)\left(4-(0)^2\right)(0)\\ +\dfrac{3600}{(1+\gamma)}(1)^2(0)^2\left(4-(0)^2\right)\\ +\left(\begin{array}{l}\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\dfrac{9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}\end{array}\right)(1)\left(4-(0)^2\right)(0)^2\\ +\left(\begin{array}{l}\dfrac{2500(1+7\gamma)}{(1+\gamma)^4}+\dfrac{18000(1+\gamma)^2(3+15\gamma)-15000(1+3\gamma)(3+15\gamma)}{12(1+\gamma)^4(1+2\gamma)}\\ +75(1+3\gamma)\left(\dfrac{6(1+\gamma)^2(1+3\gamma)-5(1+3\gamma)^2}{(1+\gamma)^2(1+2\gamma)}\right)^2\\ +\dfrac{216000(1+\gamma)^2(3+15\gamma)-180000(1+3\gamma)(3+15\gamma)}{24(1+\gamma)^2(1+2\gamma)\left(6(1+\gamma)^2-5(1+3\gamma)\right)}\\ +\dfrac{10800(1+\gamma)^2(1+3\gamma)-9000(1+3\gamma)^2}{2(1+\gamma)^2(1+2\gamma)^2}+\dfrac{2700(1+3\gamma)}{(1+2\gamma)^2}\end{array}\right)(0)^4\end{array}\right]. \tag{4.147}$$

By simple calculations, the result is

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{248832(1+3\gamma)}\left[\frac{2700(1+3\gamma)}{(1+2\gamma)^2}\left(4-(0)^2\right)^2\right]$$

$$= \frac{1}{248832(1+3\gamma)}\left[\frac{2700(1+3\gamma)}{(1+2\gamma)^2}(16)\right]$$

$$= \frac{1}{248832(1+3\gamma)}\left[\frac{43200(1+3\gamma)}{(1+2\gamma)^2}\right]$$

$$= \frac{43200}{248832(1+2\gamma)^2}. \tag{4.148}$$

Finally, the inequality for the second Hankel determinant is given by

$$\left|a_2a_4 - a_3^2\right| \le \frac{25}{144(1+2\gamma)^2}. \tag{4.149}$$

Proof of Theorem 4.3.3 completes.

Setting $\gamma = 0$, the subsequent corollary is obtained:

Corollary 4.3.3.

Let $f \in S_{4,L}^*$. *Then,*

$$\left|a_2a_4 - a_3^2\right| \le \frac{25}{144}. \tag{4.150}$$

The inequality of (4.150) from Corollary 4.3.3 aligns to the sharp inequality obtained by Shi et al. (2022) for the class of $S_{4,\mathrm{L}}^*$.

4.3.4 Toeplitz Determinant

Following the derivation of the inequality for the second Hankel determinant, the focus shifts to the Toeplitz determinant. Three specific cases $T_2(2)$, $T_3(1)$ and $T_3(2)$ are examined, and the corresponding bounds are established for functions in the class $\mathrm{M}_{\gamma,4L}$.

Theorem 4.3.4.

Let $f \in \mathrm{M}_{\gamma,4L}$ be the expression given in the form (1.1). If $0 \le \gamma \le 1$, then

$$\left|T_2(2)\right| \le \frac{125}{144(1+\gamma)^2}. \tag{4.151}$$

Proof.

Assumed that $f \in \mathrm{M}_{\gamma,4L}$. Then, the expression for the Toeplitz determinant $T_2(2)$ is given by equation (4.70). Utilising equations (4.110) and (4.114), the squares of $|a_2|$ and $|a_3|$ are computed, yielding the following expressions

$$\left|a_2^2\right| \le \left(\frac{5}{6(1+\gamma)}\right)^2$$

$$= \frac{25}{36(1+\gamma)^2}. \tag{4.152}$$

$$\left|a_3^2\right| \le \left(\frac{5}{12(1+\gamma)}\right)^2$$

$$= \frac{25}{144(1+\gamma)^2}. \tag{4.153}$$

Substituting equations (4.152) and (4.153) into equation (4.70) gives

$$\left|T_2(2)\right| \leq \frac{25}{36(1+\gamma)^2} + \frac{25}{144(1+\gamma)^2}$$

$$= \frac{125}{144(1+\gamma)^2}. \tag{4.154}$$

Proof of Theorem 4.3.4 completes.

