UNIVERSITI TEKNOLOGI MARA

BIPOLAR NEUTROSOPHIC WITH DOMBI-HERONIAN MEAN OPERATORS FOR MULTI-CRITERIA DECISION-MAKING APPLICATIONS

SITI NURHIDAYAH YAACOB

MSc

June 2026

UNIVERSITI TEKNOLOGI MARA

BIPOLAR NEUTROSOPHIC WITH DOMBI-HERONIAN MEAN OPERATORS FOR MULTI-CRITERIA DECISION-MAKING APPLICATIONS

SITI NURHIDAYAH YAACOB

Thesis submitted in fulfilment
of the requirements for the degree of
Master of Science
(Mathematics)

Faculty of Computer and Mathematical Sciences

June 2026

CONFIRMATION BY PANEL OF EXAMINERS

I certify that a Panel of Examiners has met on 4 November 2025 to conduct the final examination of Siti Nurhidayah Binti Yaacob on her Master of Science thesis entitled "Bipolar Neutrosophic with Dombi Heronian Mean Operator for Multi-criteria Decision-making Applications" in accordance with Universiti Teknologi MARA Act 1976 (Akta 173). The Panel of Examiners recommends that the student be awarded the relevant degree. The Panel of Examiners was as follows:

Fariza Hanis Abdul Razak, PhD
Associate Professor
Faculty of Computer and Mathematical Sciences
Universiti Teknologi MARA
(Chairman)

Haji Zamali Haji Tarmudi, PhD
Associate Professor
Faculty of Computer and Mathematical Sciences
Universiti Teknologi MARA
(Internal Examiner)

Ku Muhammad Na'im Ku Khalif, PhD
Associate Professor
Centre of Mathematical Sciences
Universiti Malaysia Pahang Al-Sultan Abdullah
(External Examiner)

PROFESSOR DR HJH ZURAEDA IBRAHIM
Dean
Institute of Postgraduates Studies
Universiti Teknologi MARA

Date: 15 June 2026

AUTHOR'S DECLARATION

I declare that the work in this thesis was carried out in accordance with the regulations of Universiti Teknologi MARA. It is original and is the results of my own work, unless otherwise indicated or acknowledged as referenced work. This thesis has not been submitted to any other academic institution or non-academic institution for any degree or qualification.

I, hereby, acknowledge that I have been supplied with the Academic Rules and Regulations for Postgraduate, Universiti Teknologi MARA, regulating the conduct of my study and research.

Name of Student	:	Siti Nurhidayah Yaacob
Student ID. No.	:	2022694412
Programme	:	Master of Science (Mathematics) – CS752
Faculty	:	Faculty of Computer and Mathematical Sciences
Thesis Title	:	Bipolar Neutrosophic with Dombi-Heronian Mean Operators for Multi-criteria Decision-making Applications.
Signature of Student	:	…………………………………………………..
Date	:	15 June 2026

ABSTRACT

Aggregation operators (AOs) play a crucial role in multi-criteria decision-making (MCDM) by providing methods to combine multiple evaluations into a single representative value. Dombi operations, based on Dombi t-norms and t-conorms, offer flexible parameters to address varying degrees of uncertainty and aggregation requirements. At the same time, the bipolar neutrosophic set (BNS) enhances decision-making by capturing positive, negative, and indeterminate information, making it particularly effective under uncertain or incomplete data. However, existing Dombi-based operators within the BNS framework do not consider interrelationships between input arguments. To address this limitation, this study introduces a novel aggregation operator known as the bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean (BND-IGWHM), which combines Dombi operations, the Heronian mean, and BNS. Furthermore, the Decision-Making Trial and Evaluation Laboratory (DEMATEL) method is employed to analyze cause-and-effect relationships in complex decision problems. Recognizing its limitations in capturing interdependencies among arguments, we propose a DEMATEL method enhanced with the BND-IGWHM operator, offering a more comprehensive decision-making methodology. To demonstrate the applicability and effectiveness of the proposed operators, we integrate them into multi-criteria decision-making (MCDM) processes using both a score-based ranking method and a Decision-Making Trial and Evaluation Laboratory (DEMATEL)-based method. A comparative analysis is then performed to evaluate the performance of the proposed BND-IGWHM-DEMATEL method against existing aggregation methods. The results demonstrate that the DEMATEL-based method employing the BND-IGWHM operator offers greater flexibility and improved aggregation accuracy, leading to more effective decision-making in uncertain and bipolar environments.

ACKNOWLEDGEMENT

Firstly, I wish to thank God for giving me the opportunity to embark on my master and for completing this long and challenging journey successfully. My gratitude and thanks go to my supervisor Dr. Hazwani binti Hashim and Dr. Nor Hashimah binti Sulaiman.

My appreciation goes to my family and friends for their unwavering moral support. I truly appreciate their assistance, encouragement, and constant support. A special thanks is also extended to everyone who contributed directly and indirectly to completing this work

Finally, this thesis is dedicated to my father and mother for the vision and determination to educate me. This piece of victory is dedicated to both of you. Alhamdulillah.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

Symbols

i	Criteria i
j	Criteria j
F	Fuzzy Set
B	Bipolar Fuzzy Set
N	Neutrosophic Set
S	Single-Valued Neutrosophic Set
A	Bipolar Neutrosophic Set
K	Set of Experts
$\tilde{A}$	Set of Alternatives
C	Set of Criteria
$S(A)$	Score Function
$a(A)$	Accuracy Function
$c(A)$	Uncertainty Function
$\mu(A)$	Deneutrosophication of BNS
w	Weight of Vector
p	Parameter of Heronian Mean
q	Parameter of Heronian Mean
γ	Parameter in Dombi TN and Dombi TCN

$T_S(x)$	Truth-Membership Function
$I_S(x)$	Indeterminacy-Membership Function
$F_S(x)$	Falsity-Membership Function
X^K	Direct-Relation Matrices
A'	Crisp Number Matrix
I	Identity Matrix
D	Normalized Matrix
T	Total-Relation Matrix
$\tilde{R}_i$	The Sum of the Row of the Total-Relation Matrix
$\tilde{C}_j$	The Sum of the Column of the Total-Relation Matrix
$\tilde{R}_i + \tilde{C}_j$	Adding $\tilde{R}_i$ and $\tilde{C}_j$
$\tilde{R}_i - \tilde{C}_j$	Subtracting $\tilde{R}_i$ and $\tilde{C}_j$

LIST OF ABBREVIATIONS

Abbreviations

AHP	Analytical Hierarchy Process
AM	Arithmetic Mean
ANP	Analytic Network Process
AO	Aggregation Operator
BFS	Bipolar Fuzzy Set
BIFS	Bipolar Intuitionistic Fuzzy Set
BM	Bonferroni Mean
BND-GWHM	Bipolar Neutrosophic Dombi-Based Generalized Weighted Heronian Mean
BNDHM	Bipolar Neutrosophic Dombi-Based Heronian Mean
BND-IGWHM	Bipolar Neutrosophic Dombi-Based Improved Generalized Weighted Heronian Mean
BNN	Bipolar Neutrosophic Number
BNS	Bipolar Neutrosophic Set
DEMATEL	Decision Making Trial and Evaluation Laboratory
DM	Decision-Maker
ELECTRE	Elimination and Choice Translating Reality
FS	Fuzzy Set
GM	Geometric Mean
GWHM	Generalized Weighted Heronian Mean

HM	Heronian Mean
IFS	Intuitionistic Fuzzy Set
IGWHM	Improved Generalized Weighted Heronian Mean
MCDM	Multi-Criteria Decision-Making
NS	Neutrosophic Set
OWA	Ordered Weighted Average
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluation
SBRM	Score-Based Ranking Method
SVNN	Single-Valued Neutrosophic Number
SVNS	Single-Valued Neutrosophic Set
SWM	Solid Waste Management
TCN	Dombi T-Conorm
TN	Dombi T-Norm
TOPSIS	Technique For Order Preference by Similarity to Ideal Solution
VIKOR	Vlsekriterijumska Optimizacija I Kompromisno Resenje

CHAPTER 1
INTRODUCTION

1.1 Research Background

Multi-criteria decision-making is a process for dealing with complex decision-making issues, including multiple criteria and alternative options. The process of MCDM involves selecting criteria, alternatives, and aggregation methods to choose the best alternative based on weights or outranking ultimately (Tuğrul et al., 2020). In MCDM, decision problems with multiple criteria are handled using pairwise or non-pairwise procedures. The pairwise method is frequently applied when two criteria must be directly compared. Each alternative is typically evaluated using any possible combination of the two criteria. However, non-pairwise deals with decision-making issues involving more than two criteria. The analytic hierarchy process (AHP) is an example of MCDM using pairwise, while elimination and choice expressing reality (ELECTRE) is an example of non-pairwise.

By applying pairwise approaches, the decision-makers' preferences can be obtained and incorporated into the decision-making process in an organized way. Among the various pairwise methods in MCDM, the Decision-Making Trial and Evaluation Laboratory (DEMATEL) is widely known and employed. The DEMATEL method is a method that allows for identifying causal relationships within a complex system based on expert assessments. It uses a step-by-step procedure to analyze the data collected by the expert and the respondent. To clarify cause and effect in a complex decision-making problem, DEMATEL effectively analyzes the direct or indirect interrelations of various aspects (Yazdi et al., 2020). As a result of this approach, the different relationships between the criteria can be leveraged to evaluate the overall criteria accurately, leading to more informed and accurate decisions.

Several extensions of the DEMATEL method have been described in the literature, such as fuzzy DEMATEL that allows for an inaccurate description of the criteria or judgments, and neutrosophic DEMATEL, which allows for the consideration of indeterminate viewpoints regarding the criteria or conclusions. The idea of a fuzzy set (FS) introduced by Zadeh (1965) corresponds to the situation in which there is no precisely defined set, and there are increasing applications in various fields. Combining

FS with DEMATEL can be helpful in decision making contexts where the criteria are uncertain or incompletely known. Research on DEMATEL under FS has been applied in various fields such as supplier selection (Chang et al, 2011), business analytics (Yalcin et al., 2022), and supply chain (Deepu & Ravi, 2021). However, FS do not cater the elements of indeterminacy and inconsistent information. To overcome this limitation, Smarandache (1998) introduced the concept of neutrosophic set (NS). Since the appearance of NS, DEMATEL method has been combined with NS. neutrosophic DEMATEL can consider multiple objectives, criteria, and opinions and enables users to assess and adjust the relative importance of criteria and objectives until a consensus is reached. DEMATEL has been extended for better decision making under different environments. When it comes to making decisions, bipolarity plays an important role as it considers both the positive and negative sides of a situation. Therefore, Deli et al. (2015) introduced bipolar neutrosophic set (BNS). Rahim et al., (2020) integrated BNS into DEMATEL by assigning the negative and positive functions in expressing the DEMATEL's linguistic scale.

The DEMATEL method consists of components which includes the structural analysis, score function, and aggregation operator (AO). These elements are combined to create a thorough understanding of the system being studied and to offer information for making decisions. A structural analysis identifies the relationships between elements, and the score function assigns numerical values. AOs are one of the important components in DEMATEL that combines several evaluations from decision-makers into single comprehensive matrices. The aggregation process allows comparing many options and decision-making based on multiple parameters. AO is the process of combining the values of a group of attributes into one representative value for the complete set of attributes. The aggregation of individual preferences or judgments changes the relative weight of expert expertise and knowledge.

In the classical DEMATEL, the aggregation operators used are arithmetic mean (AM) and geometric mean (GM). The critical characteristics of AM emphasize group opinion, while GM focuses on individual opinions (Park & Park, 2013). Based on the literature, various decent aggregation operators have been proposed, such as ordered weighted average, ordered weighted geometric, power averaging, Choquet integral, Bonferroni mean, Heronian mean (HM), prioritized AO, prioritized OWA, harmonic mean, and logarithmic hybrid. HM operator received extensive attention from researchers in the aggregation research direction. The HM was first introduced by

Beliakov et al. (2007) to cater the interrelationship between input arguments. In the topic of MCDM, the HM operator has been utilized as a widespread compromise criterion to estimate the optimal solution between two (or more) criteria. Thus far, some studies have developed and improved the HM operator to be applied in decision making problems (Fan et al., 2019; Hashim et al., 2022; Rao et al., 2023).

Aside from that, Dombi operators offer an operational method in MCDM that allow decision-makers to express their preferences more flexibly. The Dombi is one of the operational methods used in MCDM that aggregates criteria based on Dombi t-norm and tconorm operators and serve as an alternative to the algebraic product and sum operator (Dombi, 1982). The Dombi rule has applications in various fields, such as finance, engineering, and environmental management. It is a powerful method for handling decision-making problems with multiple criteria by integrating different perspectives of decision-makers and demonstrating flexibility in operational parameters.

MCDM studies have been applied in various fields. One of the most significant fields in environmental studies is Solid Waste Management (SWM). Nowadays, the SWM generated in Malaysia has increased significantly over the past few years. Disposing of this waste has also contributed to environmental pollution and degradation, leading to health concerns for the population. As a result, municipal solid waste disposal without consideration has evolved into a widespread issue, particularly in developing nations. Malaysia has a non-existence of appropriate policy on a proper structure and system to manage and dispose the solid waste (Moh & Manaf, 2017). The difficulty of SWM in Malaysia is complex due to various issues such as lacks in community participation and commitment, responsibility sense, weak of organizational, ineffective education, funding issues and so on. Thus, the MCDM approach, specifically DEMATEL is suitable to clarify the interdependence of the criteria and provide the solution of SWM problems. The proposed Dombi Heronian mean operator is used to unify the opinions of decision makers since multiple decision makers with different levels of experience in SWM.

1.2 Problem Statement

The Dombi t-norm (TN) and Dombi t-conorm (TCN) have been widely utilized to generate flexible operation rules by enabling the adjustment of a parameter to manage the weight given to inconsistent data. Mahmood et al. (2020) proposed Dombi bipolar neutrosophic aggregation operators. However, these operators do not take into account the interrelationship between input arguments. To manage such an issue, the Heronian mean (HM) operator is used to cater the interrelationship in aggregating decision makers opinions. In aggregating information, interrelationship is important to generate comprehensive values of different factors. Dombi operations are used to replace the algebraic operations employed in the HM operator. Considering this gap, in this study, the HM operator is combined with Dombi operation to aggregate bipolar neutrosophic information.

Decision Making Trial and Evaluation Laboratory (DEMATEL) is a well-known method for solving complex decision-making problems by analysing factors through cause-and-effect relationships among interrelated criteria. One of the key components in the DEMATEL method is the aggregation operator, which combines multiple evaluations from various decision makers into a single representative assessment. Abdullah and Rahim (2020) integrated bipolar neutrosophic sets (BNS) into the DEMATEL framework, where the arithmetic mean was applied to aggregate decision makers' judgments. The traditional way of aggregating multiple decision makers' opinions is either by averaging mean or geometric mean. However, these approaches treat all input arguments as independent and fail to consider the interrelationships among criteria during the aggregation process, which may lead to the loss of important structural information in complex systems.

Moreover, although various aggregation methods have been proposed to assess and rank alternatives, many of them are not sufficiently effective when applied to real-world decision-making problems characterized by high levels of uncertainty, vagueness, and inconsistent expert opinions. In such situations, traditional aggregation operators may not accurately reflect the practical behavior of decision makers, resulting in less reliable prioritization outcomes. This limitation becomes more critical in bipolar neutrosophic environments, where both positive and negative information must be processed simultaneously, further increasing the complexity of the decision structure. Consequently, a significant gap remains in developing aggregation operators that can simultaneously handle uncertainty, capture interrelationships among criteria, and support causal analysis within the DEMATEL framework. To address this gap, this study proposes a new aggregation approach based on Dombi operators and the Heronian mean under bipolar neutrosophic settings, aiming to improve the reliability and applicability of MCDM solutions in complex real-world problems.

1.3 Research Questions

The questions of this research are

1. How can the Heronian mean operator be effectively incorporated into the Dombi-based aggregation operator in the context of bipolar neutrosophic sets?
2. How can the proposed BNDHM aggregation operators be effectively applied within MCDM problem under bipolar neutrosophic environments?
3. How does the performance of the proposed BNDHM-based decision-making method in comparison to the existing aggregation methods in terms applicability, particularly in decision-making problems?

1.4 Research Objectives

The objectives of this research are:

1. To propose a bipolar neutrosophic Dombi-based Heronian mean (BNDHM) aggregation operators.
2. To integrate the proposed BNDHM operator in multi-criteria decision-making (MCDM) namely the score-based ranking method and DEMATEL-based method.
3. To compare the performance and applicability of the proposed BNDHM-based MCDM method with existing aggregation methods in decision-making problems.

1.5 Scope of the Study

This research focuses on introducing a new aggregation operator, the bipolar neutrosophic Heronian mean operator based on the Dombi operation, aiming to advance the application of aggregation operators in decision-making. Subsequently, the study integrates this operator with the DEMATEL method. The research utilizes secondary data on solid waste management case study from Abdullah et al. (2019) to illustrate the applicability and effectiveness of the proposed operators BNDHM-based MCDM method.

1.6 Significance of Study

This study has been a significant contribution in the realm of decision-making, providing methodologies that take into consideration the uncertainties elements in decision-making process. This study explored new aggregation operators where the HM operator is combined with the Dombi operation under the BNS environment. Besides, this research is significant in the aggregate opinions of group decision-makers on interdependent factors. The DEMATEL method helps in the selection of the best factor based on causes and effects groups, allowing for better decision-making.

1.7 Thesis Outline

This thesis comprises seven chapters. The first chapter provides insight into the research field and its objectives. It describes the research background, problem statement, objectives, significance of the study, and thesis outline well. Chapter 2 comprehensively reviews the literature on multi-criteria decision-making (MCDM) methods, focusing on the decision-making trial and evaluation laboratory (DEMATEL) method. It also examines various sets, and their generalizations and reviews related studies on aggregation operators. Additionally, this chapter discusses a case study on solid waste management. The final section identifies gaps in the existing literature. Subsequently, Chapter 3 presents the preliminaries of related set theories and methods, defining key concepts, properties, and fundamentals. It begins with fundamental concepts and definitions of fuzzy sets, bipolar fuzzy sets, neutrosophic sets, single-valued neutrosophic sets, and bipolar neutrosophic sets (BNS). Next, it explores the properties of the BNS, followed by discussions on the Heronian mean, Dombi operations of BNSs, and the fundamental concept of DEMATEL. These preliminaries and theoretical concepts will be applied when conducting the study.

Chapter 4 introduces new aggregation operators for bipolar neutrosophic sets by incorporating the Heronian mean operator into Dombi. These operators are the bipolar neutrosophic Dombi-based generalized weighted Heronian mean (BND-GWHM) and bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean (BND-IGWHM). The chapter provides details of each aggregation operator's development and properties. Afterwards, it presents an MCDM method based on these operators, including a numerical illustration and a comparison with existing approaches. Afterwards, Chapter 5 examines the integration of the proposed bipolar neutrosophic Dombi-based Heronian mean (BNDHM) aggregation operators into a MCDM framework. Specifically, the chapter introduces a score-based ranking method and a DEMATEL-based method. The chapter illustrates the applicability and effectiveness of the proposed approach through a detailed example addressing an MCDM problem.

Chapter 6 presents a comparative analysis of the two proposed MCDM methods. It also examines the impact of various parameters and compares the results with those of existing approaches. The final chapter summarizes all the contents and accomplishments of the thesis. The objective and outcomes of the study are summarized in this chapter. Contributions of the research are then discussed. Finally, the chapter addresses the research limitations and offers recommendations for future studies.

1.8 Conclusion

This chapter offers an overview of the study by discussing the research background and highlighting the key issues. It also emphasizes the three main objectives designed to address these issues. The thesis outline and research framework are given in Figure 1.1 and Figure 1.2, respectively

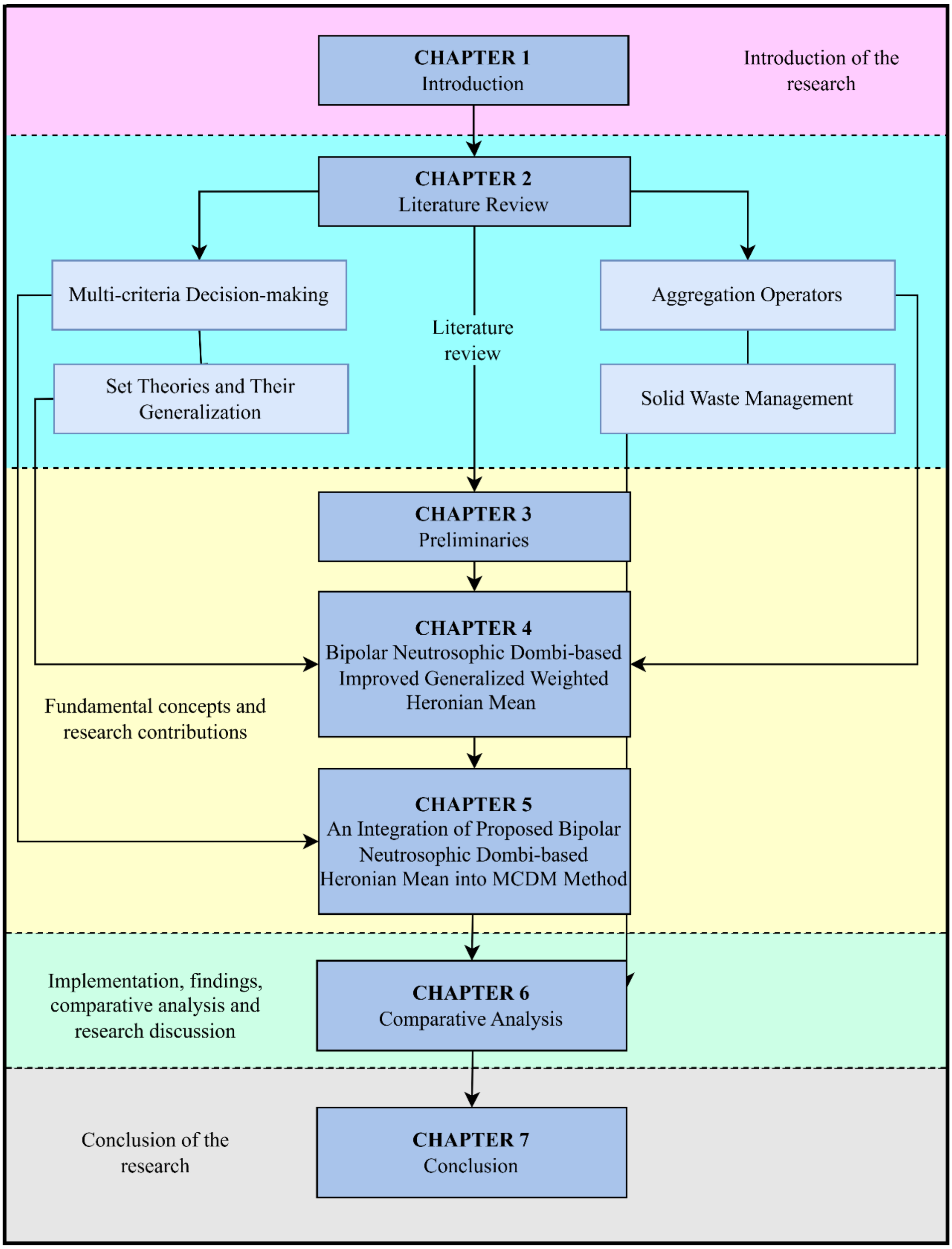

Figure 1.1 Thesis Outline

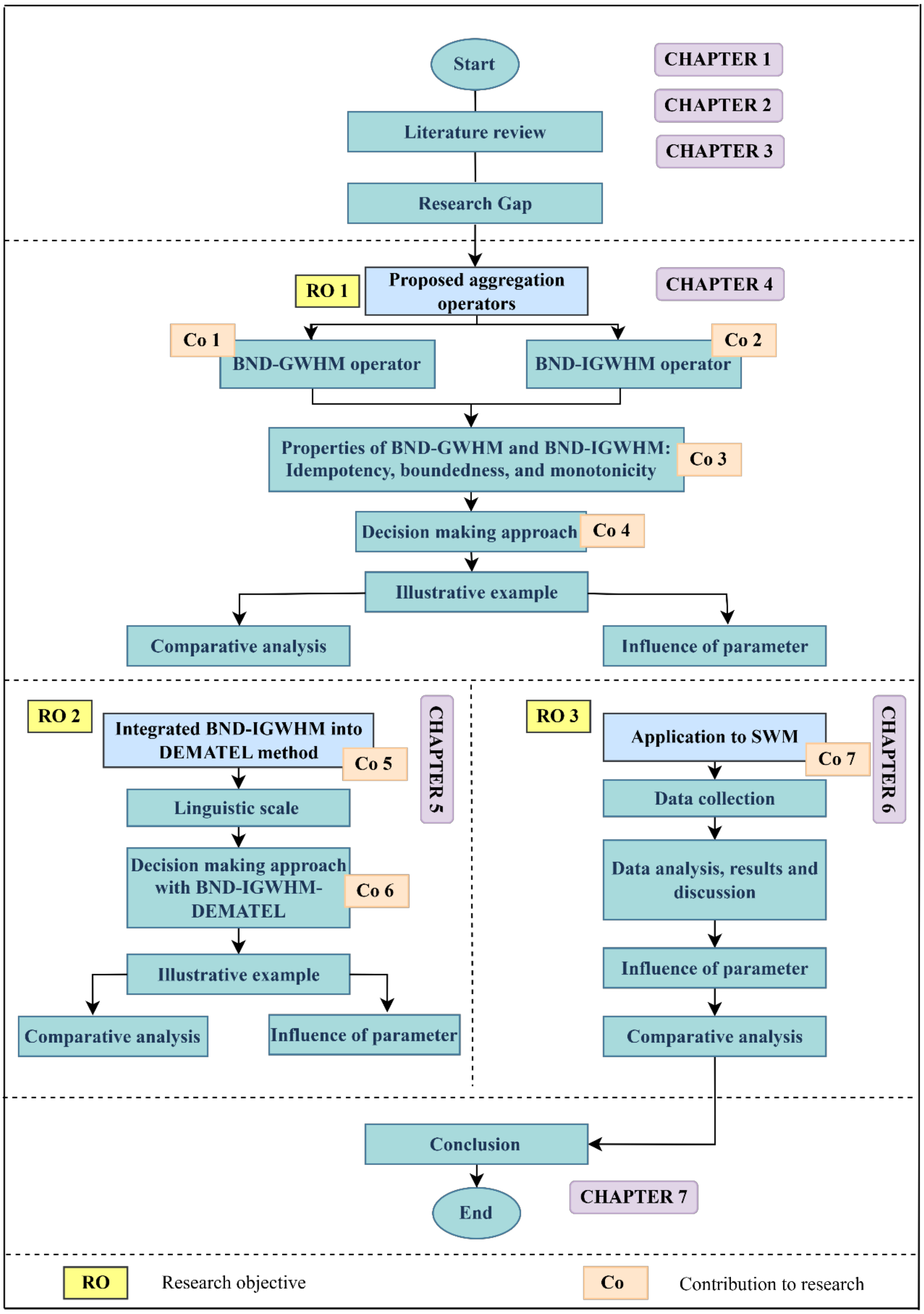

Figure 1.2 Research Framework

CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

This chapter reviews some relevant topics in six sections. Section 2.2 reviews MCDM methods. Section 2.3 discusses sets and their generalization, particularly in MCDM. Section 2.4 provides a review of studies about aggregation operators, and section 2.5 identifies literature gaps. Next, a conclusion is provided at the end of this chapter.

2.2 Multi-criteria Decision-making

A multi-criteria decision-making (MCDM) involves decision-making situations characterized by multiple, often competing criteria. It is a systematic approach for evaluating and selecting alternatives based on various criteria. According to Leake (2000), MCDM can be further divided into two main subfields based on the number of alternatives taken into consideration: multi-attribute decision-making (MADM) and multi-objective decision-making (MODM). The MADM can evaluate discrete decision spaces with predetermined alternatives, while MODM is preferable for continuous decision problems with the unpredictability of alternatives. The MODM method generates solutions based on mathematical optimization models such as linear programming or evolutionary algorithms. As a result, both methods offer decision-makers systematic ways to deal with complicated decision issues involving several criteria or objectives.

MCDM method can be classified as a pairwise and nonpairwise technique. The pairwise comparison approach involves comparing two factors against one another in order to make a choice between them. Examples of MCDM method using pairwise technique is (AHP), analytic network process (ANP) and decision-making trial and evaluation laboratory (DEMATEL). Meanwhile, non-pairwise methods deal with decision-making issues involving more than two criteria. The elimination and choice expressing reality (ELECTRE), technique for order of preference by similarity to ideal solution (TOPSIS), vise kriterijumsa optimizacija i kompromisno kesenje (VIKOR) are example of MCDM method using non-pairwise method. Every MCDM method has its specialty and advantages in solving MCDM problems. However, pairwise methods reduce the comparisons required to manage complexity in a real-world decision scenario.

As for clarifying the interrelationships of criteria, the DEMATEL method is the best method for solving the MCDM problem. The Decision-Making Trial and Evaluation Laboratory (DEMATEL) method was proposed by Battelle Geneva Institute as a way of analyzing these relationships (Gabus & Fontela, 1973). The cause-and-effect relationship diagram of the DEMATEL method transforms complex causal relationships into a visible structure that helps the decision-makers to make the right decision by understanding the mutual influence or impact between different elements. The method has tremendous research achievements that have been effectively implemented to tackle challenging MCDM problems, such as business analytics (Yalcin et al., 2022), coastal erosion (Awang et al., 2019), and supply chain (Deepu & Ravi, 2021). Next, Asadi et al. (2022) introduced a multicriteria model that can help managers work together to determine what makes a smart city smart using the DEMATEL method.

Furthermore, DEMATEL has been expanded to allow for better decision-making in various environments because imprecise and uncertain information exists in real-world situations. Many academics include fuzzy DEMATEL in their research to deal with the situations. In the literature, Ghadami et al. (2019) conducted a study utilizing the fuzzy DEMATEL approach to rate the accrediting categories, subcategories, and hospital requirements. As a result of their research, hospitals can improve operations by taking appropriate action and paying close attention to the core management team. In addition, Mahmoudi et al. (2019) also carried out the elements of heart failure self-care in uncertain circumstances using fuzzy DEMATEL. Abdullah et al. (2019) recently developed a DEMATEL tool based on straight-line fuzzy numbers for waste disposal. Zhu et al. (2020) applied a combination of DEMATEL and VIKOR techniques to identify critical failure paths in dam systems. Besides, Salehi et al. (2021) conducted a study on the sugarcane supply chain in Khuzestan province using a combined ANP-DEMATEL method. To manage the complexity of MCDM problems, Abdullah et al. (2023) proposed an integrated fuzzy DEMATEL–fuzzy TOPSIS approach for analyzing smart manufacturing technologies.

2.3 Set Theories and Their Generalizations

The decision-making process often requires managing complex and uncertain data. Traditional crisp sets, defined by clear boundaries and binary membership, usually fail to sufficiently capture the inherent vagueness and ambiguity in real-world decision problems. Therefore, Zadeh (1965) introduced the fuzzy set (FS) theory, presenting a more flexible framework for a smoother representation of uncertain information, especially in real-life problems. However, FSs were limited in dealing with complicated information since they only had one membership degree. To improve FS theory, Atanassov (1986) studied the theory of an intuitionistic fuzzy set (IFS), which characterized the membership and non-membership functions between 0 and 1. Atanassov and Gargov (1989) later proposed the interval-valued intuitionistic fuzzy set (IVIFS) idea. However, IFS and IVIFS have the limitation that they can only represent contradictory information in real-life decision-making (DM) situations.

Later, Smarandache (1998) introduced the neutrosophic set (NS) as an extension of fuzzy sets (FSs), which incorporated the third parameter, the neutrosophic set. As a result, decision-makers gain a more flexible framework for modelling complex decision problems, accommodating indeterminacy or inconsistency. NS can represent membership, non-membership, and indeterminacy functions, empowering decision-makers to make informed and robust decisions in an increasingly uncertain and complex environment. Wang et al. (2004) expanded upon this concept with interval neutrosophic sets (INS). In 2010, the authors introduced single-valued neutrosophic sets (SVNS) to apply NS to practical DM problems. SVNS are a powerful tool for DM, as they combine the advantages of classical and fuzzy set theories into a single framework and allow for more complex and accurate modelling of decision problems, leading to better decision-making outcomes. Additionally, Liu et al. (2018,a) proposed a new single-valued neutrosophic number (SVNN) with DEMATEL multi-criteria model for the selection of the transport service provider, and (Abdel-Basset et al., 2018) suggested a hybrid approach of NSs and DEMATEL method for developing supplier selection criteria. Also, Awang et al. (2018) applied Shapley weighting vector into SVNS-DEMATEL. Kilic et al. (2021) developed an approach for evaluating leanness, and in the Philippines, Mamites et al. (2022) studies about factors affecting university teaching quality based on neutrosophic DEMATEL. Sathyan et al. (2023) investigated fuzzy DEMATEL to identify responsiveness enablers in automotive supply chains.

In real-life scenarios, decision-making processes often involve expressing positive and negative opinions regarding a specific context. Zhang (1994) addressed this issue by introducing bipolarity into FS theory, which led to the development of the bipolar fuzzy set (BFS). According to Bosc and Pivert (2013), "bipolarity" refers to the human mind's ability to consider both positive and negative factors in decision-making. Positive information refers to viable, satisfactory, or acceptable options, whereas negative information refers to unworkable or prohibited options. In decision-making processes, positive preferences indicate desires, while negative preferences indicate limitations. Recognizing the benefits of both BFS and NS, Deli et al. (2015) introduced the concept of a bipolar neutrosophic set (BNS). A BNS has positive and negative degrees of membership, which are designed specifically to manage bipolar, inconsistent, and ambiguous data. Recently, the concept of a BNS has been integrated with DEMATEL (BNS-DEMATEL) due to its ability to handle uncertain and conflicting information since it considers both the positive and negative sides of the problem. However, few studies about the BNS-DEMATEL exist in the literature, such as urban sustainability (Abdullah & Rahim., 2020) and green supply chain (Saleh et al., 2023). As an overview, Figure 2.1 depicts the chronological evolution from classical set theory to the bipolar neutrosophic set.

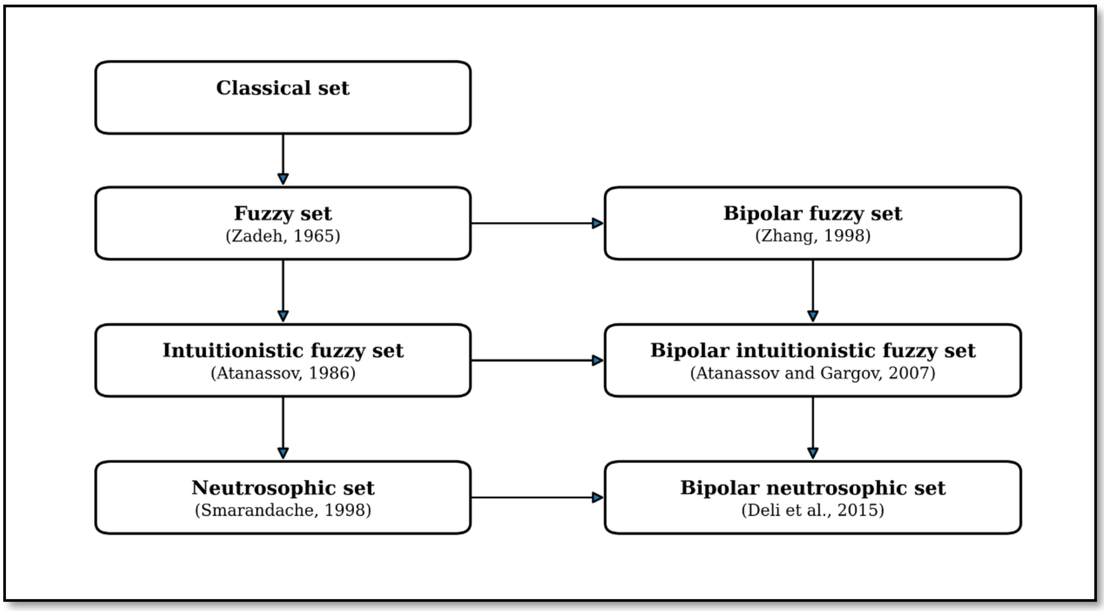

Figure 2.1 The Chronological Development of Bipolar Neutrosophic Set

2.4 Aggregation Operators

Aggregation operators are mathematical functions that merge multiple inputs into a single output. They play a pivotal role in diverse fields, such as decision-making (Petrović et al., 2023; Khan & Gulistan, 2023), data analysis (Riaz et al., 2022), and artificial intelligence (Torra et al., 2018). These operators are essential for synthesizing information from various sources. The use of aggregate operators is becoming increasingly important, especially in group decision-making contexts. The arithmetic mean and geometric mean are two fundamental aggregation operators commonly employed to aggregate data in a crisp environment. The arithmetic mean emphasizes group opinion, while the geometric mean focuses on individual opinions (Park & Park, 2013). Also, several advanced aggregation operators have been proposed, including the ordered weighted average (Yager, 1988), ordered weighted geometric (Chiclana et al., 2002), power averaging (Yager, 2001), Choquet integral (Choquet, 1953), and Bonferroni mean (Bonferroni, 1950), others.

Most current aggregation operators assume that criteria are mutually independent based on the assumption of their independence. These limitations can be encountered by the Heronian mean (HM), Choquet integral, and Bonferroni mean (BM) operators. However, the Bonferroni mean, and Choquet integral ignores the interrelationship within itself. The HM operator is significant in measuring the interrelationship of input arguments and itself without repeated calculation. Consider the following situation to illustrate how the interrelationship between input arguments works in the HM operator (Liu & You, 2018): when investing in a project, there are some attributes that need to be considered such as geological location (C_1), production risk (C_2), market risk analysis (C_3), and social environment (C_4). For a set of criteria $C_i (i = 1,2,3,4)$, the HM operator consider the interrelationship between C_1 and C_1, C_2, C_3, C_4; C_2 and C_2, C_3, C_4; C_3 and C_3, C_4; C_4 and C_4. The HM operator can solve the interrelationship of both different criteria values $C_i, C_j (i < j)$ and the criteria value C_i itself.

Moreover, the research on HM operators has been extended under neutrosophic environments. To indicate the truth, indeterminate, and falsity membership degree in HM aggregation operator, Li et al. (2016) proposed single valued neutrosophic set information with HM operator and applied it in the case study of air quality. Fan et al.

(2019) employed an HM operator to solve MCDM problems under a bipolar neutrosophic set environment. Afterward, Lin et al. (2022) explored picture fuzzy (PF) HM operators in MADM, and Hu et al. (2022) proposed a decision-making method using interval-valued intuitionistic fuzzy weighted HM to handle complex problems with uncertainty. Hashim et al. (2022) constructed HM under interval neutrosophic vague set. Mishra et al. (2023) studied urban climate change policy for transportation using Fermatean fuzzy Heronian mean operators, which use Heronian mean aggregation to improve decision accuracy and reliability. Later, Kakati et al. (2024) developed the Fermatean fuzzy Archimedean HM to estimate sustainable urban transport solutions. Ali et al. (2024) explored Archimedean Heronian mean operators based on complex IFSs.

Apart from that, the HM operator is a common practice in decision-making processes, where multiple inputs or criteria are combined to determine an overall outcome. Typically, algebraic operational rules like intersection and union are used to aggregate data in HM operator. However, complex relationships between criteria are always present in real-world decision problems. As a result, it is critical for an aggregation operator to capture these complex interrelationships and produce more accurate aggregation results. Thus, researchers concentrated on using union and intersection based on t-operators because they provide an optimal unique solution. Until now, research on aggregation operators used different norm (TM) and t-conorm (TCN) such as algebraic operation, Einstein operations, Hamacher operations, Dombi operations and just to name a few. These operational laws have been used in combination with the HM to facilitate decision-making. For instance, Peng et al. (2018) developed a linguistic intuitionistic MCDM using Frank Heronian mean operators. Similarly, Liu et al. (2018,b) have extended the Frank HM operator to partitioned HM operators. In another study, Anusha and Sireesha (2022) combined the HM with the Einstein algorithm to develop an interval-valued intuitionistic fuzzy Einstein Heronian mean operator for decision-making cases. Also, Peng et al. (2018) introduced single-valued neutrosophic hesitant fuzzy geometric Choquet integral HM operator. Moreover, Li and Yang (2021) developed the HM based on Hamacher operations under a multi-valued picture fuzzy uncertain linguistic environment. Along with the previously mentioned operators, the Dombi operation offers enhanced flexibility through its tunable parameter, which enables the aggregation process to be adjusted to reflect different levels of strictness or compensation among the input arguments.

The Dombi operational law is a widely recognized approach to decision-making that has been combined with the HM to achieve superior accuracy and consistency in decision-making. Dombi (1982) proposed the Dombi T-norm and T-conorm operations, which demonstrate the benefit of good flexibility with the operational parameter. Liu et al. (2017) expanded Dombi operations to intuitionistic fuzzy sets (IFSs) and suggested some intuitionistic fuzzy (IF) Dombi Bonferroni mean operators for MAGDM problems. To cope with the aggregation of single valued neutrosophic number, (Chen & Ye, 2017) presented the single valued neutrosophic Dombi weighted arithmetic average (SVNDWAA) operator and the SVN Dombi weighted geometric average (SVNDWGA) operator and pointed in their properties. Later, Liu et al. (2019) explored Dombi T-norm operational laws in the context of linguistic neutrosophic and proposed a 2-tuple linguistic neutrosophic power Heronian mean (2-TLNPHM) operator. Recently, Liu et al. (2020) proposed some HM fused operators with IVIFNs, and Mahmood et al. (2020) suggested some bipolar neutrosophic based on Dombi operations. Fan et al. (2019) developed the single-valued triangular neutrosophic Dombi prioritized normalized Bonferroni mean (SVTNDPNBM) operator and applied to the problem of the green supplier selection, which proves its feasibility and stability. In 2020, Wu et al. (2020) employed the Dombi HM to perform the ecological value evaluation of forest ecological tourism demonstration. In addition, Jana et al. (2021) present the MCDM method for selecting the best contractor for a road construction company based on single-valued trapezoidal neutrosophic number Dombi aggregation. Recently, Alballa et al. (2024) studied a 2-tuple linguistic Pythagorean fuzzy Dombi aggregation operation to examine the MCDM problem. Besides, Önden et al. (2024) introduced a novel fuzzy Dombi Bonferroni weighted assessment to integrate Geographic Information System (GIS) output with capacity calculations.

2.5 Identification of Research Gaps

Aggregation operators (AOs) are crucial in multi-criteria decision-making, as they integrate various types of evaluation data from decision-makers. According to the literature, numerous AOs have been developed for this purpose. However, most studies rely on traditional methods like arithmetic mean and geometric mean for data aggregation, which did not take into account for the interrelationships among input arguments. Therefore, the study proposes combining Heronian mean operators with Dombi operations, replacing conventional algebraic methods to capture these interrelationships better. Table 2.1 presents the findings and gaps in the existing AOs based on published literature.

The literature review on the DEMATEL method highlights its effectiveness in analyzing the factors through cause-and-effect relationships and provides a more flexible framework for decision analysis. However, the existing research on the DEMATEL method in the context of the neutrosophic set primarily relies on arithmetic operations such as addition and subtraction. However, this approach may be insufficient in fully capturing complex interrelationships between input arguments. To address this gap, this study aims to develop a modification that enhances the DEMATEL method's ability to clarify the interrelationships between input arguments during the aggregation process. Table 2.2 summarizes the findings and gaps identified in the literature review on DEMATEL methods.