Setting $\gamma = 0$ and $\gamma = 1$, the subsequent corollary is obtained:

Corollary 4.3.4.

Let $f \in \mathrm{M}_{0,4L}$. Then,

$$\left|T_2(2)\right| \leq \frac{125}{144}. \tag{4.155}$$

Let $f \in \mathrm{M}_{1,4L}$. Then,

$$\left|T_2(2)\right| \leq \frac{125}{576}. \tag{4.156}$$

Theorem 4.3.5.

Let $f \in \mathrm{M}_{\gamma,4L}$ be the expression given in the form (1.1). If $0 \leq \gamma \leq 1$, then

$$\left|T_3(1)\right| \leq 1 + \frac{25}{36}\left[\frac{2}{(1+\gamma)^2} + \frac{1}{4(1+2\gamma)^2}\max\left\{1, \frac{5(3+5\gamma)}{6(1+\gamma)^2}\right\}\right]. \tag{4.157}$$

Proof.

Assumed that $f \in \mathrm{M}_{\gamma,4L}$. Then, the expression for the Toeplitz determinant $T_3(1)$ is given by equation (4.76). Equations (4.100) and (4.104) are applied to obtain the necessary expressions of $a_3 - 2a_2^2$ as

$$a_3 - 2a_2^2 = \frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right) - 2\left(\frac{5}{12(1+\gamma)}c_1\right)^2$$

$$= \frac{5}{288}\left(\frac{12}{(1+2\gamma)}c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right) - 2\left(\frac{25}{144(1+\gamma)^2}c_1^2\right)$$

$$= \frac{5}{288}\left(\frac{12}{(1+2\gamma)}\left(c_2 - \frac{(1+2\gamma)}{12}\left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}\right)c_1^2\right)\right) - \frac{25}{72(1+\gamma)^2}c_1^2$$

$$= \frac{5}{24(1+2\gamma)}\left(c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{12(1+\gamma)^2}\right)c_1^2\right) - \frac{25}{72(1+\gamma)^2}c_1^2$$

$$= \frac{5}{24(1+2\gamma)}\left[\left(c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma)}{12(1+\gamma)^2}\right)c_1^2\right) - \frac{5(1+2\gamma)}{3(1+\gamma)^2}c_1^2\right]$$

$$= \frac{5}{24(1+2\gamma)}\left[c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma) + 20(1+2\gamma)}{12(1+\gamma)^2}\right)c_1^2\right]. \qquad (4.158)$$

Upon taking the modulus on both sides, the inequality becomes

$$\left|a_3 - 2a_2^2\right| = \frac{5}{24(1+2\gamma)}\left|c_2 - \left(\frac{6(1+\gamma)^2 - 5(1+3\gamma) + 20(1+2\gamma)}{12(1+\gamma)^2}\right)c_1^2\right|. \qquad (4.159)$$

By applying Lemma 3.2 with $\mu = \dfrac{6(1+\gamma)^2 - 5(1+3\gamma) + 20(1+2\gamma)}{12(1+\gamma)^2}$ results in

$$\left|a_3 - 2a_2^2\right| \le 2\left|\frac{5}{24(1+2\gamma)}\right| \max\left\{1, \left|2\left(\frac{6(1+\gamma)^2 - 5(1+3\gamma) + 20(1+2\gamma)}{12(1+\gamma)^2}\right) - 1\right|\right\}$$

$$= \frac{5}{12(1+2\gamma)} \max\left\{1, \left|\left(\frac{6(1+\gamma)^2 - 5(1+3\gamma) + 20(1+2\gamma)}{6(1+\gamma)^2}\right) - 1\right|\right\}$$

$$= \frac{5}{12(1+2\gamma)} \max\left\{1, \left|\frac{20(1+2\gamma) - 5(1+3\gamma)}{6(1+\gamma)^2}\right|\right\}. \qquad (4.160)$$