Table 2.1
Research gaps on Aggregation Operators

AO	Set	Author	Findings	Gaps
Weighted HM	SVNS	Li et al. (2016)	-Proposed single-valued neutrosophic information with HM -Applied in the case study of air quality	-Did not consider the bipolarity -Did not consider the flexibility through a parameter
Heronian mean	BNS	Fan et al. (2019)	-Solve MCDM problem under BNS environment which can handle bipolarity -Consider the interrelationship between argument	-Uses algebraic product and algebraic sum in the operation -Did not consider the flexibility through a parameter
Heronian mean	Picture fuzzy	Lin et al. (2022)	-Studied generalized HM under picture fuzzy and prove some properties -Solve multiple attribute decision-making problem	Did not consider the bipolarity -Only uses three membership degrees (positive, neutral, and negative)
Weighted HM	IVIFS	Liu et al. (2020)	-Introduced the membership function and non-membership function of INVS -Applied in the case study of investment decision	-Did not consider both positive and negative memberships -Did not consider the flexibility through a parameter
Heronian mean	IVFFSs	Mishra et al. (2023)	-AOs can capture the relationships between IVFFNs -Application in transportation sector	-Limited flexibility in membership representation -Did not consider the flexibility through a parameter
Archimedean HM	Fermatean fuzzy sets	Kakati et al. (2024)	-Handle ambiguity and dissatisfaction in real-life -Proposed MADM approach to in urban transport systems	-Did not consider the bipolarity -Less flexibility in handling complex aggregations
Dombi BM	IFS and IF	Liu et al. (2017)	-Studied Dombi operation under IFS and introduced IF into Dombi BM -Allow flexibility through parameter adjustment	-Did not consider both positive and negative memberships
Dombi	SVNS	Chen and Ye (2017)	-Explored Dombi operation under SVNS environment -Solved decision-making problem	-Did not consider the bipolarity -Aggregate using arithmetic mean and geometric mean
Dombi	2-TLNPHM	Liu et al. (2018)	-Explored Dombi operation in the context of linguistic neutrosophic -Consider interrelationship among two input argument -Developed an algorithm for solving MAGDM problem	Limited handling of bipolarity -Only represent truth, indeterminacy, and falsity and not explicitly separate positive and negative memberships
Dombi	BNS	Mahmood et al. (2020)	-Can handle bipolarity and indeterminacy -Allow flexibility through parameter	-Did not capture interrelationship between input argument
Dombi	SVTNS	Jana et al. (2021)	-Presented the MCDM method for finding the best contractor	-Limited flexibility in handling complex uncertainty -Does not explicitly separate positive and negative memberships
Dombi	2-TLPFS	Alballa et al. (2024)	-Studied 2-tuple linguistic Phythagorean fuzzy with Dombi operation and applied in MCDM problem	-Less flexible in handling bipolar uncertainties -Did not consider the interrelationship between argument

Table 2.2
Research gaps on DEMATEL and its variants

AO	Author	Findings	Gaps
DEMATEL	Gabus and Fontela (1973)	-Can use NRM to represent the relative relationships between the criteria. -Able to organize the criteria into cause-and-effect categories -Able to rank the criteria according to the important of degree	-Struggles to handle uncertain and complex decision environments -Ignore the presence of extreme values due to bias from decision-makers (DMs) Unable to assign appropriate weights to the criteria
Fuzzy-DEMATEL	Ghadami et al. (2019), Mahmoudi et al. (2019), Abdullah et al. (2019), and Sathyan et al. (2023)	-Able to evaluate the interrelations of the criteria -Able to organize the criteria into cause-and-effect categories -Able to rank the criteria according to the important of degree	-Did not take into account the interdependencies among the evaluation criteria -DMs were unable to express their opposing preference honestly Unable to assign appropriate weights to the criteria and DMs
NS-DEMATEL	Liu et al. (2018), Abdel-Basset et al. (2018), Awang et al. (2018)	-Able to handle uncertainty and indeterminacy in decision-making -Able to organize the criteria into cause-and-effect categories -Able to rank the criteria according to the important of degree	-Ignore the presence of extreme values due to bias from decision-makers (DMs) -Unable to assign appropriate weights to the criteria
BNS-DEMATEL	Abdullah and Rahim, (2020) and Saleh et al. (2023)	-Can handle bipolarity and indeterminacy in decision-making -Able to organize the criteria into cause-and-effect categories -Able to rank the criteria according to the important of degree	-Did not consider the overall interrelationship of different input argument
Integration of DEMATEL and VIKOR	Zhu et al. (2020)	-Able to determine the causal relationship among the criteria -Able to evaluate the interrelations of the criteria -Capable of identifying the optimal solution by finding the one closest to the ideal solution	-Unable to assign weights of criteria and decision-makers -Ignore the presence of extreme values due to bias from DMs
Integration of DEMATEL and ANP	Salehi et al. (2020)	-Able to determine the causal relationship among the criteria -Able to evaluate the interrelations of the criteria -Able to rank the criteria according to the important of degree	-Ignore the presence of extreme values due to bias from DMs -The 9-point scale for pairwise comparison is lengthy and can be confusing
Integration of DEMATEL and TOPSIS	Abdullah et al. (2023)	-Able to mitigate the inherent complexity of MCDM problems -Able to rank the criteria according to the important of degree	-Unable to assign weights of criteria and DMs -Ignore the presence of extreme values due to bias from decision-makers (DMs)

2.6 Conclusion

This chapter has reviewed the literature relating on the study such as MCDM, sets, and aggregation operators. Subsequently, research gaps are identified, and some issues have been highlighted. Based upon the literature review, the existing aggregations operators rely mainly on arithmetic-based methods that overlook interrelationship among input argument and the current application on DEMATEL method in NS environment use basic operations that fail to capture complex dependencies. On that account, this study aims to proposes some solution to overcome these issues in the following chapter.

CHAPTER 3
PRELIMINARIES

3.1 Introduction

This chapter provides six sections containing the preliminaries of related set theories and methods involving concepts' definitions, properties, and fundamentals. Section 3.2 presents fundamental concepts and definitions, related to fuzzy set, bipolar fuzzy set, neutrosophic set, singled-valued neutrosophic set, and bipolar neutrosophic set (BNS). Next, in Section 3.3 provides the properties of the bipolar neutrosophic set, followed by the Heronian mean in Section 3.4. Section 3.5 defines the Dombi operations of BNSs, while the last section of this chapter demonstrates the fundamental concept of DEMATEL.

3.2 Fundamental Basic Concepts and Definitions of Sets

3.2.1 Fuzzy Set and Bipolar Fuzzy Set

This section reviews some definitions and properties related to fuzzy set theory proposed by Zadeh (1965) and bipolar fuzzy set introduced by Zhang (1998) , which will be used throughout the study.

Definition 3.1. Fuzzy Set (Zadeh, 1965)

Let X be a nonempty set. A fuzzy set F in X is described membership functions $\tilde{\mu}_F(x): X \rightarrow [0,1]$. Then, a fuzzy set (FS) is defined as follows:

$$F = \{\langle x, \tilde{\mu}_F(x) \rangle : x \in X\} \tag{3.1}$$

Definition 3.2. Bipolar Fuzzy Set (BFS) (Zhang, 1994)

Let X be a nonempty set. A bipolar fuzzy set B in X is described as $\tilde{\mu}_B(x): X \rightarrow [0,1]$.

$$B = \{\langle x, \tilde{\mu}_B^{+}(x), \tilde{\mu}_B^{-}(x)\rangle : x \in X\} \tag{3.2}$$

where $\tilde{\mu}_B^{+}(x) : X \to [0,1]$ is a positive membership degree and $\tilde{\mu}_B^{-}(x) : X \to [0,1]$ is a negative membership of $x \in X$.

3.2.2 Neutrosophic Set

This section reviews some definitions and properties related to neutrosophic set proposed by Smarandache (1998) and single-valued neutrosophic set studied by Wang et al. in 2010, and bipolar neutrosophic set developed by Deli et al. (2010) which will be used throughout the study.

Definition 3.3. Neutrosophic Set (NS) (Smarandache, 1998)

Assume X be a universal set. A neutrosphic set N in X is divided into three membership functions: namely the membership function true $T_N(x)$, indeterminate $I_N(x)$ and false $F_N(x)$, each of which is included in a real standard or non-standard subset of $]-0,1+[$. Then, a NS is defined as follows:

$$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\} \tag{3.3}$$

Definition 3.4. Single-valued Neutrosophic Set (SVNS) (Wang et al., 2010)

Assume X a universal set. A single-valued neutrosphic set S in X which is described by a truth-membership function $T_S(x)$, an indeterminacy-function $I_S(x)$, and a falsity-membership function $F_S(x)$. Then, a SVNS is defined as follows:

$$S = \{\langle x, T_S(x), I_S(x), F_S(x)\rangle : x \in X\} \tag{3.4}$$

where the functions $T_S(x), I_S(x), F_S(x) \in [0,1]$ satisfy the condition $0 \le T_S(x) + I_S(x)$ $+ F_S(x) \le 3$ for $x \in X$.

Definition 3.5. Bipolar Neutrosophic Set (BNS) (Deli et al., 2015)

Let X be a nonempty set. A bipolar neutrosophic set A is defined as:

$$A=\left\{\left\langle x,T^{+}\left(x\right),I^{+}\left(x\right),F^{+}\left(x\right),T^{-}\left(x\right),I^{-}(x),F^{-}\left(x\right)\right\rangle :x\in X\right\}, \tag{3.5}$$

where $T^{+},I^{+},F^{+}:X\rightarrow\left[1,0\right]$ and $T^{-},I^{-},F^{-}:X\rightarrow\left[1,0\right]$. Hence, the $T^{+}\left(x\right)$, $I^{+}\left(x\right)$, $F^{+}\left(x\right)$ represents positive membership degree, while $T^{-}\left(x\right)$, $I^{-}\left(x\right)$, $F^{-}\left(x\right)$ represents negative membership degree of an element $x\in X$.

3.3 Properties of the Bipolar Neutrosophic Set (BNS)

The properties of BNS are essential to allow the combination and modification of BNS, especially in solving complex problems involving uncertainty. Based on Deli et al. (2015), the properties of the BNS can be defined as follows:

Definition 3.6. The Inequality Property (Deli et al., 2015)

Let $A_1=\left\{x,\left\langle t_1^{+}(x),i_1^{+}(x),\ f_1^{+}(x),t_1^{-}(x),i_1^{-}(x),f_1^{-}(x)\right\rangle\right\}$ and $A_2=\left\{x,\left\langle t_2^{+}(x),i_2^{+}(x),f_2^{+}(x),\right.\right.$ $\left.\left.t_2^{-}(x),i_2^{-}(x),f_2^{-}(x)\right\rangle\right\}$ be two BNSs. Then, the inequality property is defined as $A_1\subseteq A_2$ if and only if

$$t_1^{+}(x)\le t_2^{+}(x),i_1^{+}(x)\le i_2^{+}(x),f_1^{+}(x)\le f_2^{+}(x), \tag{3.6}$$

and

$$t_1^{-}(x)\le t_2^{-}(x),i_1^{-}(x)\le i_2^{-}(x),f_1^{-}(x)\le f_2^{-}(x) \tag{3.7}$$

for all $x\in X$.

Definition 3.7. The Equality Property (Deli et al., 2015)

Let $A_1=\left\{x,\left\langle t_1^{+}(x),i_1^{+}(x),\ f_1^{+}(x),t_1^{-}(x),i_1^{-}(x),f_1^{-}(x)\right\rangle\right\}$ and $A_2=\left\{x,\left\langle t_2^{+}(x),i_2^{+}(x),f_2^{+}(x),\right.\right.$ $\left.\left.t_2^{-}(x),i_2^{-}(x),f_2^{-}(x)\right\rangle\right\}$ be two BNSs. Then, the equality property is defined as $A_1=A_2$ if and only if

$$t_1^{+}(x)=t_2^{+}(x),\ i_1^{+}(x)=i_2^{+}(x),\ f_1^{+}(x)=f_2^{+}(x), \tag{3.8}$$

and

$$t_1^-(x) = t_2^-(x),\ i_1^-(x) = i_2^-(x),\ f_1^-(x) = f_2^-(x) \tag{3.9}$$

for all $x \in X$.

Definition 3.8. Union of BNSs (Deli et al., 2015) Let $A_1 = \{x, \langle t_1^+(x), i_1^+(x), f_1^+(x), t_1^-(x), i_1^-(x), f_1^-(x) \rangle\}$ and $A_2 = \{x, \langle t_2^+(x), i_2^+(x), f_2^+(x), t_2^-(x), i_2^-(x), f_2^-(x) \rangle\}$ be two BNSs. Then, their union is defined as:

$$A_1 \cup A_2 = \left\{ x, \left\langle \begin{pmatrix} \max\left(t_1^+(x), t_2^+(x)\right), \dfrac{\left(i_1^+(x) + i_2^+(x)\right)}{2}, \min\left(f_1^+(x), f_2^+(x)\right) \\ \min\left(t_1^-(x), t_2^-(x)\right), \dfrac{\left(i_1^-(x) + i_2^-(x)\right)}{2}, \max\left(f_1^-(x), f_2^-(x)\right) \end{pmatrix} \right\rangle \right\} \tag{3.10}$$

for all $x \in X$.

Definition 3.9. Intersection of BNSs (Deli et al., 2015)

Let $A_1 = \{x, \langle t_1^+(x), i_1^+(x), f_1^+(x), t_1^-(x), i_1^-(x), f_1^-(x) \rangle\}$ and $A_2 = \{x, \langle t_2^+(x), i_2^+(x), f_2^+(x), t_2^-(x), i_2^-(x), f_2^-(x) \rangle\}$ be two BNSs. Then, their intersection is defined as:

$$A_1 \cap A_2 = \left\{ x, \left\langle \begin{pmatrix} \min\left(t_1^+(x), t_2^+(x)\right), \dfrac{\left(i_1^+(x) + i_2^+(x)\right)}{2}, \max\left(f_1^+(x), f_2^+(x)\right) \\ \max\left(t_1^-(x), t_2^-(x)\right), \dfrac{\left(i_1^-(x) + i_2^-(x)\right)}{2}, \min\left(f_1^-(x), f_2^-(x)\right) \end{pmatrix} \right\rangle \right\} \tag{3.11}$$

for all $x \in X$.

Definition 3.10. Complement of BNS (Deli et al., 2015)

Let $A = \{x, \langle t^+(x), i^+(x), f^+(x), t^-(x), i^-(x), f^-(x) \rangle\}$ be a BNS in X. Then, the complement of A is denoted by A^C and is defined as:

$$t_{A^C}^+(x) = \{1^+\} - t_A^+(x),\ i_{A^C}^+(x) = \{1^+\} - i_A^+(x),\ f_{A^C}^+(x) = \{1^+\} - f_A^+(x) \tag{3.12}$$

and

$$t_{A^C}^-(x) = \{1^-\} - t_A^-(x),\ i_{A^C}^-(x) = \{1^-\} - i_A^-(x),\ f_{A^C}^-(x) = \{1^-\} - f_A^-(x) \tag{3.13}$$

for all $x \in X$.

Definition 3.11. Let $A_1 = \left\langle t_1^+, i_1^+, f_1^+, t_1^-, i_1^-, f_1^- \right\rangle$ be a BNS, then the score, accuracy, and certain function of A_1 are represented by $s(A_1)$, $a(A_1)$, and $c(A_1)$, respectively such that:

$$s(A_1) = \frac{t_1^+ + 1 - i_1^+ + 1 - f_1^+ + 1 + t_1^- - i_1^- - f_1^-}{6}; \tag{3.14}$$

$$a(A_1) = t_1^+ - f_1^+ + t_1^- - f_1^-; \tag{3.15}$$

$$c(A_1) = t_1^+ - f_1^+ \tag{3.16}$$

Definition 3.12. Deneutrosophication of BNS (Abdullah & Rahim, 2020)

Let $A_1 = \left\langle t_1^+, i_1^+, f_1^+, t_1^-, i_1^-, f_1^- \right\rangle$ be a BNS. Then, the deneutrosophication of A_{ij} is denoted as follows:

$$\mu(A_1) = 1 - \left[\frac{1}{2} \sqrt{\left(\frac{(1-t_1^+)^2 + (i_1^+)^2 + (f_1^+)^2}{3} \right) + \left(\frac{(1-(-t_1^-))^2 + (-i_1^-)^2 + (-f_1^-)^2}{3} \right)} \right] \tag{3.17}$$

where $\mu(A_1) \in [0,1]$.

3.4 Heronian Mean (HM) Operator

The generalized weighted Heronian mean and improved generalized Heronian mean proposed by Yu and Wu (2012) and Liu (2012). The definition of HM operator is given as follows:

Definition 3.13. Heronian mean (Sýkora, 2009)

Let HM operator of dimension n be a mapping HM, such that

$$HM(x_1, x_2, \ldots, x_n) = \frac{2}{n(n+1)} \sum_{i=j=1}^{n} \sqrt{x_i x_j} \tag{3.18}$$

Then, the function HM is called Heronian mean (HM) operator.

Definition 3.14. Generalized weighted Heronian mean (Yu & Wu, 2012)

Let $w_i = (w_1, w_2, ..., w_n)^T$ be the weight vector of a collection of non-negative numbers $x_i (i = 1, 2, ..., n)$, and satisfying $p, q \geq 0$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then,

$$GWHM^{p,q}(x_1, x_2, ..., x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \left((w_i x_i)^p \otimes (w_j x_j)^q \right) \right)^{\frac{1}{p+q}} \tag{3.19}$$

which $GWHM^{p,q}$ is called a generalized weighted Heronian mean (GWHM) operator.

Definition 3.15. Improved generalized weighted Heronian mean (Liu, 2012)

Let $w_i = (w_1, w_2, ..., w_n)^T$ be the weight vector of a collection of non-negative numbers $x_i (i = 1, 2, ..., n)$, and satisfying $p, q \geq 0$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then,

$$IGWHM^{p,q}(x_1, x_2, ..., x_n) = \frac{\left(\sum_{i=1}^{n} \sum_{j=i}^{n} w_i \, w_j \, x_i^{\,p} \otimes x_j^{\,q} \right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^{n} \sum_{j=i}^{n} w_i \, w_j \right)^{\frac{1}{p+q}}} \tag{3.20}$$

which $IGWHM^{p,q}$ is called an improved generalized weighted Heronian mean (IGWHM) operator. In the following, the Dombi operations of BNSs presented.

3.5 Dombi Operations of BNSs

Mahmood et al. (2020) proposed the Dombi operations of bipolar neutrosophic sets, and the following are their properties:

Definition 3.16. Dombi t-norm and t-conorm (Dombi, 1982)

Let γ be a positive real number and $x, y \in [0,1]$. Then, Dombi t-norm (TN) and t-conorm (TCN) among x and y are denoted as $T_{D,\gamma}(x, y)$ and $T^{*}_{D,\gamma}(x, y)$ respectively, such that

$$T_{D,\gamma}(x,y)=\left(1+\left(\left(\frac{1-x}{x}\right)^{\gamma}+\left(\frac{1-y}{y}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1} \tag{3.21}$$

$$T^{*}_{D,\gamma}(x,y)=1-\left(1+\left(\left(\frac{x}{1-x}\right)^{\gamma}+\left(\frac{y}{1-y}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1} \tag{3.22}$$

On the basis of the Dombi t-norm and t-conorm, Mahmood et al. (2020) defined the following operational rules of BNS.

Definition 3.17. Dombi operations of bipolar neutrosophic sets.

Let $A_1=\langle t_1^{+},i_1^{+},f_1^{+},t_1^{-},i_1^{-},f_1^{-}\rangle$ and $A_2=\langle t_2^{+},i_2^{+},f_2^{+},t_2^{-},i_2^{-},f_2^{-}\rangle$ be two BNSs and γ,δ be two positive real numbers. Then, the operations of Dombi sums $(A_1\oplus_D A_2)$ and product $(A_1\otimes_D A_2)$ operation are defined as follows:

(a) $A_1\oplus_D A_2$

$$=\left\langle 1-\left(1+\left(\left(\frac{t_1^{+}}{1-t_1^{+}}\right)^{\gamma}+\left(\frac{t_2^{+}}{1-t_2^{+}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1},\left(1+\left(\left(\frac{1-i_1^{+}}{i_1^{+}}\right)^{\gamma}+\left(\frac{1-i_2^{+}}{i_2^{+}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1},\right. \tag{3.23}$$

$$\left(1+\left(\left(\frac{1-f_1^{+}}{f_1^{+}}\right)^{\gamma}+\left(\frac{1-f_2^{+}}{f_2^{+}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1},-\left(1+\left(\left(\frac{1-t_1^{-}}{-t_1^{-}}\right)^{\gamma}+\left(\frac{1-t_2^{-}}{-t_2^{-}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left.\left(1+\left(\left(\frac{-i_1^{-}}{1+i_1^{-}}\right)^{\gamma}+\left(\frac{-i_2^{-}}{1+i_2^{-}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1}-1,\left(1+\left(\left(\frac{-f_1^{-}}{1+f_1^{-}}\right)^{\gamma}+\left(\frac{-f_2^{-}}{1+f_2^{-}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1}-1\right\rangle$$

(b) $A_1\otimes_D A_2$

$$=\left\langle\left(1+\left(\left(\frac{1-t_1^{+}}{t_1^{+}}\right)^{\gamma}+\left(\frac{1-t_2^{+}}{t_2^{+}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1},1-\left(1+\left(\left(\frac{i_1^{+}}{1-i_1^{+}}\right)^{\gamma}+\left(\frac{i_2^{+}}{1-i_2^{+}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{-1},\right. \tag{3.24}$$

$$1-\left(1+\left(\left(\frac{f_1^+}{1-f_1^+}\right)^\gamma+\left(\frac{f_2^+}{1-f_2^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},\left(1+\left(\left(\frac{-t_1^-}{1+t_1^-}\right)^\gamma+\left(\frac{-t_2^-}{1+t_2^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$\left.-\left(1+\left(\left(\frac{1+i_1^-}{-i_1^-}\right)^\gamma+\left(\frac{1+i_2^-}{-i_2^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},-\left(1+\left(\left(\frac{1+f_1^-}{-f_1^-}\right)^\gamma+\left(\frac{1+f_2^-}{-f_2^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1}\right\rangle$$

(c) $\delta \cdot A_1$

$$=\left\langle 1-\left(1+\left(\delta\left(\frac{t_1^+}{1-t_1^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},\left(1+\left(\delta\left(\frac{1-i_1^+}{i_1^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},\left(1+\left(\delta\left(\frac{1-f_1^+}{f_1^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},\right. \tag{3.25}$$

$$\left.-\left(1+\left(\delta\left(\frac{1+t_1^-}{-t_1^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},\left(1+\left(\delta\left(\frac{-i_1^-}{1+i_1^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1}-1,\left(1+\left(\delta\left(\frac{-f_1^-}{1+f_1^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1}-1\right\rangle$$

(d) A_1^{δ}

$$=\left\langle\left(1+\left(\delta\left(\frac{1-t_1^+}{t_1^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},1-\left(1+\left(\delta\left(\frac{i_1^+}{1-i_1^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},1-\left(1+\left(\delta\left(\frac{f_1^+}{1-f_1^+}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},\right. \tag{3.26}$$

$$\left.\left(1+\left(\delta\left(\frac{-t_1^-}{1+t_1^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1}-1,-\left(1+\left(\delta\left(\frac{1+i_1^-}{-i_1^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1},-\left(1+\left(\delta\left(\frac{1+f_1^-}{-f_1^-}\right)^\gamma\right)^{\frac{1}{\gamma}}\right)^{-1}\right\rangle$$

3.6 Fundamental Concept of DEMATEL Method

This section offers a comprehensive overview of the DEMATEL method, which utilizes a standard scale with precise, absolute numbers. The DEMATEL method is particularly effective for understanding and analyzing complex systems, as well as the relationships and interactions between their elements. As a structural modelling approach, DEMATEL can identify the interdependencies among known factors within intricate systems, making it invaluable for examining cause-and-effect relationships among system components. Traditionally, classical DEMATEL consists of six key steps. These steps include defining the problem and linguistic scale, constructing the initial direct-relation matrix, normalizing the matrix, determining the total relation matrix, and visualizing the results to identify prominent factors and their influence.

In the classical DEMATEL, the initial direct influence matrix is assessed using integer scale from 0 to 4, with 0= "no influence", 1= "weak influence", 2= "medium influence", 3= "strong influence", and 4= "very strong influence" to describe the direct influence of factor. Besides that, according to published literature, Ismail et al. (2023) modified the 5-term linguistic scale into 7-term linguistic scale in the study of Pythagorean neutrosophic with DEMATEL. In addition, the modified DEMATEL introduces a novel approach by incorporating the $\lambda -$ Scale, which is based on Euler' series. This fuzzy linguistic scale provides more realistic and symmetrical terms for building the initial influence matrix. Also, the modified DEMATEL aims to enhance the applicability of the mathematical series in addressing real-world problems by involving experts and criteria associated with decision-making (Ebrahimi, 2023).

Moreover, a key step in the DEMATEL method involves deriving vectors R_i and C_j, which are obtained by summing the rows and columns of the total relation matrix (Si et al., 2018). The horizontal axis $R_i + C_j$, named "Prominence", shows the ranking of each criterion, while the vertical axis $R_i - C_j$, called "Relation", measures the cause-effect of the group. The statements indicate that a factor belongs to the causal group if the value of $R_i - C_j$ is positive and to the effect group if the value of $R_i - C_j$ is negative. In Figure 3.1, the casual diagram is illustrated by mapping the dataset $\left(R_i + C_j, R_i - C_j\right)$.

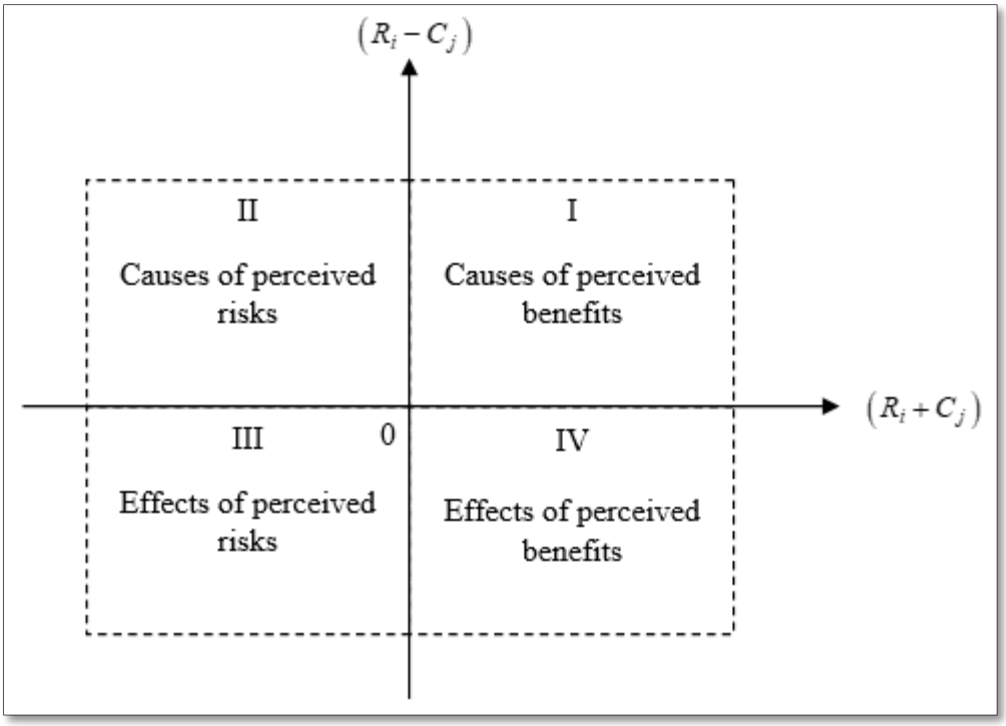

Figure 3.1 The Causal Diagram

According to the $R_i + C_j$ calculation, the network relationship map is divided into four quadrants as shown in Figure 3.2. Factors can be divided into the following categories based on quadrant positions of $R_i + C_j$:

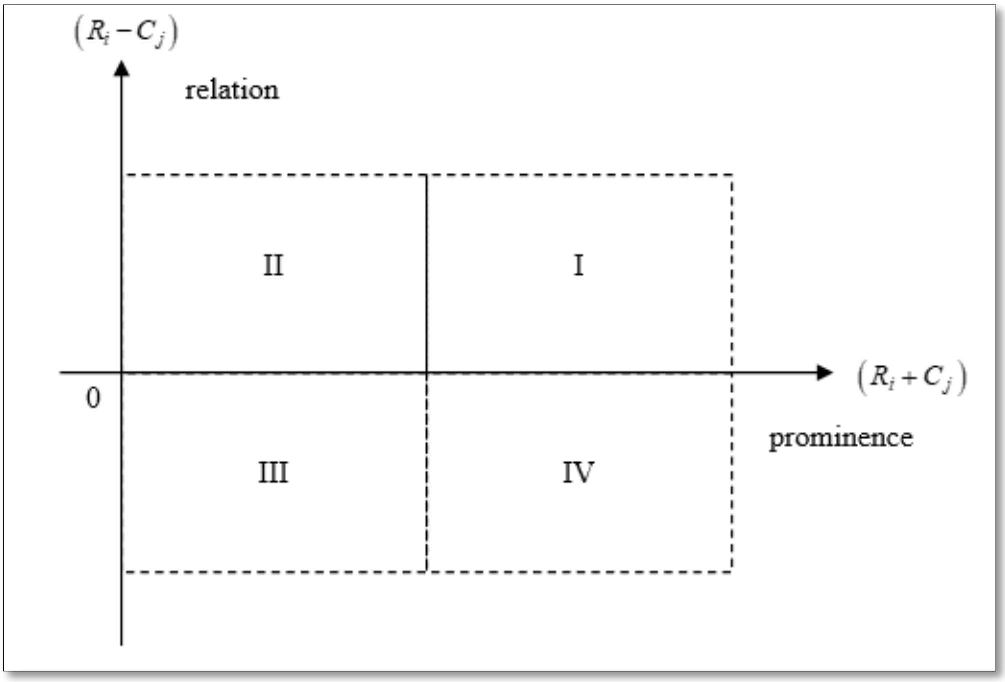

Figure 3.2 Network Relationship Map

The details of four quadrants in the network relationship map is as follows:

i) It is in Quadrant I that the most visible and interconnected elements of vital importance can be found.

ii) Quadrant II also includes autonomous driving elements that are less significant but nonetheless have strong ties.

iii) Quadrant III is composed of separate elements that are unknown and have weak ties to one another.

iv) Quadrant III is composed of separate elements that are unknown and have weak ties to one another.

Figure 3.3 illustrates the DEMATEL method, which allows us to gain insight into cause-and-effect relationships within a system.

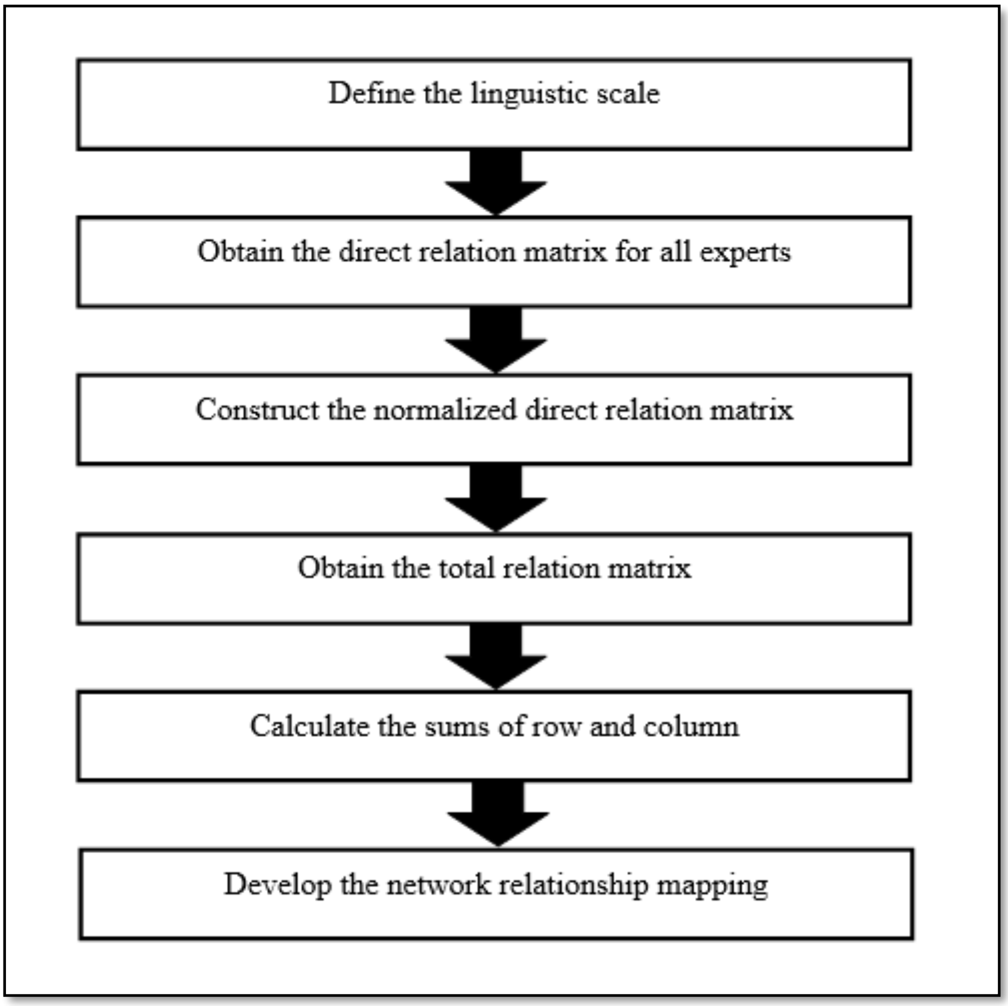

Figure 3.3 Steps in the DEMATEL Method

CHAPTER 4
BIPOLAR NEUTROSOPHIC DOMBI-BASED IMPROVED GENERALIZED WEIGHTED HERONIAN MEAN

4.1 Introduction

This chapter presents the development of new aggregation operators in bipolar neutrosophic set (BNS) environment. This generalization incorporates the Heronian mean (HM) operator into Dombi. This chapter develops to achieve the first objective of this research. Section 4.2 begins with the development of the new aggregation operators, which are the bipolar neutrosophic Dombi-based generalized weighted Heronian mean (BND-GWHM) and bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean (BND-IGWHM) aggregation operators, as well as the properties of each of them. In Section 4.3, an MCDM method is developed based on proposed operators. The proposed methodology is illustrated numerically in Section 4.4. Furthermore, this section discusses the impact of different parameter values on the ranking results and makes comparisons with existing methods. A summary is presented at the end of this chapter.

4.2 Proposed Dombi-based Heronian Mean Operators for BNSs

Aggregation operators (AOs) are crucial for information fusion, particularly in multi-criteria decision-making (MCDM) problems, as they provide a comprehensive overview of data from multiple decision-makers and criteria. Commonly, algebraic operational laws like the algebraic product and algebraic sum are applied for the aggregation process. However, these algebraic methods have limitations in flexibility when handling uncertain information. They focus on producing deterministic results that often fail to capture the inherent uncertainty in many decision-making scenarios. To address these limitations effectively, we proposed the combination of the Heronian mean aggregation operator and Dombi with a BNS environment specifically two AOs are proposed namely BND-GWHM and BND-IGWHM operator. Several properties of both operators are presented, and the proofs of some of these properties are also explained in detail.

4.2.1 BND-GWHM Operator and Its Properties

In this subsection, the bipolar neutrosophic Dombi-based generalized weighted Heronian mean operator is introduced. Subsequently, a thorough mathematical analysis is carried out to establish its fundamental properties, namely idempotency, monotonicity, and boundedness.

Definition 4.1 Let $p,q \geq 0$ and $A_i = \left\langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \right\rangle (i=1,2,...,n)$ be a collection of BNSs with weight $w_i = (w_1, w_2, ..., w_n)^T$, such that $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. Then, the bipolar neutrosophic Dombi-based generalized weighted Heronian mean (BND-GWHM) is defined as follows:

$$BND-GWHM^{p,q}\left(A_1, A_2, ..., A_n\right) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \left(\left(w_i A_i\right)^p \otimes \left(w_j A_j\right)^q\right)\right)^{\frac{1}{p+q}} \tag{4.1}$$

Theorem 1. Let $p,q,\gamma \geq 0$ and $A_i = \left\langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \right\rangle (i=1,2,...,n)$ be a collection of BNSs. The fused value produced by the BND-GWHM operators is also a BNS, and let

$$G_i = \frac{t_i^+}{1-t_i^+},\ H_i = \frac{1-i_i^+}{i_i^+},\ I_i = \frac{1-f_i^+}{f_i^+},\ J_i = \frac{1+t_i^-}{-t_i^-},\ K_i = \frac{-i_i^-}{1+i_i^-},\ L_i = \frac{-f_i^-}{1+f_i^-},$$

and

$$G_j = \frac{t_j^+}{1-t_j^+},\ H_j = \frac{1-i_j^+}{i_j^+},\ I_j = \frac{1-f_j^+}{f_j^+},\ J_j = \frac{1+t_j^-}{-t_j^-},\ K_j = \frac{-i_j^-}{1+i_j^-},\ L_j = \frac{-f_j^-}{1+f_j^-}.$$

Then, Equation (4.1) also yields the following value.

$$BND-GWHM^{p,q}\left(A_1, A_2, ..., A_n\right)$$

$$= \left\langle \left(1 + \left(\frac{n(n+1)}{2(p+q)} \times \frac{1}{\sum_{i=1}^{n} \sum_{j=i}^{n} \left(p/\left(w_i G_i^{\gamma}\right) + q/\left(w_j G_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}, \right. \tag{4.2}$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iH_i^{\gamma}\right)+q/\left(w_jH_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iI_i^{\gamma}\right)+q/\left(w_jI_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iJ_i^{\gamma}\right)+q/\left(w_jJ_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iK_i^{\gamma}\right)+q/\left(w_jK_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iL_i^{\gamma}\right)+q/\left(w_jL_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}\Bigg\rangle$$

Proof: By using Equation (3.25), we obtain

$$w_iA_i=\Bigg\langle 1-\frac{1}{1+\left(w_i\left(G_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(w_i\left(H_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(w_i\left(I_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{-1}{1+\left(w_i\left(J_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}},$$

$$\frac{1}{1+\left(w_i\left(K_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}-1,\frac{1}{1+\left(w_i\left(L_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}-1\Bigg\rangle$$

and

$$w_jA_j=\Bigg\langle 1-\frac{1}{1+\left(w_j\left(G_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(w_j\left(H_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(w_j\left(I_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{-1}{1+\left(w_j\left(J_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}},$$

$$\frac{1}{1+\left(w_j\left(K_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}-1,\frac{1}{1+\left(w_j\left(L_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}-1\Bigg\rangle$$

Then, by using Equation (3.26), we get

$$\left(w_iA_i\right)^p = \left\langle \frac{1}{1+\left(p/\left(w_iG_i^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(p/\left(w_iH_i^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(p/\left(w_iI_i^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, \right.$$

$$\left. \frac{1}{1+\left(p/\left(w_iJ_i^{\gamma}\right)\right)^{\frac{1}{\gamma}}}-1, \frac{-1}{1+\left(p/\left(w_iK_i^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, \frac{-1}{1+\left(p/\left(w_iL_i^{\gamma}\right)\right)^{\frac{1}{\gamma}}} \right\rangle$$

and

$$\left(w_jA_j\right)^q = \left\langle \frac{1}{1+\left(q/\left(w_jG_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(q/\left(w_jH_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(q/\left(w_jI_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, \right.$$

$$\left. \frac{1}{1+\left(q/\left(w_jJ_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}-1, \frac{-1}{1+\left(q/\left(w_jK_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, \frac{-1}{1+\left(q/\left(w_jL_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}} \right\rangle$$

Thereafter, using Equation (3.24) we get

$$\left(w_iA_i\right)^p \otimes \left(w_jA_j\right)^q =$$

$$\left\langle \frac{1}{1+\left(p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(p/\left(w_iH_i^{\gamma}\right)+q/\left(w_jH_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, 1- \right.$$

$$\frac{1}{1+\left(p/\left(w_iI_i^{\gamma}\right)+q/\left(w_jI_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(p/\left(w_iJ_i^{\gamma}\right)+q/\left(w_jJ_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}-1,$$

$$\left. \frac{-1}{1+\left(p/\left(w_iK_i^{\gamma}\right)+q/\left(w_jK_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}}, \frac{-1}{1+\left(p/\left(w_iL_i^{\gamma}\right)+q/\left(w_jL_j^{\gamma}\right)\right)^{\frac{1}{\gamma}}} \right\rangle$$

By employing Equation (3.23),

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\left(w_iA_i\right)^p \otimes \left(w_jA_j\right)^q\right) =$$

$$\left\langle 1-\left(1+\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},\right.$$

$$\left(1+\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iH_i^{\gamma}\right)+q/\left(w_jH_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iI_i^{\gamma}\right)+q/\left(w_jI_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iJ_i^{\gamma}\right)+q/\left(w_jJ_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iK_i^{\gamma}\right)+q/\left(w_jK_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$\left.\left(1+\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iL_i^{\gamma}\right)+q/\left(w_jL_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1\right\rangle$$

and by using Equation (3.25),

$$\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\left(w_iA_i\right)^{p}\otimes\left(w_jA_j\right)^{q}\right)$$

$$=\left\langle 1-\left(1+\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},\right.$$

$$\left(1+\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iH_i^{\gamma}\right)+q/\left(w_jH_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iI_i^{\gamma}\right)+q/\left(w_jI_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iJ_i^{\gamma}\right)+q/\left(w_jJ_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iK_i^{\gamma}\right)+q/\left(w_jK_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$\left(1+\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iL_i^{\gamma}\right)+q/\left(w_jL_j^{\gamma}\right)}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1\Bigg\rangle$$

Therefore, using Equation (3.26) we get

$$\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\left(w_iA_i\right)^{p}\otimes\left(w_jA_j\right)^{q}\right)\right)^{\frac{1}{p+q}}$$

$$=\Bigg\langle\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iH_i^{\gamma}\right)+q/\left(w_jH_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iI_i^{\gamma}\right)+q/\left(w_jI_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iJ_i^{\gamma}\right)+q/\left(w_jJ_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iK_i^{\gamma}\right)+q/\left(w_jK_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iL_i^{\gamma}\right)+q/\left(w_jL_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}\Bigg\rangle$$

Thus, Equation (4.2) is proven.

The following example illustrates the application of the proposed operator BND-GWHM.

Example 1. Let $A_1=(0.4,0.5,0.3,-0.6,-0.4,-0.5)$, $A_2=(0.6,0.1,0.2,-0.4,-0.3,-0.2)$, and $A_3=(0.8,0.6,0.5,-0.3,-0.2,-0.1)$ be a collection of BNSs, and $p=2,q=1,\gamma=3,$ $w=(0.2,0.5,0.3)$. Then, we employ the BND-GWHM operator to fuse three BNSs.

First, we calculated the values $BND-GWHM^{p,q}(A_1,A_2,A_3)$ using Equation (4.2).

The equation for t^+ is stated as

$$t^+=\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$=\left(1+\left(\frac{n(n+1)}{2(p+q)\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$=\left(1+\left(\frac{3(3+1)}{2(2+1)\times\sum_{i=1}^{3}\sum_{j=i}^{3}\left(2/\left(w_iG_i^{3}\right)+1/\left(w_jG_j^{3}\right)\right)^{-1}}\right)^{\frac{1}{3}}\right)^{-1}$$

Note that: $\sum_{i=1}^{3}\sum_{j=i}^{3}\left(2/\left(w_iG_i^{3}\right)+1/\left(w_jG_j^{3}\right)\right)^{-1}$

$$=\left(2/\left(w_1G_1^{3}\right)+1/\left(w_1G_1^{3}\right)\right)^{-1}+\left(2/\left(w_1G_1^{3}\right)+1/\left(w_2G_2^{3}\right)\right)^{-1}+$$

$$\left(2/\left(w_1G_1^{\,3}\right)+1/\left(w_3G_3^{\,3}\right)\right)^{-1}+\left(2/\left(w_2G_2^{\,3}\right)+1/\left(w_2G_2^{\,3}\right)\right)^{-1}+$$

$$\left(2/\left(w_2G_2^{\,3}\right)+1/\left(w_3G_3^{\,3}\right)\right)^{-1}+\left(2/\left(w_3G_3^{\,3}\right)+1/\left(w_3G_3^{\,3}\right)\right)^{-1}$$

$$=\left(2/\left(0.2\left(\frac{0.4}{1-0.4}\right)^3\right)+1/\left(0.2\left(\frac{0.4}{1-0.4}\right)^3\right)\right)^{-1}+$$

$$\left(2/\left(0.2\left(\frac{0.4}{1-0.4}\right)^3\right)+1/\left(0.5\left(\frac{0.6}{1-0.6}\right)^3\right)\right)^{-1}+$$

$$\left(2/\left(0.2\left(\frac{0.4}{1-0.4}\right)^3\right)+1/\left(0.3\left(\frac{0.8}{1-0.8}\right)^3\right)\right)^{-1}+$$

$$\left(2/\left(0.5\left(\frac{0.6}{1-0.6}\right)^3\right)+1/\left(0.5\left(\frac{0.6}{1-0.6}\right)^3\right)\right)^{-1}+$$

$$\left(2/\left(0.5\left(\frac{0.6}{1-0.6}\right)^3\right)+1/\left(0.3\left(\frac{0.8}{1-0.8}\right)^3\right)\right)^{-1}+$$

$$\left(2/\left(0.3\left(\frac{0.8}{1-0.8}\right)^3\right)+1/\left(0.3\left(\frac{0.8}{1-0.8}\right)^3\right)\right)^{-1}$$

$$=7.8491$$

Hence,

$$t^+=\left(1+\left(\frac{3(3+1)}{2(2+1)\times 7.8491}\right)^{\frac{1}{3}}\right)^{-1}$$

$$=0.6120$$

In the same way, we also acquired the values for i^+, f^+, t^-, i^-, and f^-. Finally, $BND-GWHM^{2,1}(A_1,A_2,A_3)=(0.6120,0.2026,0.3468,-0.4781,-0.2362,-0.2528)$.