It follows that

$$\left|a_3 - 2a_2^2\right| \le \frac{5}{12(1+2\gamma)} \max\left\{1, \frac{5(3+5\gamma)}{6(1+\gamma)^2}\right\}. \qquad (4.161)$$

Substituting equations (4.114), (4.152), and (4.161) into equation (4.74) yields

$$\left|T_3(1)\right| \le 1 + 2\left|a_2^2\right| + \left|a_3\right|\left|a_3 - 2a_2^2\right|$$

$$= 1 + 2\left(\frac{25}{36(1+\gamma)^2}\right) + \frac{5}{12(1+2\gamma)}\left(\frac{5}{12(1+2\gamma)} \max\left\{1, \frac{5(3+5\gamma)}{6(1+\gamma)^2}\right\}\right)$$

$$= 1 + \frac{25}{18(1+\gamma)^2} + \frac{25}{144(1+2\gamma)^2} \max\left\{1, \frac{5(3+5\gamma)}{6(1+\gamma)^2}\right\}. \qquad (4.162)$$

Hence, the bound of Toeplitz determinant $T_3(1)$ can be expressed as

$$\left|T_3(1)\right| \le 1 + \frac{25}{36}\left[\frac{2}{(1+\gamma)^2} + \frac{1}{4(1+2\gamma)^2} \max\left\{1, \frac{5(3+5\gamma)}{6(1+\gamma)^2}\right\}\right]. \qquad (4.163)$$

Proof of Theorem 4.3.5 completes.

Setting $\gamma = 0$ and $\gamma = 1$, the subsequent corollary is obtained:

Corollary 4.3.5.

Let $f \in \mathrm{M}_{0,4L}$. *Then,*

$$\left|T_3(1)\right| \le \frac{271}{96}. \tag{4.164}$$

Let $f \in \mathrm{M}_{1,4L}$. *Then,*

$$\left|T_3(1)\right| \le 1.3794. \tag{4.165}$$

Theorem 4.3.6.

Let $f \in \mathrm{M}_{\gamma,4L}$ *be the expression given in the form (1.1). If* $0 \le \gamma \le 1$*, then*

$$\left|T_3(2)\right| \le \frac{125(2+5\gamma)}{1296(1+\gamma)(1+3\gamma)}\left(\frac{5(1+2\gamma)^2+(1+\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\right). \tag{4.166}$$

Proof.

Assumed that $f \in \mathrm{M}_{\gamma,4L}$. Then, the expression for the Toeplitz determinant $T_3(2)$ is given by equation (4.85). The expression $|a_2 - a_4|$ is solved, and upon applying the triangle inequality and substituting equations (4.110) and (4.122), it follows that

$$\begin{aligned} |a_2 - a_4| &\le |a_2| + |a_4| \\ &= \frac{5}{6(1+\gamma)} + \frac{5}{18(1+3\gamma)} \\ &= \frac{15(1+3\gamma)+5(1+\gamma)}{18(1+\gamma)(1+3\gamma)} \\ &= \frac{20+50\gamma}{18(1+\gamma)(1+3\gamma)}. \end{aligned} \tag{4.167}$$

Upon simplification, the expression becomes

$$|a_2|+|a_4|\le\frac{5(2+5\gamma)}{9(1+\gamma)(1+3\gamma)}. \tag{4.168}$$

Substituting the outcomes of equations (4.149), (4.154), and (4.168) into equation (4.85) yields the following result

$$\begin{aligned}|T_3(2)|&\le|a_2-a_4|\left(\left|a_2^2-a_3^2\right|+\left|a_2a_4-a_3^2\right|\right)\\&=\frac{5(2+5\gamma)}{9(1+\gamma)(1+3\gamma)}\left(\frac{125}{144(1+\gamma)^2}+\frac{25}{144(1+2\gamma)^2}\right).\end{aligned} \tag{4.169}$$

Hence, the Toeplitz determinant $T_3(2)$ takes the form

$$|T_3(2)|\le\frac{125(2+5\gamma)}{1296(1+\gamma)(1+3\gamma)}\left(\frac{5(1+2\gamma)^2+(1+\gamma)^2}{(1+\gamma)^2(1+2\gamma)^2}\right). \tag{4.170}$$

Proof of Theorem 4.3.6 completes.