Theorem 2. (Idempotency property)

Let $A_i=\left\langle t_i^+,i_i^+,f_i^+,t_i^-,i_i^-,f_i^-\right\rangle$ $(i=1,2,...,n)=A=\left\langle t^+,i^+,f^+,t^-,i^-,f^-\right\rangle$ be a collection of BNSs, so $G=G_i=G_j=\dfrac{t^+}{1-t^+}$, then $i=j$

Proof: Suppose $BND-GWHM^{p,q}(A_1,A_2,...,A_n)=\left\langle t_\alpha^+,i_\alpha^+,f_\alpha^+,t_\alpha^-,i_\alpha^-,f_\alpha^-\right\rangle$, we can obtain that

$$t_{\alpha}^{+}=\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iG_i^{\gamma}\right)+q/\left(w_jG_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$=\left(1+\left(\frac{n(n+1)}{2(p+q)}\times 1/\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_jw_jG^{\gamma}}{w_jp+w_jq}\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$=\frac{1}{1+\left(\frac{n(n+1)}{2(p+q)}\times 1/\left(\left(\frac{n(n+1)}{2(p+q)}\right)\left(\frac{w_jG^{\gamma}}{p+q}\right)\left(\frac{p+q}{w_j}\right)\right)\right)^{\frac{1}{\gamma}}}$$

$$=\frac{1}{1+\left(G^{\gamma}\right)^{\frac{1}{\gamma}}}$$

$$=\frac{1}{1+G}$$

So, $BND-GWHM^{p,q}\left(A_1,A_2,...,A_n\right)=A$. Theorem 2 is proven.

Theorem 3. (Monotonicity property)

Let $A_i=\left\langle t^{+},i^{+},f^{+},t^{-},i^{-},f^{-}\right\rangle$ $(i=1,2,...,n)$ and $A_i*=\left\langle t^{+*},i^{+*},f^{+*},t^{-*},i^{-*},f^{-*}\right\rangle$ $(i=1,2,...,n)$ be two collections of BNSs. If $t^{+}\le t^{+*},i^{+}\ge i^{+*},f^{+}\ge f^{+*}$ and $t^{-}\ge t^{-*},i^{-}\le i^{-*},f^{-}\le f^{-*}$, then

$$BND-GWHM^{p,q}\left(A_1,A_2,...,A_n\right)\le BND-GWHM^{p,q}\left(A_1^{*},A_2^{*},...,A_n^{*}\right)$$

Proof: For $t^{+}\le t^{+*},i^{+}\ge i^{+*},f^{+}\ge f^{+*}$ and $t^{-}\ge t^{-*},i^{-}\le i^{-*},f^{-}\le f^{-*}$, it is obvious that

$$\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{t_i^{+}}{1-t_i^{+}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{t_j^{+}}{1-t_j^{+}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}\leq$$

$$\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{t_i^{+*}}{1-t_i^{+*}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{t_j^{+*}}{1-t_j^{+*}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

Similarly,

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{1-i_i^{+}}{i_i^{+}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{1-i_j^{+}}{i_j^{+}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}\geq$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{1-i_i^{+*}}{i_i^{+*}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{1-i_j^{+*}}{i_j^{+*}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{1-f_i^{+}}{f_i^{+}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{1-f_j^{+}}{f_j^{+}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}\geq$$

$$1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{1-f_i^{+*}}{f_i^{+*}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{1-f_j^{+*}}{f_j^{+*}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{1+t_i^{-}}{-t_i^{-}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{1+t_j^{-}}{-t_j^{-}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}-1\geq$$

$$\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{1+t_i^{-*}}{-t_i^{-*}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{1+t_j^{-*}}{-t_j^{-*}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{-i_i^-}{1+i_i^-}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{-i_j^-}{1+i_j^-}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}\le$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{-i_i^{-*}}{1+i_i^{-*}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{-i_j^{-*}}{1+i_j^{-*}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

And

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{-f_i^-}{1+f_i^-}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{-f_j^-}{1+f_j^-}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}\le$$

$$-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(1/\left(p/\left(w_i\left(\frac{-f_i^{-*}}{1+f_i^{-*}}\right)^{\gamma}\right)+q/\left(w_j\left(\frac{-f_j^{-*}}{1+f_j^{-*}}\right)^{\gamma}\right)\right)\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}.$$

This is proven Theorem 3.

Theorem 4. (Boundedness property)

Let $A_i = \left\langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \right\rangle \left(i = 1, 2, 3, ..., n\right)$ be a collection of BNSs and if

$$A^- = \left\langle min\left(t^+\right), max\left(i^+\right), max\left(f^+\right), max\left(t^-\right), min\left(i^-\right), min\left(f^-\right)\right\rangle$$

$$A^+ = \left\langle max\left(t^{+*}\right), min\left(i^{+*}\right), min\left(f^{+*}\right), min\left(t^{-*}\right), max\left(i^{-*}\right), max\left(f^{-*}\right)\right\rangle$$

where $\gamma > 0$,

Proof: Based on Theorem 2 and 3, the following can be obtained:

$$A^+ = BND - GWHM^{p,q}(A^+, A^+, ..., A^+), \quad A^- = BND - GWHM^{p,q}(A^-, A^-, ..., A^-).$$

As such,

$$BND - GWHM^{p,q}(A^-, A^-, ..., A^-) \leq BND - GWHM^{p,q}(A_1, A_2, ..., A_n)$$
$$\leq BND - GWHM^{p,q}\ (A^+, A^+, ..., A^+)$$

Hence, $A^- \leq BND - GWHM^{p,q}\left(A_1, A_2, ..., A_n\right) \leq A^+$.

This proves Theorem 4.

4.2.2 BND-IGWHM Operator and Its Properties

By considering the latest developments in Heronian mean operators, we propose a novel aggregation operator, namely the bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean. Subsequently, a thorough mathematical analysis is carried out to establish its fundamental properties, namely idempotency, monotonicity, and boundedness.

Definition 4.2 Let $p,q \geq 0$ and $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2,...,n)$ be a collection of BNSs with weight $w_i = (w_1, w_2, ..., w_n)^T$, and satisfying $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. Then, the bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean (BND-IGWHM) is defined as follows:

$$BND-IGWHM^{p,q}\left(A_1, A_2, ..., A_n\right) = \frac{\left(\sum_{i=1}^n \sum_{j=i}^n w_i\, w_j\, A_i^p \otimes A_j^q\right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^n \sum_{j=i}^n w_i\, w_j\right)^{\frac{1}{p+q}}} \tag{4.3}$$

Theorem 5. Let $p,q \geq 0$ and $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2,...,n)$ be a collection of BNSs. The fused value produced by the BND-IGWHM operators is also a BNS, and let

$$A_i = \frac{1-t_i^+}{t_i^+},\ B_i = \frac{i_i^+}{1-i_i^+},\ C_i = \frac{f_i^+}{1-f_i^+},\ D_i = \frac{-t_i^-}{1+t_i^-},\ E_i = \frac{1+i_i^-}{-i_i^-},\ F_i = \frac{1+f_i^-}{-f_i^-},$$

and

$$A_j = \frac{1-t_j^+}{t_j^+},\ B_j = \frac{i_j^+}{1-i_j^+},\ C_j = \frac{f_j^+}{1-f_j^+},\ D_j = \frac{-t_j^-}{1+t_j^-},\ E_j = \frac{1+i_j^-}{-i_j^-},\ F_j = \frac{1+f_j^-}{-f_j^-}$$

Then, the value derived from Equation (4.3) is represented as:

$$BND-IGWHM^{p,q}\left(A_1, A_2, ..., A_n\right)$$

$$= \left\langle \left(1 + \left(\left(\frac{\sum_{i=1}^n \sum_{j=i}^n w_i\, w_j}{p+q}\right) \times \left(\sum_{i=1}^n \sum_{j=i}^n \left(w_i\, w_j / \left(pA_i^\gamma\right) + \left(qA_j^\gamma\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}, \right. \tag{4.4}$$

$$1-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j \,/\left(\left(pB_i^{\gamma}\right)+\left(qB_j^{\gamma}\right)\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j \,/\left(\left(pC_i^{\gamma}\right)+\left(qC_j^{\gamma}\right)\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j \,/\left(\left(pD_i^{\gamma}\right)+\left(qD_j^{\gamma}\right)\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j \,/\left(\left(pE_i^{\gamma}\right)+\left(qE_j^{\gamma}\right)\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j \,/\left(\left(pF_i^{\gamma}\right)+\left(qF_j^{\gamma}\right)\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}\Bigg\rangle$$

Proof: By using Equation (3.26), we obtain

$$A_i^{\,p} = \Bigg\langle \frac{1}{1+\left(p\left(A_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(p\left(B_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(p\left(C_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}},$$

$$\frac{1}{1+\left(p\left(D_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}-1, \frac{-1}{1+\left(p\left(E_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{-1}{1+\left(p\left(F_i\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\Bigg\rangle$$

and

$$A_j^{\,q} = \Bigg\langle \frac{1}{1+\left(q\left(A_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(q\left(B_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(q\left(C_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}},$$

$$\frac{1}{1+\left(q\left(D_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}-1, \frac{-1}{1+\left(q\left(E_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{-1}{1+\left(q\left(F_j\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\Bigg\rangle.$$

By using Equation (3.24), we get

$$A_i^{\,p} \otimes A_j^{\,q} = \left\langle \frac{1}{1+\left(pA_i^{\gamma}+qA_j^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(pB_i^{\gamma}+qB_j^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(pC_i^{\gamma}+qC_j^{\gamma}\right)^{\frac{1}{\gamma}}}, \right.$$

$$\left. \frac{1}{1+\left(pD_i^{\gamma}+qD_j^{\gamma}\right)^{\frac{1}{\gamma}}}-1, \frac{-1}{1+\left(pE_i^{\gamma}+qE_j^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{-1}{1+\left(pF_i^{\gamma}+qF_j^{\gamma}\right)^{\frac{1}{\gamma}}} \right\rangle.$$

Next, using Equation (3.25) we get

$$w_i\, w_j\, A_i^{p}\, A_j^{q} =$$

$$\left\langle 1-\frac{1}{1+\left(\frac{w_i w_j}{pA_i^{\gamma}+qA_j^{\gamma}}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\frac{w_i w_j}{pB_i^{\gamma}+qB_j^{\gamma}}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\frac{w_i w_j}{pC_i^{\gamma}+qC_j^{\gamma}}\right)^{\frac{1}{\gamma}}}, \right.$$

$$\left. \frac{-1}{1+\left(\frac{w_i w_j}{pD_i^{\gamma}+qD_j^{\gamma}}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\frac{w_i w_j}{pE_i^{\gamma}+qE_j^{\gamma}}\right)^{\frac{1}{\gamma}}}-1, \frac{1}{1+\left(\frac{w_i w_j}{pF_i^{\gamma}+qF_j^{\gamma}}\right)^{\frac{1}{\gamma}}}-1 \right\rangle$$

Afterwards, using Equation (3.23) we get

$$\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n} w_i\, w_j\, A_i^{p}\, A_j^{q} =$$

$$\left\langle 1-\left(1+\left(\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n}\left(\frac{w_i w_j}{pA_i^{\gamma}+qA_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}, \left(1+\left(\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n}\left(\frac{w_i w_j}{pB_i^{\gamma}+qB_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}, \right.$$

$$\left(1+\left(\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n}\left(\frac{w_i w_j}{pC_i^{\gamma}+qC_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}, -\left(1+\left(\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n}\left(\frac{w_i w_j}{pD_i^{\gamma}+qD_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left. \left(1+\left(\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n}\left(\frac{w_i w_j}{pE_i^{\gamma}+qE_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1, \left(1+\left(\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=i}^{n}\left(\frac{w_i w_j}{pF_i^{\gamma}+qF_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1 \right\rangle$$

Thereafter, based on Equation (3.25),

$$\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\, A_i^{\,p}\, A_j^{\,q}\right)$$

$$= \left\langle\, 1-\left(1+\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pA_i^{\gamma}+qA_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},\right.$$

$$\left(1+\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pB_i^{\gamma}+qB_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pC_i^{\gamma}+qC_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pD_i^{\gamma}+qD_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i\, w_j}{pE_i^{\gamma}+qE_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$\left.\left(1+\left(\frac{1}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\right)}\times\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i\, w_j}{pF_i^{\gamma}+qF_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)^{-1}-1\,\right\rangle$$

Therefore, based on Equation (3.26) we get

$$\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j\, A_i^{\,p}\, A_j^{\,q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}\right)^{\frac{1}{p+q}}$$

$$= \left\langle \left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right)\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pA_i^{\gamma}+qA_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},\right.$$

$$1-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right)\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pB_i^{\gamma}+qB_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right)\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pC_i^{\gamma}+qC_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right)\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pD_i^{\gamma}+qD_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}-1,$$

$$-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right)\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pE_i^{\gamma}+qE_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left.-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right)\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pF_i^{\gamma}+qF_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}\right\rangle.$$

Thus, Equation (4.4) is proven.

Example 2 illustrates the implementation of the operator BND-IGWHM on BNSs.

Example 2. Assume $A_1 = \left(0.4, 0.5, 0.3, -0.6, -0.4, -0.5\right)$, $A_2 = (0.6, 0.1, 0.2, -0.4, -0.3, -0.2)$ and $A_3 = \left(0.8, 0.6, 0.5, -0.3, -0.2, -0.1\right)$ be collection of BNSs, and $p = 2$, $q = 1$, $\gamma = 3$, $w = \left(w_1, w_2, w_3\right) = \left(0.2, 0.5, 0.3\right)$. Then we employ the BND-IGWHM operator to fuse three BNSs.

First, we used Equation (4.4) to determine the valued of $BND - IGWHM^{p,q}\left(A_1, A_2, A_3\right) = \left(t^+, i^+, f^+, t^-, i^-, f^-\right)$.

For t^+, the equation is defined as

$$t^+ = \left(1 + \left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q}\right) \times \frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{w_i w_j}{pA_i^{\gamma} + qA_j^{\gamma}}\right)}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$= \left(1 + \left(\frac{\sum_{i=1}^{3}\sum_{j=i}^{3} w_i w_j}{(2+1) \times \sum_{i=1}^{3}\sum_{j=i}^{3}\left(\frac{w_i w_j}{(2)A_i^3 + (1)A_j^3}\right)}\right)^{\frac{1}{3}}\right)^{-1}$$

$$= \left(1 + \left(\frac{w_1 w_1 + w_1 w_2 + w_1 w_3 + w_2 w_2 + w_2 w_3 + w_3 w_3}{(2+1) \times \left(\frac{w_1 w_1}{(2)A_1^3 + (1)A_1^3} + \frac{w_1 w_2}{(2)A_1^3 + (1)A_2^3} + \ldots + \frac{w_3 w_3}{(2)A_3^3 + (1)A_3^3}\right)}\right)^{\frac{1}{3}}\right)^{-1}$$

$$= \left(1 + \left(\frac{(0.2 \times 0.2) + (0.2 \times 0.5) + (0.2 \times 0.3) + (0.5 \times 0.5) + (0.5 \times 0.3) + (0.3 \times 0.3)}{(2+1) \times \left(\frac{(0.2 \times 0.2)}{(2)A_1^3 + (1)A_1^3} + \ldots + \frac{(0.3 \times 0.3)}{(2)A_3^3 + (1)A_3^3}\right)}\right)^{\frac{1}{3}}\right)^{-1}$$

Afterwards, substitute the value of A_1, A_2, and A_3 which written as:

$$A_1 = \frac{1 - 0.4}{0.4}, \quad A_2 = \frac{1 - 0.6}{0.6}, \quad A_3 = \frac{1 - 0.8}{0.8}$$

Then,

$$t^{+}=\left(1+\left(\frac{0.69}{(2+1)\times(0.0040+0.0142+0.0089+0.2813+0.2466+1.9200)}\right)^{\frac{1}{3}}\right)^{-1}$$

$$=0.6883$$

Also, we obtained the values for i^{+}, f^{+}, t^{-}, i^{-}, and f^{-} in the same manner. Thus, $BND-IGWHM^{2,1}(A_1,A_2,A_3)=(0.6883, 0.1348, 0.2490, -0.3871, -0.3021, -0.2933)$.

Theorem 6. (Idempotency property)

Let $A_i=\left\langle t_i^{+},i_i^{+},f_i^{+},t_i^{-},i_i^{-},f_i^{-}\right\rangle$ $(i=1,2,...,n)$ be a collection of BNSs, if all BNSs are equal, that is $A_i=A=\left\langle t^{+},i^{+},f^{+},t^{-},i^{-},f^{-}\right\rangle$, then

$$BND-IGWHM^{p,q}\left(A_1,A_2,...,A_n\right)=A$$

Proof: When $A_i=A=\left\langle t^{+},i^{+},f^{+},t^{-},i^{-},f^{-}\right\rangle$ for every $(i=1,2,...,n)$ according to Definition 3.4, we can obtain that

$$BND-IGWHM^{p,q}\left(A_i\right)=\frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j\,A^{p}\,A^{q}\right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j\right)^{\frac{1}{p+q}}}$$

$$=\frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j\,A^{p+q}\right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j\right)^{\frac{1}{p+q}}}$$

$$=A\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j}\right)^{\frac{1}{p+q}}$$

$$=A$$

Theorem 7. (Monotonicity property)

Let $A_i=\left\langle t^{+},i^{+},f^{+},t^{-},i^{-},f^{-}\right\rangle$ $(i=1,2,...,n)$ and $A_i*=\left\langle t^{+*},i^{+*},f^{+*},t^{-*},i^{-*},f^{-*}\right\rangle$ $(i=1,2,...,n)$ be two collections of BNSs. If $t^{+}\leq t^{+*}, i^{+}\geq i^{+*}, f^{+}\geq f^{+*}$ and

$t^- \geq t^{-*}, i^- \leq i^{-*}, f^- \leq f^{-*}$ then

$$BND-IGWHM^{p,q}\left(A_1, A_2, ..., A_n\right) \leq BND-IGWHM^{p,q}\left(A_1^*, A_2^*, ..., A_n^*\right).$$

This theorem is proved similarly to Theorem 3.

Theorem 8. (Boundedness property)

Let $A_i = \left\langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \right\rangle$ $\left(i = 1, 2, 3, ..., n\right)$ where $\gamma > 0$, be a collection of BNSs and

$$A^- = \left\langle min\left(t^+\right), max\left(i^+\right), max\left(f^+\right), max\left(t^-\right), min\left(i^-\right), min\left(f^-\right)\right\rangle$$

$$A^+ = \left\langle max\left(t^{+*}\right), min\left(i^{+*}\right), min\left(f^{+*}\right), min\left(t^{-*}\right), max\left(i^{-*}\right), max\left(f^{-*}\right)\right\rangle$$

$$\max\left(i_i^{-*}\right) =$$

$$-1/\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\frac{1+\max\limits_i(i_i^-)}{-\max\limits_i(i_i^-)}\right)^{\gamma}\times 1/\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j\left(\frac{1}{p+q}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),$$

$$\max\left(f_i^{-*}\right) =$$

$$-1/\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\frac{1+\max\limits_i(f_i^-)}{-\max\limits_i(f_i^-)}\right)^{\gamma}\times 1/\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j\left(\frac{1}{p+q}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),$$

Then, $\boldsymbol{A^- \leq BND-IGWHM^{p,q}\left(A_1, A_2, ..., A_n\right) \leq A^+}$

Clearly, the BND-IGWHM operator holds these properties. Hence, we omit them.

4.3 Summary

This chapter successfully developed new aggregation operators. In particular, two aggregation operators are developed: BND-GWHM and BND-IGWHM. The properties of both operators such as idempotency, monotonicity and boundedness property are derived accordingly. These properties ensure that the operators behave consistently and are well-suited for handling uncertainty in complex decision environments.

Building upon these developments, the next chapter focuses on incorporating the proposed operators into a multi-criteria decision-making (MCDM) problem. Their practical applicability will be demonstrated through an illustrative example, highlighting their effectiveness in ranking alternatives.

CHAPTER 5
THE INTEGRATION OF PROPOSED BIPOLAR NEUTROSOPHIC DOMBI-BASED HERONIAN MEAN OPERATOR INTO MCDM METHODS

5.1 Introduction

This chapter addresses objective two, which focuses on developing procedures for multi-criteria decision-making (MCDM) problems. Section 5.2 introduces two procedures in MCDM based on the proposed aggregation operators. The BND-GWHM and BND-IGWHM operators are applied within a score-based ranking method (SBRM), as detailed in Subsection 5.2.1. Subsequently, the proposed aggregation operator, BND-IGWHM, is incorporated into a DEMATEL-based method, discussed in Subsection 5.2.2. Illustrative examples are provided for MCDM procedures. A summary is presented at the end of the chapter.

5.2 The Procedure in MCDM based on Proposed BND-GWHM and BND-IGWHM Operators

In this study, the implementation of aggregation operators is fundamental to effectively managing bipolar neutrosophic information within two distinct decision-making frameworks, namely the SBRM and the DEMATEL-based method. Within the SBRM, these operators aggregate the weighted bipolar neutrosophic evaluations of criteria to produce a unified assessment for each alternative, which is then converted into scalar score values to enable accurate ranking. Meanwhile, in the DEMATEL-based method, aggregation operators are employed to combine expert evaluations into a consolidated bipolar neutrosophic direct influence matrix, preserving the uncertainty and bipolarity of the data. This matrix is then processed to reveal the cause-and-effect relationships among criteria, enabling a deeper understanding of their interdependencies.

5.2.1 Score-Based Ranking Method (SBRM)

One of the widely recognized methods in multi-criteria decision-making (MCDM) is the score-based ranking method, which involves aggregating criteria values and ranking alternatives based on their computed scores. In this section, we introduce the application of the proposed BND-GWHM and BND-IGWHM operators within this traditional framework. The incorporation of these advanced aggregation operators is intended to address limitations associated with conventional aggregation methods, particularly in handling uncertainty, interrelationships among criteria, and bipolar information.

The use of BND-GWHM and BND-IGWHM operators enhances the robustness and adaptability of the aggregation process across various decision-making contexts. These operators offer improved flexibility in representing the positive and negative aspects of each alternative, ultimately leading to more reliable decision outcomes. Thus, applying these operators within the SBRM maintains the simplicity and clarity of MCDM procedure while significantly enhancing their capacity to handle more complex and uncertain data.

The following is an example of MCDM problem with bipolar neutrosophic information. Let $A=\left\{\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_m\right\}$ be a set of alternatives and $C=\left\{C_1,C_2,...,C_n\right\}$ be a set of criteria with the weight vector given as $w=(w_1,w_2,...,w_n)^T$ satisfying $w_i\in\left[0,1\right]$ and $\sum_{i}^{n}w_i=1$. Based on the BND-GWHM and BND-IGWHM operators, we develop an algorithm of the MCDM process. The algorithm of the SBRM is shown in Figure 5.1.

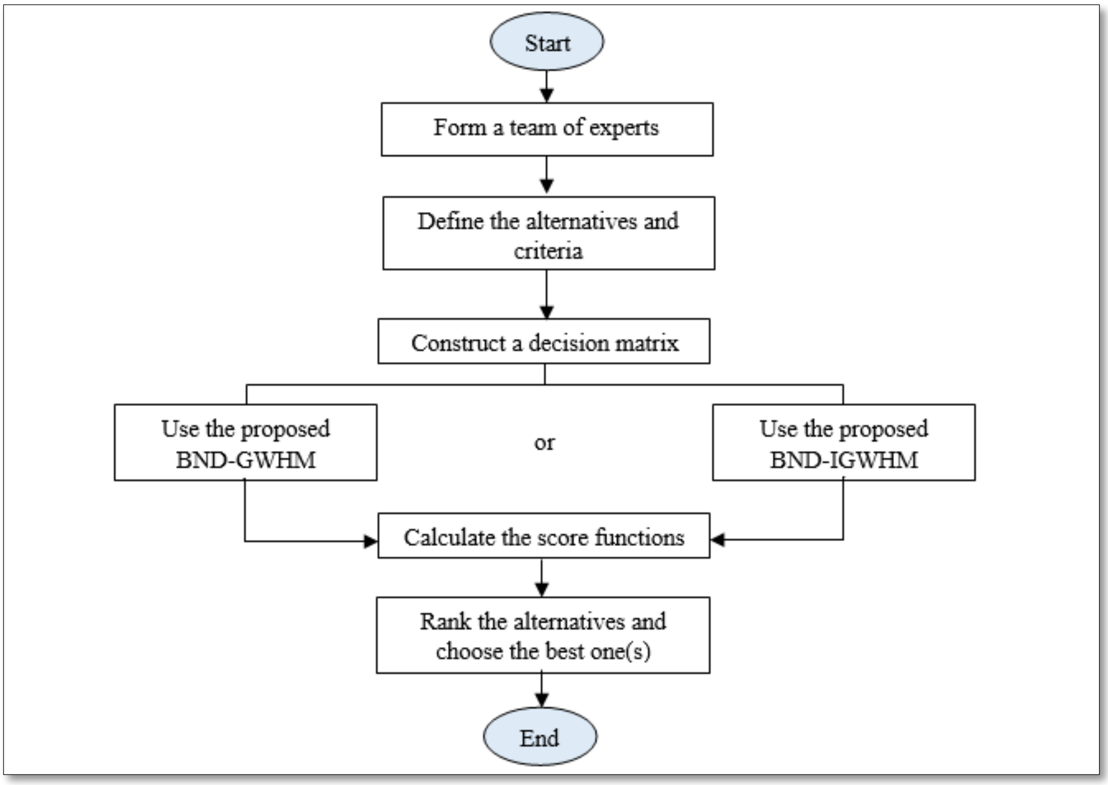

Figure 5.1 Flowchart of the SBRM Algorithm

Algorithm

Step 1: Construct a decision matrix in terms of BNSs. If BNS is denoted as $A_{ij} = \left\langle t_{ij}^{+}, i_{ij}^{+}, f_{ij}^{+}, t_{ij}^{-}, i_{ij}^{-}, f_{ij}^{-} \right\rangle$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ then general form of the decision matrix, where every column refers to criteria and a row refers to alternatives, is as follows:

$$D = \begin{bmatrix} A_{12} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix} \tag{5.1}$$

Step 2: Determine the aggregated values for each alternative $A_i (i = 1, 2, 3, 4)$ using the BND-GWHM and BND-IGWHM operators. Each alternative for every aggregation operator can be expressed in the following manner:

$$
\begin{array}{c} \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \vdots \\ \tilde{A}_m \end{array}
\begin{array}{c} C_j \\ \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \end{array}
\tag{5.2}
$$

Step 3: Determine the score value of $s\left(A_i\right)$ for $A_i (i=1,2,...,n)$ using Equation (3.16) and restated as follows

$$
s\left(A_i\right)=\frac{t_i^{+}+1-i_i^{+}+1-f_i^{+}+1+t_i^{-}-i_i^{-}-f_i^{-}}{6} \tag{5.3}
$$

Then, the score values for each alternative can be written as:

$$
\begin{array}{c} \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \vdots \\ \tilde{A}_m \end{array}
\begin{array}{c} C_j \\ \begin{bmatrix} s\left(A_1\right) \\ s\left(A_2\right) \\ \vdots \\ s\left(A_m\right) \end{bmatrix} \end{array}
\tag{5.4}
$$

Step 4: Rank all the alternatives that correspond to $s\left(A_i\right)$. In general, the alternatives with the highest score value will be the most preferable.

In the following, a numerical example is presented to illustrate using the proposed MCDM approach.

5.2.1.1 Illustrative Example

Suppose an investment company seeks to invest its funds in the most suitable option. As alternative options, an automobile company ($\tilde{A}_1$), a green energy company ($\tilde{A}_2$), a computer company ($\tilde{A}_3$) and a biotech company ($\tilde{A}_4$) may be considered. Alternatives are evaluated based on a set of criteria, including market conditions (C_1), growth factor (C_2), return on investment (ROI) (C_3), and environmental impact (C_4). To solve this problem, the weight vector of the four criteria is defined as $w=(0.30,0.20,0.30,0.20)^T$.

Step 1: The BNSs decision matrix depend on criteria, C for each alternative, $\tilde{A}$ are shown in Table 5.1.

Table 5.1
BNSs Decision Matrix

Alternative	Criteria			
	C_1	C_2	C_3	C_4
$\tilde{A}_1$	$(0.5,0.6,0.2,-0.7,-0.4,-0.6)$	$(0.5,0.4,0.4,-0.4,-0.8,-0.6)$	$(0.7,0.5,0.5,-0.8,-0.6,-0.7)$	$(0.7,0.4,0.1,-0.5,-0.3,-0.8)$
$\tilde{A}_2$	$(0.9,0.6,0.5,-0.7,-0.7,-0.1)$	$(0.8,0.6,0.7,-0.7,-0.4,-0.3)$	$(0.8,0.5,0.6,-0.7,-0.5,-0.2)$	$(0.7,0.5,0.4,-0.5,-0.9,-0.1)$
$\tilde{A}_3$	$(0.9,0.4,0.2,-0.6,-0.3,-0.1)$	$(0.5,0.3,0.2,-0.4,-0.7,-0.4)$	$(0.5,0.9,0.8,-0.6,-0.2,-0.5)$	$(0.3,0.7,0.5,-0.3,-0.2,-0.5)$
$\tilde{A}_4$	$(0.9,0.7,0.2,-0.8,-0.6,-0.1)$	$(0.5,0.7,0.3,-0.6,-0.8,-0.4)$	$(0.4,0.5,0.5,-0.2,-0.7,-0.1)$	$(0.4,0.8,0.3,-0.5,-0.6,-0.4)$

Step 2: Determine the aggregated values from BND-GWHM and BND-IGWHM operators of alternatives $\tilde{A}_i (i = 1,2,3,4)$ (assume $p = q = 1, \gamma = 1$).

The aggregated values based on BND-GWHM operator is calculated using Equation (4.2) as follows:

$$t^{+}{}_{11} = \left(1 + \left(\frac{n(n+1)}{2(p+q)} \times \frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_i G_i^{\gamma}\right) + q/\left(w_j G_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$= \left(1 + \left(\frac{4(4+1)}{2(1+1)} \times \frac{1}{\sum_{i=1}^{4}\sum_{j=i}^{4}\left(1/\left(w_i G_i^{\gamma}\right) + 1/\left(w_j G_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{1}}\right)^{-1}$$

Where the value of $\sum_{i=1}^{4}\sum_{j=i}^{4}\left(1/\left(w_i G_i^{1}\right) + 1/\left(w_j G_j^{1}\right)\right)^{-1}$ is calculated as:

$$\left(1/\left(0.3\left(\frac{0.5}{1-0.5}\right)^{1}\right) + 1/\left(0.3\left(\frac{0.5}{1-0.5}\right)^{1}\right)\right)^{-1} + \left(1/\left(0.3\left(\frac{0.5}{1-0.5}\right)^{1}\right) + 1/\left(0.2\left(\frac{0.5}{1-0.5}\right)^{1}\right)\right)^{-1} +$$

$$\left(1/\left(0.3\left(\frac{0.5}{1-0.5}\right)^{1}\right) + 1/\left(0.3\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1} + \left(1/\left(0.3\left(\frac{0.5}{1-0.5}\right)^{1}\right) + 1/\left(0.2\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1} +$$

$$\left(1/\left(0.2\left(\frac{0.5}{1-0.5}\right)^{1}\right)+1/\left(0.2\left(\frac{0.5}{1-0.5}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.2\left(\frac{0.5}{1-0.5}\right)^{1}\right)+1/\left(0.3\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1}+$$
$$\left(1/\left(0.2\left(\frac{0.5}{1-0.5}\right)^{1}\right)+1/\left(0.2\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{0.7}{1-0.7}\right)^{1}\right)+1/\left(0.3\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1}+$$
$$\left(1/\left(0.3\left(\frac{0.7}{1-0.7}\right)^{1}\right)+1/\left(0.2\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.2\left(\frac{0.7}{1-0.7}\right)^{1}\right)+1/\left(0.2\left(\frac{0.7}{1-0.7}\right)^{1}\right)\right)^{-1}$$

Hence $\boldsymbol{t}^{+}{}_{11}=0.2776$.

$$i^{+}{}_{11}=1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iH_i^{\gamma}\right)+q/\left(w_jH_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}$$
$$=1-\left(1+\left(\frac{4(4+1)}{2(1+1)}\times\frac{1}{\sum_{i=1}^{4}\sum_{j=i}^{4}\left(1/\left(w_iH_i^{1}\right)+1/\left(w_jH_j^{1}\right)\right)^{-1}}\right)^{\frac{1}{1}}\right)^{-1}$$

Where the value of $\sum_{i=1}^{4}\sum_{j=i}^{4}\left(1/\left(w_iH_i^{1}\right)+1/\left(w_jH_j^{1}\right)\right)^{-1}$ is calculated as:

$$\left(1/\left(0.3\left(\frac{1-0.6}{0.6}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.6}{0.6}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{1-0.6}{0.6}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+$$
$$\left(1/\left(0.3\left(\frac{1-0.6}{0.6}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{1-0.6}{0.6}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+$$
$$\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)\right)^{-1}+$$
$$\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)\right)^{-1}+$$
$$\left(1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}$$

Hence $\boldsymbol{i}^{+}{}_{11}=0.7862$.

$$f^{+}_{11}=1-\left(1+\left(\frac{n(n+1)}{2(p+q)}\times\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}\left(p/\left(w_iI_i^{\gamma}\right)+q/\left(w_jI_j^{\gamma}\right)\right)^{-1}}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$=1-\left(1+\left(\frac{4(4+1)}{2(1+1)}\times\frac{1}{\sum_{i=1}^{4}\sum_{j=i}^{4}\left(1/\left(w_iI_i^{1}\right)+q/\left(w_jI_j^{1}\right)\right)^{-1}}\right)^{\frac{1}{1}}\right)^{-1}$$

Where the value of $\sum_{i=1}^{4}\sum_{j=i}^{4}\left(1/\left(w_iI_i^{1}\right)+q/\left(w_jI_j^{1}\right)\right)^{-1}$ is calculated as:

$$\left(1/\left(0.3\left(\frac{1-0.2}{0.2}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.2}{0.2}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{1-0.2}{0.2}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+$$

$$\left(1/\left(0.3\left(\frac{1-0.2}{0.2}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{1-0.2}{0.2}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.1}{0.1}\right)^{1}\right)\right)^{-1}+$$

$$\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)\right)^{-1}+$$

$$\left(1/\left(0.2\left(\frac{1-0.4}{0.4}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.1}{0.1}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)+1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)\right)^{-1}+$$

$$\left(1/\left(0.3\left(\frac{1-0.5}{0.5}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.1}{0.1}\right)^{1}\right)\right)^{-1}+\left(1/\left(0.2\left(\frac{1-0.1}{0.1}\right)^{1}\right)+1/\left(0.2\left(\frac{1-0.1}{0.1}\right)^{1}\right)\right)^{-1}$$

Hence $\boldsymbol{f}^{+}_{11}=0.5771$.

The calculation for negative membership degree is similar to the previous one. The overall aggregated matrix is shown in Table 5. In addition, Equation (4.4) is used for the aggregation of BND-IGWHM operator. The calculation for positive membership degree is as follows:

$$t^{+}_{11}=\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_i\,w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\,w_j/\left(pA_i^{\gamma}\right)+\left(qA_j^{\gamma}\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}$$

where, $\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j$

$$
\begin{aligned}
&= w_1 w_1 + w_1 w_2 + w_1 w_3 + w_1 w_4 + w_2 w_2 + w_2 w_3 + w_2 w_4 + w_3 w_3 + w_3 w_4 + w_4 w_4 \\
&= (0.3\times 0.3) + (0.3\times 0.2) + (0.3\times 0.3) + (0.3\times 0.2) + (0.2\times 0.2) + (0.2\times 0.3) + \\
&\quad (0.2\times 0.2) + (0.3\times 0.3) + (0.3\times 0.2) + (0.2\times 0.2) \\
&= 0.63
\end{aligned}
$$

$$
= \left\langle \left(1 + \left(\left(\frac{0.63}{1+1} \right) \times \left(\sum_{i=1}^{4}\sum_{j=i}^{4} \left(w_i w_j / \left((1)A_i^1 \right) + \left((1)A_j^1 \right) \right) \right)^{-1} \right)^{\frac{1}{1}} \right)^{-1} \right.
$$

The value of $\sum_{i=1}^{4}\sum_{j=i}^{4} \left(w_i w_j / \left((1)A_i^1 \right) + \left((1)A_j^1 \right) \right)$ is defined as follows:

$$
\frac{(0.3\times 0.3)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.5}{0.5}\right)^1} + \frac{(0.3\times 0.2)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.5}{0.5}\right)^1} + \frac{(0.3\times 0.3)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1}
$$

$$
+ \frac{(0.3\times 0.2)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1} + \frac{(0.2\times 0.2)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.5}{0.5}\right)^1} + \frac{(0.2\times 0.3)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1}
$$

$$
+ \frac{(0.2\times 0.2)}{(1)\left(\frac{1-0.5}{0.5}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1} + \frac{(0.3\times 0.3)}{(1)\left(\frac{1-0.7}{0.7}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1} + \frac{(0.3\times 0.2)}{(1)\left(\frac{1-0.7}{0.7}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1}
$$

$$
+ \frac{(0.2\times 0.2)}{(1)\left(\frac{1-0.7}{0.7}\right)^1 + (1)\left(\frac{1-0.7}{0.7}\right)^1}
$$

Thus, $t^{+}_{11} = 0.6095$.

$$
i^{+}_{11} = 1 - \left(1 + \left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}{p+q} \right) \times \left(\sum_{i=1}^{n}\sum_{j=i}^{n} \left(w_i w_j / \left(\left(pB_i^{\gamma} \right) + \left(qB_j^{\gamma} \right) \right) \right) \right)^{-1} \right)^{\frac{1}{\gamma}} \right)^{-1}
$$

$$
= 1 - \left(1 + \left(\left(\frac{0.63}{1+1} \right) \times \left(\sum_{i=1}^{4}\sum_{j=i}^{4} \left(w_i w_j / \left(\left((1)B_i^1 \right) + \left((1)B_j^1 \right) \right) \right) \right)^{-1} \right)^{\frac{1}{1}} \right)^{-1}
$$

The value of $\sum_{i=1}^{4}\sum_{j=i}^{4}\left(w_i\, w_j / \left(\left((1)B_i^1\right)+\left((1)B_j^1\right)\right)\right)$ is defined as follows:

$$\frac{(0.3\times 0.3)}{(1)\left(\frac{0.6}{1-0.6}\right)^1+(1)\left(\frac{0.6}{1-0.6}\right)^1}+\frac{(0.3\times 0.2)}{(1)\left(\frac{0.6}{1-0.6}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}+\frac{(0.3\times 0.3)}{(1)\left(\frac{0.6}{1-0.6}\right)^1+(1)\left(\frac{0.5}{1-0.5}\right)^1}$$
$$+\frac{(0.3\times 0.2)}{(1)\left(\frac{0.6}{1-0.6}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}+\frac{(0.2\times 0.2)}{(1)\left(\frac{0.4}{1-0.4}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}+\frac{(0.2\times 0.3)}{(1)\left(\frac{0.4}{1-0.4}\right)^1+(1)\left(\frac{0.5}{1-0.5}\right)^1}$$
$$+\frac{(0.2\times 0.2)}{(1)\left(\frac{0.4}{1-0.4}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}+\frac{(0.3\times 0.3)}{(1)\left(\frac{0.5}{1-0.5}\right)^1+(1)\left(\frac{0.5}{1-0.5}\right)^1}+\frac{(0.3\times 0.2)}{(1)\left(\frac{0.5}{1-0.5}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}$$
$$+\frac{(0.2\times 0.2)}{(1)\left(\frac{0.4}{1-0.4}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}$$

Thus, $\boldsymbol{t}^{+}_{11}=0.4896$.

$$f^{+}_{11}=1-\left(1+\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i\, w_j}{p+q}\right)\times\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(w_i\, w_j / \left(\left(pC_i^{\gamma}\right)+\left(qC_j^{\gamma}\right)\right)\right)\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}$$
$$=1-\left(1+\left(\left(\frac{0.63}{1+1}\right)\times\left(\sum_{i=1}^{4}\sum_{j=i}^{4}\left(w_i\, w_j / \left(\left((1)C_i^1\right)+\left((1)C_j^1\right)\right)\right)\right)^{-1}\right)^{\frac{1}{1}}\right)^{-1}$$

The value of $\sum_{i=1}^{4}\sum_{j=i}^{4}\left(w_i\, w_j / \left(\left((1)C_i^1\right)+\left((1)C_j^1\right)\right)\right)$ is defined as follows:

$$\frac{(0.3\times 0.3)}{(1)\left(\frac{0.2}{1-0.2}\right)^1+(1)\left(\frac{0.2}{1-0.2}\right)^1}+\frac{(0.3\times 0.2)}{(1)\left(\frac{0.2}{1-0.2}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}+\frac{(0.3\times 0.3)}{(1)\left(\frac{0.2}{1-0.2}\right)^1+(1)\left(\frac{0.5}{1-0.5}\right)^1}$$
$$+\frac{(0.3\times 0.2)}{(1)\left(\frac{0.2}{1-0.2}\right)^1+(1)\left(\frac{0.1}{1-0.1}\right)^1}+\frac{(0.2\times 0.2)}{(1)\left(\frac{0.4}{1-0.4}\right)^1+(1)\left(\frac{0.4}{1-0.4}\right)^1}+\frac{(0.2\times 0.3)}{(1)\left(\frac{0.4}{1-0.4}\right)^1+(1)\left(\frac{0.5}{1-0.5}\right)^1}$$

$$+\frac{(0.2\times 0.2)}{(1)\left(\frac{0.4}{1-0.4}\right)^{1}+(1)\left(\frac{0.1}{1-0.1}\right)^{1}}+\frac{(0.3\times 0.3)}{(1)\left(\frac{0.5}{1-0.5}\right)^{1}+(1)\left(\frac{0.5}{1-0.5}\right)^{1}}+\frac{(0.3\times 0.2)}{(1)\left(\frac{0.5}{1-0.5}\right)^{1}+(1)\left(\frac{0.1}{1-0.1}\right)^{1}}$$

$$+\frac{(0.2\times 0.2)}{(1)\left(\frac{0.1}{1-0.1}\right)^{1}+(1)\left(\frac{0.1}{1-0.1}\right)^{1}}$$

Thus, $\boldsymbol{f}^{+}{}_{11}=0.2636$.

The rest of aggregated values are shown in Table 5.2.

Table 5.2
The Aggregation Values Obtained from BND-IGWHM and BND-IGWHM Operator

	BND-GWHM	BND-IGWHM
$\tilde{A}_1$	$(0.2776, 0.7862, 0.5771, -0.8590, -0.2382, -0.3480)$	$(0.6095, 0.4896, 0.2636, -0.6343, -0.5458, -0.6776)$
$\tilde{A}_2$	$(0.5256, 0.8321, 0.8297, -0.8764, -0.3562, -0.0464)$	$(0.8270, 0.5494, 0.5476, -0.6630, -0.6741, -0.1605)$
$\tilde{A}_3$	$(0.3269, 0.8365, 0.7000, -0.7786, -0.1203, -0.1245)$	$(0.6891, 0.5696, 0.3690, -0.4954, -0.3336, -0.3559)$
$\tilde{A}_4$	$(0.3221, 0.8927, 0.6471, -0.7854, -0.3480, -0.0666)$	$(0.6820, 0.6583, 0.3110, -0.4619, -0.6776, -0.1982)$

Step 3: From Table 5.2, the score function can be obtained, as presented in Table 5.3

The score values for BND-GWHM operator can be calculated using Equation (3.14) as follows:

$$s\left(\tilde{A}_i\right)=\frac{t_i^{+}+1-i_i^{+}+1-f_i^{+}+1+t_i^{-}-i_i^{-}-f_i^{-}}{6}$$

$$=\frac{0.2776+1-0.7862+1-0.5771+1+(-0.8590)-(-0.2383)-(-0.3480)}{6}$$

$$=0.2736$$

Score values for the BND-IGWHM operator are calculated similarly to the BND-GWHM. Consequently, the results are displayed in Table 5.3.

Table 5.3
The Score Values of Alternatives

Alternative, $\tilde{A}_i$	Score value, $s(\tilde{A}_i)$	
	BND-GWHM	BND-IGWHM
$\tilde{A}_1$	0.2736	0.5742
$\tilde{A}_2$	0.2317	0.4836
$\tilde{A}_3$	0.2094	0.4907
$\tilde{A}_4$	0.2353	0.5211

Step 4: Rank all the alternative according to score in Table 5.3. As the results, we can get:

Table 5.4
Ranking Results

	Ranking of alternatives
BND-GWHM	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
BND-IGWHM	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$

Thus, the most optimal option using both operators is $\tilde{A}_1$.

5.2.2 DEMATEL-Based Method (DBM)

The aggregation operators are widely used in Multi-Criteria Decision-Making (MCDM), including the Decision-Making Trial and Evaluation Laboratory (DEMATEL) method. Initially introduced by Gabus and Fontela in 1973, the DEMATEL method addresses complex interrelationship problems and visualizes network relationship maps (NRM). Over the years, the DEMATEL method has been applied across various domains, including safety systems (Khatun et al., 2023) and artificial intelligence (Hu, 2023). Traditionally, DEMATEL employs crisp and fuzzy sets for linguistic evaluations provided by decision-makers or experts. However, these evaluations often neglect the positive and negative aspects crucial in real-world decision-making. To make these judgements more specific, the bipolar information can be further specified as truth, false and indeterminacy under NS theory.