Setting $\gamma=0$ and $\gamma=1$, the subsequent corollary is obtained:

Corollary 4.3.6.

Let $f\in\mathrm{M}_{0,4L}$. *Then,*

$$|T_3(2)|\le\frac{125}{108}. \tag{4.171}$$

Let $f\in\mathrm{M}_{1,4L}$. *Then,*

$$|T_3(2)|\le 0.1149. \tag{4.172}$$

CHAPTER 5
CONCLUSION

This thesis has introduced and rigorously investigated two new subclasses of analytic univalent functions within the open unit disk: the tilted starlike functions associated with epicycloid domains with $n-1$ cusps, $S^*_{T,n-1}(\alpha)$ and the convex combinations of starlike and convex functions associated with the four-leaf epicycloid domain, $\mathrm{M}_{\gamma,4L}$.

The study began by formulating these new subclasses, building upon the earlier works of Gandhi et al. (2022) on epicycloid-based mappings and Wahid et al. (2015) on tilted starlike functions. Moreover, the work of Shi et al. (2022) on four-leaf-shaped epicycloid domains motivates the present study to further generalize the function class by employing a convex combination of starlike and convex functions. This approach allows the newly defined subclass to encompass and reduce to two well-known Ma–Minda-type function classes in limiting cases, thereby offering a broader and more unified perspective in the study of univalent functions.

The new subclasses were constructed using the principle of subordination involving mapping functions that map the open unit disk onto epicycloid domains. Hence, the first and second research objectives, which are to define the new subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$ has been successfully achieved. The key analytic properties were derived for both subclasses, including initial coefficient estimates, coefficient bounds, the Fekete–Szegö inequality, the second Hankel determinant, and the Toeplitz determinants.

The findings reveal that the inclusion of the tilted factor, $e^{i\alpha}$ introduces directional complexity, making analytic derivations more intricate, particularly for higher-order determinants. In contrast, the convex-combination $e^{i\alpha}$ subclass, retains structural complexity while offering a continuous transition between the classical starlike ($\gamma = 0$) and convex ($\gamma = 1$) cases, thereby establishing a unified analytic framework.

The results for the tilted subclass $S^*_{T,n-1}(\alpha)$ demonstrate strong agreement with established findings in the literature. The coefficient bounds presented in Corollary

4.2.1 are consistent with the sharp estimates obtained by Gandhi et al. (2022) for the class $S^*_{n-1,\mathrm{L}}$. Similarly, the Fekete–Szegö inequality (Corollary 4.2.2) and the second Hankel determinant (Corollary 4.2.3) correspond closely to the sharp results reported by Shi et al. (2022) for the same class, indicating that the tilted starlike subclass coincides with the classical starlike functions under specific parameter conditions.

Meanwhile, the convex combination subclass $\mathrm{M}_{\gamma,4L}$ also exhibits close correspondence with earlier studies. The coefficient bounds in Corollary 4.3.1 are in agreement with those obtained by Gandhi et al. (2022) for the class $S^*_{4,\mathrm{L}}$, while the Fekete–Szegö inequality in Corollary 4.3.2 and the second Hankel determinant in Corollary 4.3.3 align with the sharp results derived by Shi et al. (2022). Notably, when $\gamma = 0$ the class reduces to the starlike case, and when $\gamma = 1$, it approaches the convex case.

Hence, the third research objective, which is to establish coefficient bounds and derive coefficient inequalities for functions in the subclasses $S^*_{T,n-1}(\alpha)$ and $\mathrm{M}_{\gamma,4L}$, including the Fekete–Szegö functional and the second Hankel determinant, has been successfully achieved. The obtained results demonstrate strong consistency with established findings in the literature, while the reduction of the proposed subclasses to existing well-known classes under specific parameter conditions further confirms the validity and soundness of the present research.