Besides that, the classical DEMATEL method combines expert judgments using an arithmetic mean (Mavi & Standing, 2018). However, this averaging approach has limitations, including its failure to consider the interrelationships between input variables, potentially leading to unreliable results. The Heronian mean (HM) is one of the operators that considers these interrelationships. Liu (2012) extended HM to improve generalized weighted HM (IGWHM) to handle uncertain, imprecise, and indeterminate information in real-life problems. The Dombi operator is incorporated into the HM due to its flexibility, allowing for adaptable operations based on varying parameters. To address the interrelationship of input arguments in the aggregation process of the DEMATEL method, we have integrated the bipolar neutrosophic Dombi-based IGWHM with the DEMATEL method. The framework of the proposed BND-IGWHM-DEMATEL method is presented in Figure 5.2 below.

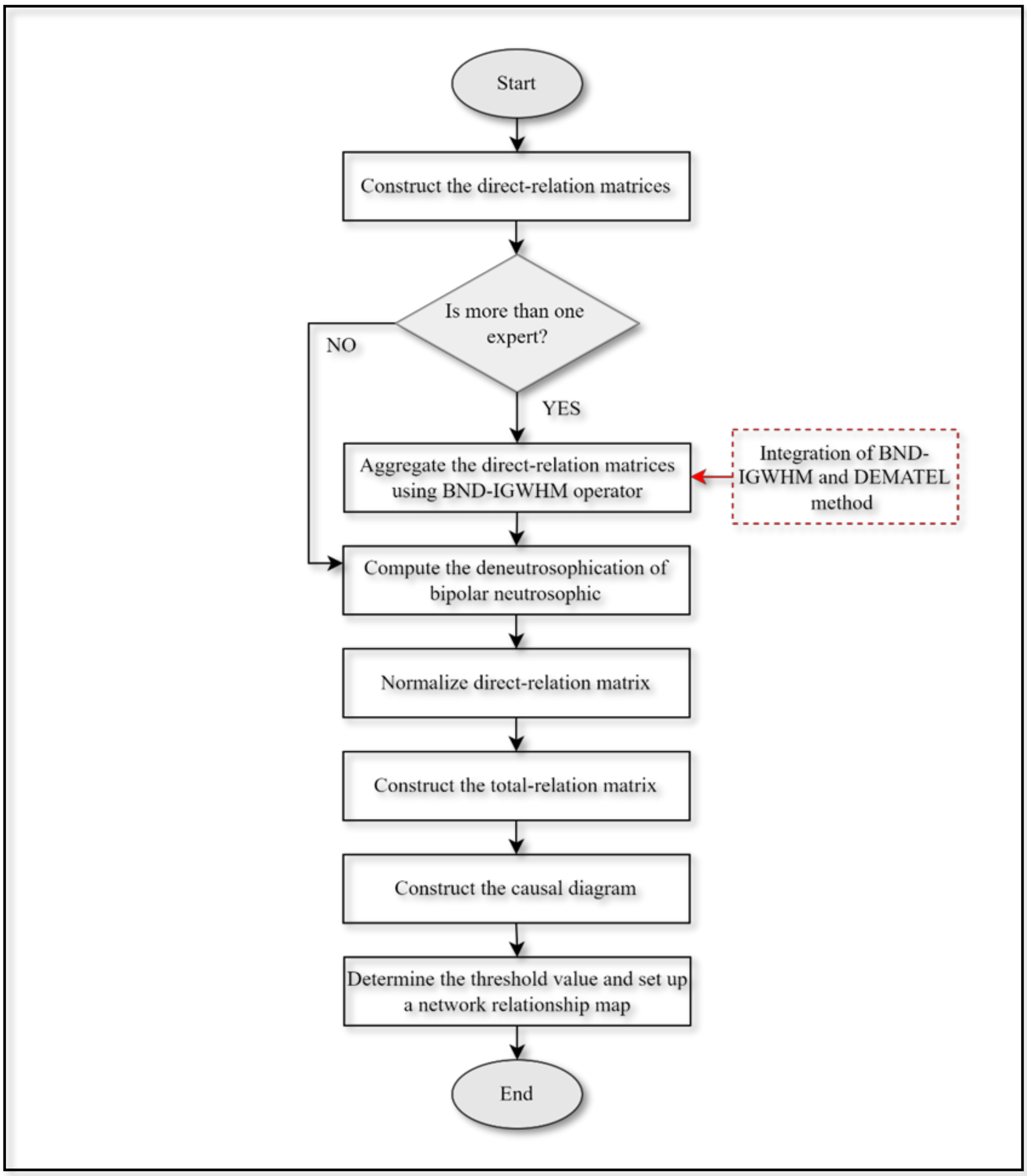

Figure 5.2 Framework of Proposed Method

Consider the MCDM problem with bipolar neutrosophic information. Assume that there are n criteria of the problem defined as $C=\{C_1,C_2,...,C_n\}$. Suppose that

$K=\{K_1,K_2,...,K_m\}$ is set of experts and $w_r=\{r=1,2,...,m\}$ is the corresponding weight of experts where $0\le w_r\le 1$ and $\sum_{r=1}^{m} w_r=1$. Then, the algorithm of the proposed BND-IGWHM-DEMATEL method is presented as follows:

Step 1: Construct the direct-relation matrices.

Let $X^K=\left(A_{ij}^K\right)_{n\times n}$ be the direct-relation matrices given as

$$X^K=\left(A_{ij}^K\right)_{n\times n}=\begin{bmatrix} 0 & A_{12}^K & \cdots & A_{1n}^K \\ A_{21}^K & 0 & \cdots & A_{2n}^K \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}^K & A_{n2}^K & \cdots & 0 \end{bmatrix} \tag{5.5}$$

where A_{ij} are the bipolar neutrosophic linguistic variables representing the degree of influence between the criteria of i and j, assigned by the K-th expert. The option of bipolar neutrosophic variables developed by Rahim (2022), ranging from "no influence (NI)" to "very high influence (VHI)", are listed in Table 5.5 below.

Table 5.5
Bipolar Neutrosophic Linguistic Variables

Variable	BNS
No Influence (NI)	$(0.10,0.80,0.90,-0.10,-0.80,-0.90)$
Low Influence (LI)	$(0.35,0.60,0.70,-0.35,-0.60,-0.70)$
Medium Influence (MI)	$(0.50,0.40,0.45,-0.50,-0.40,-0.45)$
High Influence (HI)	$(0.80,0.20,0.15,-0.80,-0.20,-0.15)$
Very High Influence (VHI)	$(0.90,0.10,0.10,-0.90,-0.10,-0.10)$

Step 2: Aggregate the direct-relation matrices.

The direct-relation matrices in BNSs are aggregated using BND-IGWHM operator as follows:

$$BND-IGWHM^{p,q}\left(A_{ij}^{1},A_{ij}^{2},...,A_{ij}^{m}\right)=\frac{\left(\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s\, A_{ij}^{k\,p}\otimes A_{ij}^{s\,q}\right)^{\frac{1}{p+q}}}{\left(\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s\right)^{\frac{1}{p+q}}}$$

$$= \left\langle \left(1+\left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q}\right)\times 1/\left(\sum_{k=1}^{n}\sum_{k=r}^{n}\left(w_k\, w_s / \left(p\left(\frac{1-t_{ij}^{+k}}{t_{ij}^{+k}}\right)^{\gamma}\right)+\left(q\left(\frac{1-t_{ij}^{+s}}{t_{ij}^{+s}}\right)^{\gamma}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1}, \tag{5.6}$$

$$1-\left(1+\left(\left(\frac{\sum_{r=1}^{m}\sum_{s=r}^{m} w_i\, w_j}{p+q}\right)\times 1/\left(\sum_{r=1}^{m}\sum_{s=r}^{m}\left(w_r\, w_s / \left(\left(p\left(\frac{i_{ij}^{+r}}{1-i_{ij}^{+r}}\right)^{\gamma}\right)+\left(q\left(\frac{i_{ij}^{+s}}{1-i_{ij}^{+s}}\right)^{\gamma}\right)\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$1-\left(1+\left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q}\right)\times 1/\left(\sum_{k=1}^{m}\sum_{s=k}^{m}\left(w_k\, w_s / \left(\left(p\left(\frac{f_{ij}^{+k}}{1-f_{ij}^{+k}}\right)^{\gamma}\right)+\left(q\left(\frac{f_{ij}^{+s}}{1-f_{ij}^{+s}}\right)^{\gamma}\right)\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$\left(1+\left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q}\right)\times 1/\left(\sum_{k=1}^{m}\sum_{s=k}^{m}\left(w_k\, w_s / \left(\left(p\left(\frac{-t_{ij}^{-k}}{1+t_{ij}^{-k}}\right)^{\gamma}\right)+\left(q\left(\frac{-t_{ij}^{-s}}{1+t_{ij}^{-s}}\right)^{\gamma}\right)\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1} -1,$$

$$-\left(1+\left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q}\right)\times 1/\left(\sum_{k=1}^{m}\sum_{s=k}^{m}\left(w_k\, w_s / \left(\left(p\left(\frac{1+i_{ij}^{-k}}{-i_{ij}^{-k}}\right)^{\gamma}\right)+\left(q\left(\frac{1+i_{j}^{-s}}{-i_{j}^{-s}}\right)^{\gamma}\right)\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1},$$

$$-\left(1+\left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q}\right)\times 1/\left(\sum_{k=1}^{m}\sum_{s=k}^{m}\left(w_k\, w_s / \left(\left(p\left(\frac{1+f_{ij}^{-k}}{-f_{ij}^{-k}}\right)^{\gamma}\right)+\left(q\left(\frac{1+f_{ij}^{-s}}{-f_{ij}^{-s}}\right)^{\gamma}\right)\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1}\right\rangle$$

Then, the aggregated direct-relation matrix is denoted as A, such that

$$A = \left(A_{ij}\right)_{n\times n} = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1n} \\ A_{21} & 0 & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & 0 \end{bmatrix} \tag{5.7}$$

where $A_{ij} = BND - IGWHM^{p,q}\left(A_{ij}^{\ 1}, A_{ij}^{\ 2}, ..., A_{ij}^{\ m}\right)$

$$= \left\langle t_{ij}^{+}, i_{ij}^{+}, f_{ij}^{+}, t_{ij}^{-}, i_{ij}^{-}, f_{ij}^{-} \right\rangle,\ i, j = 1, 2, ..., n.$$

Step 3: Compute the deneutrosophication of bipolar neutrosophic.

Deneutrosophication is the process of obtaining crisp numbers from bipolar neutrosophic numbers. The element A_{ij} from aggregated matrix, A is calculated using Equation (5.2). Let A'_{ij} be a deneutrosophication matrix, then, it is also known as the initial direct-relation matrix , A' as follows:

$$A' = \left(A'_{ij}\right)_{n\times n} = \begin{bmatrix} 0 & A'_{12} & \cdots & A'_{1n} \\ A'_{21} & 0 & \cdots & A'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A'_{n1} & A'_{n2} & \cdots & 0 \end{bmatrix} \tag{5.8}$$

Step 4: Normalize the element from direct-relation matrix.

$$D = \frac{1}{s} A', \tag{5.9}$$

where

$$s = \max\left(\max_{1\le i\le n} \sum_{j=1}^{n} A'_{ij}, \max_{1\le j\le n} \sum_{i=1}^{n} A'_{ij} \right) \tag{5.10}$$

and the matrix D can be expressed as follows:

$$D = \left(A''_{ij}\right)_{n\times n} = \begin{bmatrix} 0 & A''_{12} & \cdots & A''_{1n} \\ A''_{21} & 0 & \cdots & A''_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A''_{n1} & A''_{n2} & \cdots & 0 \end{bmatrix} \tag{5.11}$$

From Equation (5.5) and Equation (5.7), the element of

$$A''_{ij} = \frac{A'_{ij}}{\max\left(\max_{1\le i\le n}\sum_{j=1}^{n} A'_{ij}, \max_{1\le j\le n}\sum_{i=1}^{n} A'_{ij}\right)},$$

where $0 \le A''_{ij} < 1$.

Step 5: Construct the total-relation matrix.

The total-relation matrix, T can be acquired by using Equation (5.7) where I is an identity matrix, such that

$$T = D(I-D)^{-1} \tag{5.12}$$

Step 6: Construct the causal diagram.

The sum of the i^{th} rows, $\tilde{R}_i$, and the sum of j^{th} columns, $\tilde{C}_j$, of the total-relation matrix, T, are calculated using Equation (5.9) and Equation (5.10), respectively. The horizontal axis $\tilde{R}_i + \tilde{C}_j$, named "Prominence", shows the ranking of each criterion, while the vertical axis $\tilde{R}_i - \tilde{C}_j$, called "Relation", measures the cause-effect of the group. The statements indicate that a factor belongs to the causal group if the value of $\tilde{R}_i - \tilde{C}_j$ is positive and to the effect group if the value of $\tilde{R}_i - \tilde{C}_j$ is negative.

$$\tilde{R}_i = \left[\tilde{R}_i\right]_{n\times 1} = \sum_{j=1}^{n}\left[A''''_{ij}\right]_{n\times 1} \left(i = 1,2,\ldots,n\right) \tag{5.13}$$

$$\tilde{C}_j = \left[\tilde{C}_j\right]_{1\times n} = \sum_{i=1}^{n}\left[A''''_{ij}\right]_{1\times n} \left(j = 1,2,\ldots,n\right) \tag{5.14}$$

Step 7: Set up the threshold value and the network relationship map (NRM).

To explain the structural relation between criteria while keeping the complexity of the whole system at a manageable level, it is necessary to set a threshold value, t, which acts as a filter on negligible effected in the total-relation matrix. The value of t is determined by taking the average of the elements in the matrix T (Altuntas & Dereli, 2015). The elements in the matrix T are assumed to be zero if their values are less than t, meaning their influence is lower than those of the other criteria. Hence, a new total-relation matrix can be obtained, and a network relationship map can be constructed. The NRM can depict the true picture of the decision-making problem.

5.2.2.1 Illustrative Example

For the purpose of implementation, this study adopts a case analysis of solid waste management (SWM) in the context of group decision-making, utilizing secondary data from Abdullah et al. (2019). The original study, conducted in Malaysia, engaged five experts to contribute toward the formulation of a more sustainable SWM strategy. The ten evaluation criteria considered in this application are as follows: public awareness (C_1), policy and legislations (C_2), organizational capacity (C_3), collaboration and synergy (C_4), research and development (C_5), technology system and facilities (C_6), collection and transportation system (C_7), waste prevention and minimization (C_8), financial resources (C_9), and implementation and progress schedule (C_{10}).

Step 1: Construct the direct-relation matrices.

The five experts, $K = \{K_1, K_2, ..., K_5\}$ present linguistic evaluation according to bipolar neutrosophic variables developed by Rahim et al. (2020). Tables 5.6 to Table 5.10 show the results of experts' evaluation of linguistic variables. Then, we convert it into bipolar neutrosophic set variables labelled as direct-relation matrices, presented in Appendix 1.

Table 5.6
Evaluation Matrix of Expert 1

K_1	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	LI	LI	LI	LI	LI	LI	LI	LI	LI
C_2	LI	0	LI	LI	LI	LI	LI	LI	LI	LI
C_3	LI	LI	0	LI	LI	LI	LI	MI	LI	LI
C_4	LI	LI	LI	0	LI	LI	LI	LI	LI	LI
C_5	LI	LI	LI	LI	0	LI	LI	LI	LI	LI
C_6	LI	LI	LI	LI	LI	0	LI	LI	LI	LI

C_7	LI	LI	LI	LI	LI	LI	0	LI	LI	LI
C_8	LI	LI	LI	LI	LI	LI	LI	0	LI	LI
C_9	LI	LI	LI	LI	LI	LI	LI	LI	0	LI
C_{10}	LI	LI	LI	LI	LI	LI	LI	LI	LI	0

Table 5.7
Evaluation Matrix of Expert 2

K_2	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	HI	HI	HI	HI	HI	HI	HI	HI	HI
C_2	LI	0	HI	HI	HI	HI	HI	HI	HI	HI
C_3	LI	HI	0	HI	HI	HI	HI	HI	HI	HI
C_4	LI	HI	HI	0	HI	HI	HI	HI	HI	HI
C_5	LI	HI	HI	HI	0	HI	HI	HI	HI	HI
C_6	LI	HI	HI	HI	HI	0	HI	HI	HI	HI
C_7	LI	HI	HI	HI	HI	HI	0	HI	HI	HI
C_8	HI	HI	HI	HI	HI	HI	HI	0	HI	HI
C_9	LI	HI	HI	HI	HI	HI	HI	HI	0	HI
C_{10}	LI	HI	HI	HI	HI	HI	HI	HI	HI	0

Table 5.8
Evaluation Matrix of Expert 3

K_3	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	MI	HI	HI	LI	MI	MI	HI	HI	HI
C_2	HI	0	HI	HI	MI	MI	MI	HI	VHI	MI
C_3	HI	HI	0	HI	LI	LI	LI	HI	HI	HI
C_4	HI	HI	HI	0	LI	LI	LI	HI	MI	MI
C_5	LI	HI	HI	HI	0	HI	LI	HI	VHI	MI
C_6	LI	HI	HI	HI	VHI	0	MI	HI	VHI	MI
C_7	MI	HI	HI	HI	HI	MI	0	HI	VHI	MI
C_8	VHI	HI	MI	MI	MI	LI	HI	0	HI	HI
C_9	HI	HI	HI	HI	LI	HI	HI	HI	0	HI
C_{10}	HI	HI	HI	HI	LI	LI	MI	HI	HI	0

Table 5.9
Evaluation Matrix of Expert 4

K_4	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	LI	LI	VHI	LI	LI	VHI	VHI	MI	VHI
C_2	VHI	0	VHI	HI	HI	HI	HI	HI	HI	VHI
C_3	HI	HI	0	HI	VHI	HI	HI	HI	VHI	VHI
C_4	VHI	MI	VHI	0	VHI	HI	MI	HI	HI	MI
C_5	HI	HI	HI	HI	0	VHI	HI	HI	VHI	MI
C_6	HI	VHI	HI	HI	VHI	0	HI	HI	VHI	MI
C_7	VHI	HI	HI	MI	MI	HI	0	HI	HI	MI
C_8	HI	HI	HI	HI	HI	HI	HI	0	MI	MI
C_9	HI	VHI	VHI	VHI	VHI	VHI	VHI	HI	0	MI
C_{10}	VHI	VHI	VHI	HI	HI	HI	HI	HI	HI	0

Table 5.10
Evaluation Matrix of Expert 5

K_5	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	MI	HI	HI	MI	HI	MI	HI	HI	MI
C_2	MI	0	MI	MI	MI	MI	MI	HI	MI	MI
C_3	MI	MI	0	MI	HI	MI	LI	MI	HI	MI
C_4	HI	HI	MI	0	MI	MI	MI	MI	MI	HI
C_5	MI	MI	HI	MI	0	MI	MI	MI	HI	LI
C_6	HI	HI	MI	HI	MI	0	HI	HI	HI	LI
C_7	HI	MI	MI	MI	HI	HI	0	HI	MI	HI
C_8	HI	MI	MI	MI	HI	HI	MI	0	MI	HI
C_9	HI	MI	HI	HI	HI	MI	HI	HI	0	MI
C_{10}	HI	MI	MI	HI	MI	MI	HI	MI	HI	0

Step 2: Aggregate the direct-relation matrices.

The data evaluations by experts are then aggregated using the proposed BND-IGWHM operator. The weight of experts is $w=(0.1,0.2,0.3,0.2,0.2)$ and assume that $p=q=\gamma=1$.

From Table 5.6 to Table 5.10, the aggregated values based on BND-IGWHM operator with respect to the pair of criteria C_1 and C_2 are calculated as:

$$t^{+}{}_{12} = \left(1 + \left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q} \right) \times 1 / \left(\sum_{k=1}^{n}\sum_{k=r}^{n} \left(\frac{w_k\, w_s}{\left(p\left(\frac{1-t_{ij}^{+k}}{t_{ij}^{+k}} \right)^{\gamma} \right) + \left(q\left(\frac{1-t_{ij}^{+s}}{t_{ij}^{+s}} \right)^{\gamma} \right)} \right) \right) \right)^{\frac{1}{\gamma}} \right)^{-1},$$

where, $\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s$

$= w_1 w_1 + w_1 w_2 + w_1 w_3 + w_1 w_4 + w_1 w_5 + w_2 w_2 + w_2 w_3 + w_2 w_4 + w_2 w_5 + w_3 w_3 + w_3 w_4 +$

$w_3 w_5 + w_4 w_4 + w_4 w_5 + w_5 w_5$

$= (0.1\times 0.1) + (0.1\times 0.2) + (0.1\times 0.3) + (0.1\times 0.2) + (0.1\times 0.2) + (0.2\times 0.2) +$
$(0.2\times 0.3) + (0.2\times 0.2) + (0.2\times 0.2) + (0.3\times 0.3) + (0.3\times 0.2) + (0.3\times 0.2) + (0.2\times 0.2) +$
$(0.2\times 0.2) + (0.2\times 0.2)$

$= 0.61$

$$t^{+}{}_{12} = \left(1 + \left(\left(\frac{0.61}{1+1} \right) \times 1 / \left(\sum_{k=1}^{5}\sum_{k=r}^{5} \left(\frac{w_k\, w_s}{\left((1)\left(\frac{1-t_{ij}^{+k}}{t_{ij}^{+k}} \right)^{1} \right) + \left((1)\left(\frac{1-t_{ij}^{+s}}{t_{ij}^{+s}} \right)^{1} \right)} \right) \right) \right)^{\frac{1}{1}} \right)^{-1}$$

The sum of $\sum_{k=1}^{5}\sum_{k=r}^{5} \left(w_k\, w_s \Big/ \left((1)\left(\frac{1-t_{ij}^{+k}}{t_{ij}^{+k}} \right)^{1} \right) + \left((1)\left(\frac{1-t_{ij}^{+s}}{t_{ij}^{+s}} \right)^{1} \right) \right)$ is calculated as follows:

$$\frac{(0.1\times0.1)}{(1)\left(\frac{1-0.35}{0.35}\right)^{1}+(1)\left(\frac{1-0.35}{0.35}\right)^{1}}+\frac{(0.1\times0.2)}{(1)\left(\frac{1-0.35}{0.35}\right)^{1}+(1)\left(\frac{1-0.35}{0.35}\right)^{1}}+\ldots+$$

$$\frac{(0.2\times0.2)}{(1)\left(\frac{1-0.5}{0.5}\right)^{1}+(1)\left(\frac{1-0.5}{0.5}\right)^{1}}$$

Thus, $t^{+}{}_{12}=0.709$.