The outcomes related to the Toeplitz determinants for both subclasses (Corollaries 4.2.4–4.2.6 and 4.3.4–4.3.6) do not reduce to any existing subclasses reported in the literature. This common outcome underscores the novelty and originality of the present work, highlighting a potential direction for future work in establishing the sharpness of these determinant bounds. Hence, the fourth research objective, which is to compute the Toeplitz determinant $T_2(2)$, $T_3(1)$ and $T_3(2)$ for both subclasses has also been successfully achieved.

Overall, the results obtained not only reduce to known classical cases under limiting parameter values but also validate the generalisations proposed in this thesis. The derived structural formulas and sharp estimates are consistent with established results in GFT, thereby reinforcing the mathematical soundness and relevance of the proposed subclasses.

Despite the successful achievement of all research objectives, several analytical challenges were encountered throughout this study. In particular, the inclusion of the rotational factor $e^{i\alpha}$ significantly increased the algebraic complexity of the derivations, especially in obtaining higher-order coefficient estimates and determinant expressions. The interaction between the rotational effect and the epicycloid mapping required careful and often intricate manipulations, where standard techniques in Geometric Function Theory were not always directly applicable. Additionally, the analysis of the Toeplitz determinants proved to be particularly demanding, as the derived bounds could not be reduced to existing known subclasses, and establishing their sharpness remains an open problem. The convex-combination subclass further introduced non-trivial interactions between starlikeness and convexity, adding another layer of complexity to the analytical process. Nevertheless, through systematic derivation and careful application of subordination principles, all required coefficient bounds, inequalities, and determinant estimates were successfully obtained, thereby fulfilling the research objectives and confirming the validity and significance of the proposed subclasses.

In summary, this thesis contributes to the advancement of GFT by bridging geometric domain transformations with parameter-based generalisations. It provides a rigorous analytic foundation for the study of tilted and convex-combination functions associated with epicycloid domains, paving the way for future research on other non-circular geometries, functional inequalities, and extended mapping domains within GFT.

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Nur Athirah Hani Senin received her Diploma in Mathematical Sciences from Universiti Teknologi MARA (UiTM), Cawangan Johor, Kampus Segamat, in 2020. She subsequently obtained her Bachelor of Science (Hons.) in Management Mathematics from Universiti Teknologi MARA (UiTM), Cawangan Negeri Sembilan, Kampus Seremban 3, in 2023. She is currently pursuing her Master's degree by research in Mathematics at Universiti Teknologi MARA, Cawangan Melaka, Kampus Bandaraya Melaka. In July 2025, she participated in the 32nd Seminar Kebangsaan Sains Matematik (SKSM32) and presented her research titled *"Toeplitz Determinant for a Subclass of Tilted Starlike Functions Associated with the Epicycloid Domain"*. At present, she has published two research articles based on the findings of this thesis, while another one additional manuscript has been accepted for publication.

LIST OF PUBLICATIONS

Senin, N. A. H., Yunus, Y., Wahid, N. H. A. A., & Omar, R. (2026). On Convex Combinations of Starlike and Convex Functions Associated with the Epicycloid Domain. *International Journal of Neutrosophic Science, 27*(1), 234–245. https://doi.org/10.54216/IJNS.270121

Senin, N. A. H., Yunus, Y., Wahid, N. H. A. A., & Omar, R. (2025). Toeplitz Determinant for a Subclass of Tilted Starlike Functions Associated with the Epicycloid Domain. *Menemui Matematik (Discovering Mathematics), 47*(3), 52-64.

LIST OF ARTICLES ACCEPTED FOR PUBLICATION

Senin, N. A. H., Yunus, Y., Wahid, N. H. A. A., & Omar, R. (2025). On a Subclass of Tilted Starlike Functions Associated with the Epicycloid Domain. *TWMS Journal of Applied and Engineering Mathematics*.