$i^{+}{}_{12}=$

$$1-\left(1+\left(\left(\frac{\sum_{r=1}^{m}\sum_{s=r}^{m}w_i\,w_j}{p+q}\right)\times1/\left(\sum_{r=1}^{m}\sum_{s=r}^{m}\left(\frac{w_r\,w_s}{\left(\left(p\left(\frac{i_{ij}^{+r}}{1-i_{ij}^{+r}}\right)^{\gamma}\right)+\left(q\left(\frac{i_{ij}^{+s}}{1-i_{ij}^{+s}}\right)^{\gamma}\right)\right)}\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$1-\left(1+\left(\left(\frac{0.61}{1+1}\right)\times1/\left(\sum_{r=1}^{5}\sum_{s=r}^{5}\left(w_r\,w_s/\left(\left((1)\left(\frac{i_{ij}^{+r}}{1-i_{ij}^{+r}}\right)^{1}\right)+\left((1)\left(\frac{i_{ij}^{+s}}{1-i_{ij}^{+s}}\right)^{1}\right)\right)\right)\right)\right)^{\frac{1}{1}}\right)^{-1}$$

The sum of $\sum_{r=1}^{5}\sum_{s=r}^{5}\left(w_r\,w_s/\left(\left((1)\left(\frac{i_{ij}^{+r}}{1-i_{ij}^{+r}}\right)^{1}\right)+\left((1)\left(\frac{i_{ij}^{+s}}{1-i_{ij}^{+s}}\right)^{1}\right)\right)\right)$ is computed as follows:

$$=\frac{(0.1\times0.1)}{(1)\left(\frac{0.6}{1-0.6}\right)^{1}+(1)\left(\frac{0.6}{1-0.6}\right)^{1}}+\frac{(0.1\times0.2)}{(1)\left(\frac{0.6}{1-0.6}\right)^{1}+(1)\left(\frac{0.6}{1-0.6}\right)^{1}}+\ldots+$$

$$\frac{(0.2\times0.2)}{(1)\left(\frac{0.4}{1-0.4}\right)^{1}+(1)\left(\frac{0.4}{1-0.4}\right)^{1}}$$

Thus, $i^{+}{}_{12}=0.273$.

$f^{+}_{12} =$

$$1-\left(1+\left(\left(\frac{\sum_{k=1}^{m}\sum_{s=k}^{m} w_k\, w_s}{p+q}\right)\times 1\Big/\left(\sum_{k=1}^{m}\sum_{s=k}^{m}\left(\frac{w_k\, w_s}{\left(\left(p\left(\frac{f_{ij}^{+k}}{1-f_{ij}^{+k}}\right)^{\gamma}\right)+\left(q\left(\frac{f_{ij}^{+s}}{1-f_{ij}^{+s}}\right)^{\gamma}\right)\right)}\right)\right)\right)^{\frac{1}{\gamma}}\right)^{-1}$$

$$=1-\left(1+\left(\left(\frac{0.61}{1+1}\right)\times 1\Big/\left(\sum_{k=1}^{5}\sum_{s=k}^{5}\left(w_k\, w_s \Big/\left(\left((1)\left(\frac{f_{ij}^{+k}}{1-f_{ij}^{+k}}\right)^{1}\right)+\left((1)\left(\frac{f_{ij}^{+s}}{1-f_{ij}^{+s}}\right)^{1}\right)\right)\right)\right)\right)^{\frac{1}{1}}\right)^{-1}$$

The sum of $\sum_{k=1}^{5}\sum_{s=k}^{5}\left(w_k\, w_s \Big/\left(\left((1)\left(\frac{f_{ij}^{+k}}{1-f_{ij}^{+k}}\right)^{1}\right)+\left((1)\left(\frac{f_{ij}^{+s}}{1-f_{ij}^{+s}}\right)^{1}\right)\right)\right)$ is calculated as follows:

$$=\frac{(0.1\times 0.1)}{(1)\left(\frac{0.7}{1-0.7}\right)^{1}+(1)\left(\frac{0.7}{1-0.7}\right)^{1}}+\frac{(0.1\times 0.2)}{(1)\left(\frac{0.7}{1-0.7}\right)^{1}+(1)\left(\frac{0.7}{1-0.7}\right)^{1}}+\cdots+$$
$$\frac{(0.2\times 0.2)}{(1)\left(\frac{0.45}{1-0.45}\right)^{1}+(1)\left(\frac{0.45}{1-0.45}\right)^{1}}$$

Thus, $f^{+}_{12} = 0.260$.

Note that the values for t_{12}^{-}, i_{12}^{-}, and f_{12}^{-} are derived in the same manner. As a result, $x_{12} = (0.709, 0.273, 0.260, -0.623, -0.332, -0.347)$.

The final result of aggregation is presented in Table 5.11 and Table 5.12.

Table 5.11
The Aggregated Values, A

Criteria	C_1	C_2	C_3	C_4	C_5
C_1	0	(0.538, 0.400, 0.423, −0.508, −0.420, −0.477)	(0.719, 0.275, 0.228, −0.667, −0.313, −0.323)	(0.807, 0.192, 0.157, −0.790, −0.205, −0.178)	(0.494, 0.458, 0.490, −0.452, −0.491, −0.573)
C_2	(0.709, 0.273, 0.260, −0.623, −0.332, −0.374)	0	(0.769, 0.220, 0.194, −0.730, −0.244, −0.237)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.637, 0.319, 0.306, −0.603, −0.335, −0.352)
C_3	(0.660, 0.317, 0.287, −0.603, −0.355, −0.384)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	0	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.722, 0.270, 0.248, −0.619, −0.349, −0.402)
C_4	(0.758, 0.237, 0.207, −0.688, −0.290, −0.312)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.769, 0.220, 0.194, −0.730, −0.244, −0.237)	0	(0.663, 0.312, 0.317, −0.559, −0.389, −0.457)
C_5	(0.494, 0.458, 0.490, −0.452, −0.491, −0.573)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.777, 0.221, 0.171, −0.767, −0.229, −0.190)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	0
C_6	(0.580, 0.398, 0.384, −0.504, −0.454, −0.528)	(0.807, 0.192, 0.157, −0.790, −0.205, −0.178)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.777, 0.221, 0.171, −0.767, −0.229, −0.190)	(0.821, 0.172, 0.167, −0.766, −0.207, −0.218)
C_7	(0.673, 0.298, 0.302, −0.585, −0.358, −0.410)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.677, 0.292, 0.265, −0.642, −0.310, −0.313)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)
C_8	(0.824, 0.175, 0.148, −0.804, −0.192, −0.172)	(0.732, 0.254, 0.212, −0.706, −0.268, −0.249)	(0.637, 0.319, 0.306, −0.603, −0.335, −0.352)	(0.637, 0.319, 0.306, −0.603, −0.335, −0.352)	(0.700, 0.277, 0.243, −0.666, −0.294, −0.289)
C_9	(0.719, 0.275, 0.228, −0.667, −0.313, −0.323)	(0.769, 0.220, 0.194, −0.730, −0.244, −0.237)	(0.807, 0.192, 0.157, −0.790, −0.205, −0.178)	(0.807, 0.192, 0.157, −0.790, −0.205, −0.178).	(0.722, 0.270, 0.248, −0.619, −0.349, −0.402)
C_{10}	(0.758, 0.237, 0.207, −0.688, −0.290, −0.312)	(0.769, 0.220, 0.194, −0.730, −0.244, −0.237)	(0.769, 0.220, 0.194, −0.730, −0.244, −0.237)	(0.777, 0.221, 0.171, −0.767, −0.229, −0.190)	(0.603, 0.366, 0.352, −0.541, −0.410, −0.465)

Table 5.12
The Aggregated Values (Cont.)

Criteria	C_6	C_7	C_3	C_4	C_5
C_1	$(0.616, 0.348, 0.334, -0.566, -0.380, -0.420)$	$(0.689, 0.276, 0.278, -0.624, -0.312, -0.341)$	$(0.807, 0.192, 0.157, -0.790, -0.205, -0.178)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$	$(0.769, 0.220, 0.194, -0.730, -0.244, -0.237)$
C_2	$(0.637, 0.319, 0.306, -0.603, -0.335, -0.352)$	$(0.637, 0.319, 0.306, -0.603, -0.335, -0.352)$	$(0.777, 0.221, 0.171, -0.767, -0.229, -0.190)$	$(0.792, 0.1992, 0.182, -0.743, -0.230, -0.230)$	$(0.689, 0.276, 0.278, -0.624, -0.312, -0.341)$
C_3	$(0.603, 0.366, 0.352, -0.541, -0.410, -0.465)$	$(0.580, 0.398, 0.384, -0.504, -0.454, -0.528)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$	$(0.807, 0.192, 0.157, -0.790, -0.205, -0.178)$	$(0.769, 0.220, 0.194, -0.730, -0.244, -0.237)$
C_4	$(0.603, 0.366, 0.352, -0.541, -0.410, -0.465)$	$(0.522, 0.421, 0.447, -0.485, -0.448, -0.517)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$	$(0.637, 0.319, 0.306, -0.603, -0.335, -0.352)$	$(0.637, 0.319, 0.306, -0.603, -0.335, -0.352)$
C_5	$(0.769, 0.220, 0.194, -0.730, -0.244, -0.237)$	$(0.603, 0.366, 0.352, -0.541, -0.410, -0.465)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$	$(0.848, 0.152, 0.137, -0.826, -0.169, -0.161)$	$(0.538, 0.400, 0.423, -0.508, -0.420, -0.477)$
C_6	0	$(0.700, 0.277, 0.243, -0.666, -0.294, -0.289)$	$(0.777, 0.221, 0.171, -0.767, -0.229, -0.190)$	$(0.848, 0.152, 0.137, -0.826, -0.169, -0.161)$	$(0.538, 0.400, 0.423, -0.508, -0.420, -0.477)$
C_7	$(0.700, 0.277, 0.243, -0.666, -0.294, -0.289)$	0	$(0.777, 0.221, 0.171, -0.767, -0.229, -0.190)$	$(0.792, 0.199, 0.182, -0.743, -0.230, -0.230)$	$(0.637, 0.319, 0.306, -0.603, -0.335, -0.352)$
C_8	$(0.673, 0.316, 0.275, -0.600, -0.370, -0.410)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$	0	$(0.677, 0.292, 0.265, -0.642, -0.310, -0.313)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$
C_9	$(0.769, 0.220, 0.194, -0.730, -0.244, -0.237)$	$(0.807, 0.192, 0.157, -0.790, -0.205, -0.178)$	$(0.777, 0.221, 0.171, -0.767, -0.229, -0.190)$	0	$(0.677, 0.292, 0.265, -0.642, -0.310, -0.313)$
C_{10}	$(0.603, 0.366, 0.352, -0.541, -0.410, -0.465)$	$(0.700, 0.277, 0.243, -0.666, -0.294, -0.289)$	$(0.732, 0.254, 0.212, -0.706, -0.268, -0.249)$	$(0.777, 0.221, 0.171, -0.767, -0.229, -0.190)$	0

Step 3: Compute the deneutrosophication of bipolar neutrosophic.

Based on Table 5.11 and Table 5.12, the deneutrosophication of bipolar neutrosophic can be obtained using Equation (5.2). For example, the value of $\mu(C_{12})$ is calculated as follows:

$$\mu(C_{12})=1-\left[\frac{1}{2}\sqrt{\left(\frac{(1-t_{12}^{+})^2+(i_{12}^{+})^2+(f_{12}^{+})^2}{3}\right)+\left(\frac{(1-(-t_{12}^{-}))^2+(-i_{12}^{-})^2+(-f_{12}^{-})^2}{3}\right)}\right]$$

$$=1-\left[\frac{1}{2}\sqrt{\left(\frac{(1-0.538)^2+(0.400)^2+(0.423)^2}{3}\right)+\left(\frac{(1-(0.508))^2+(0.420)^2+(0.477)^2}{3}\right)}\right]$$

$=0.481.$

The remaining elements are calculated using similar computations, and the results are presented as the initial direct-relation matrix in Table 5.13.

Table 5.13

Initial Direct-Relation Matrix, A'

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	0.481	0.485	0.470	0.469	0.487	0.492	0.470	0.482	0.480
C_2	0.491	0	0.480	0.482	0.489	0.489	0.489	0.473	0.479	0.492
C_3	0.488	0.482	0	0.482	0.490	0.486	0.481	0.482	0.470	0.480
C_4	0.484	0.482	0.480	0	0.492	0.486	0.477	0.482	0.489	0.489
C_5	0.469	0.482	0.473	0.482	0	0.480	0.486	0.482	0.464	0.481
C_6	0.481	0.470	0.482	0.473	0.476	0	0.486	0.473	0.464	0.481
C_7	0.492	0.482	0.482	0.488	0.482	0.486	0	0.473	0.479	0.489
C_8	0.467	0.482	0.489	0.489	0.486	0.488	0.482	0	0.488	0.482
C_9	0.485	0.480	0.470	0.470	0.490	0.480	0.470	0.473	0	0.488
C_{10}	0.484	0.480	0.480	0.473	0.486	0.486	0.486	0.482	0.473	0

Step 4: Normalize direct-relation matrix.

Based on Table 5.13, the normalized direct-relation matrix can be obtained using Equation (5.5) and Equation (5.6). The results of the normalized matrix are shown in Table 5.14.

Table 5.14

The Normalized Direct-Relation Matrix, D

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	0.1102	0.1110	0.1075	0.1074	0.1116	0.1126	0.1075	0.1103	0.1099
C_2	0.1124	0	0.1099	0.1103	0.1119	0.1119	0.1119	0.1082	0.1096	0.1126
C_3	0.1118	0.1103	0	0.1103	0.1123	0.1112	0.1102	0.1103	0.1075	0.1099
C_4	0.1109	0.1103	0.1099	0	0.1126	0.1112	0.1093	0.1103	0.1119	0.1119
C_5	0.1074	0.1103	0.1082	0.1103	0	0.1099	0.1112	0.1103	0.1061	0.1102
C_6	0.1102	0.1075	0.1103	0.1082	0.1089	0	0.1113	0.1082	0.1061	0.1102
C_7	0.1127	0.1103	0.1103	0.1117	0.1103	0.1113	0	0.1082	0.1096	0.1119
C_8	0.1070	0.1103	0.1119	0.1119	0.1113	0.1117	0.1103	0	0.1117	0.1103
C_9	0.1110	0.1099	0.1075	0.1075	0.1123	0.1099	0.1075	0.1082	0	0.1117
C_{10}	0.1109	0.1099	0.1099	0.1082	0.1112	0.1112	0.1113	0.1103	0.1082	0

Step 5: Construct the total-relation matrix.

The total-relation matrix, T, can be obtained using Equation (5.8), where I represents the identify matrices. Table 5.15 shows the total relation matrices T of the criteria.

Table 5.15

The Total Relation Matrix, T

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	11.22	11.27	11.27	11.24	11.36	11.38	11.34	11.19	11.19	11.37
C_2	11.44	11.28	11.38	11.35	11.48	11.50	11.45	11.30	11.30	11.48
C_3	11.38	11.33	11.23	11.30	11.42	11.44	11.40	11.25	11.25	11.43
C_4	11.43	11.37	11.37	11.25	11.47	11.49	11.44	11.30	11.30	11.48
C_5	11.28	11.23	11.23	11.20	11.22	11.34	11.30	11.15	11.15	11.33
C_6	11.25	11.20	11.20	11.17	11.29	11.21	11.27	11.12	11.12	11.30
C_7	11.41	11.35	11.35	11.32	11.45	11.47	11.32	11.28	11.27	11.45
C_8	11.41	11.36	11.36	11.33	11.45	11.47	11.42	11.18	11.28	11.45
C_9	11.30	11.24	11.24	11.21	11.34	11.36	11.31	11.17	11.07	11.34
C_{10}	11.36	11.30	11.30	11.27	11.40	11.42	11.37	11.23	11.22	11.30

Step 6: Construct the causal diagram.

The values of $\tilde{C}_j$ and $\tilde{R}_i$ are derived using Equation (5.9) and (5.10), respectively as follows:

$$\begin{aligned}\tilde{R}_1 &= 11.22+11.27+11.27+11.24+11.36+11.38+11.34+11.19+11.19+11.37\\ &=112.83\end{aligned}$$

$$\begin{aligned}\tilde{C}_1 &= 11.22+11.44+11.38+11.43+11.28+11.25+11.41+11.41+11.30+11.36\\ &=113.49\end{aligned}$$

$$\begin{aligned}\tilde{R}_1+\tilde{C}_1 &= 112.83+113.49\\ &=226.32\end{aligned}$$

$$\begin{aligned}\tilde{R}_1-\tilde{C}_1 &= 112.83-113.49\\ &=-0.656\end{aligned}$$

Based on these values, the prominence, $\tilde{R}_i-\tilde{C}_j$ and the relation, $\tilde{R}_i+\tilde{C}_j$ are determined and the results are displayed in Table 5.16.

Table 5.16

Prominence and Relation Degree of Each Criterion

Criteria	$\tilde{R}_i$	$\tilde{C}_j$	$\tilde{R}_i + \tilde{C}_j$	Rank	$\tilde{R}_i - \tilde{C}_j$	Category
C_1	112.83	113.49	226.32	6	-0.656	Effect group
C_2	113.95	112.92	226.87	3	1.024	Cause group
C_3	113.43	112.93	226.36	5	0.497	Cause group
C_4	113.90	112.63	226.53	4	1.275	Cause group
C_5	112.42	113.89	226.31	7	-1.466	Effect group
C_6	112.12	114.08	226.20	8	-1.960	Effect group
C_7	113.69	113.62	227.31	1	0.066	Cause group
C_8	113.71	112.18	225.89	9	1.532	Cause group
C_9	112.58	112.13	224.71	10	0.448	Cause group
C_{10}	113.17	113.93	227.09	2	-0.759	Effect group

Next, based on Table 5.16, the causal diagram is constructed to visually illustrate the interrelationships and influence dynamics within the system.

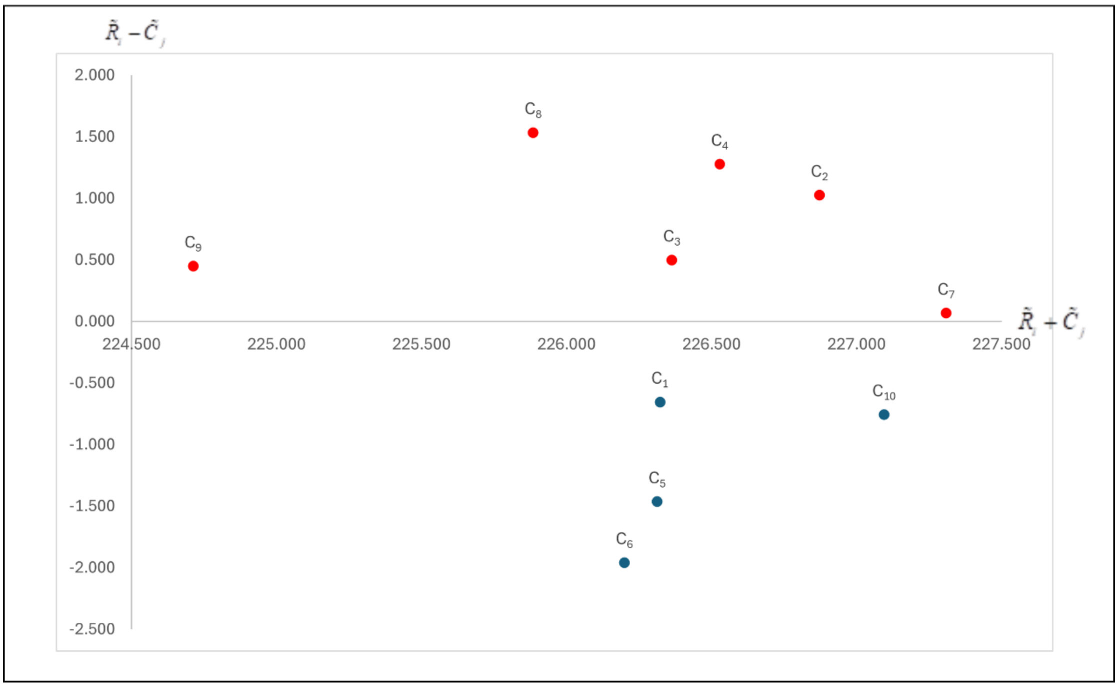

Figure 5.3 The Causal Diagram

Figure 5.3 shows causal relationship maps based on their $\tilde{R}_i + \tilde{C}_j$ and $\tilde{R}_i - \tilde{C}_j$ values. The red dots represent cause group also known as strong influencers. This group can cause change in other areas. This include policy and legislations (C_2), organizational capacity (C_3), collaboration and synergy (C_4), collection and transportation system (C_7), waste prevention and minimization (C_8), and financial resources (C_9). Improving these red criteria can help improve the SWM system.

Meanwhile, the blue dots represent effect group which known as influenced by others. The effect group are public awareness (C_1), research and development (C_5), technology system and facilities (C_6), and implementation and progress schedule (C_{10}). These criteria do not drive the SWM system but improve as a result of enhancements in the cause group.

In conclusion, effective improvement of the SWM system should begin with the cause group, as it directly influences and enhances the performance of the effect group.

Step 7: Determine the threshold value and set up a network relationship map (NRM).

The threshold value is determined based on the average of the total relation matrix, which is *t*=11.318. A new total relational matrix is then formed. If the corresponding entries of the total relation matrix are greater than *t,* then the element of the new total relation matrix is 1. Otherwise, the element is assigned as 0.

Table 5.17

The New Total - Relation Matrix

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	0	0	0	0	1	1	1	0	0	1
C_2	1	0	1	1	1	1	1	0	0	1
C_3	1	1	0	0	1	1	1	0	0	1
C_4	1	1	1	0	1	1	1	0	0	1
C_5	0	0	0	0	0	1	0	0	0	1
C_6	0	0	0	0	0	0	0	0	0	0
C_7	1	1	1	1	1	1	1	0	0	1
C_8	1	1	1	1	1	1	1	0	0	1
C_9	0	0	0	0	1	1	0	0	0	1
C_{10}	1	0	0	0	1	1	1	0	0	0

Then, a network relationship map (NRM) is constructed based on Table 5.17 to visualize the relationship between criteria. The relationship between influences is illustrated in Figure 5.4. Contrastingly, the causal diagram presented in Figure 5.3 implies that the collection and transportation system (C_7), emerges as the most pivotal influencing criterion, given its highest intensity of relationship with other factors. This assertion can be further examined by analyzing the obtained values of the $\tilde{R}_i - \tilde{C}_j$ measure.

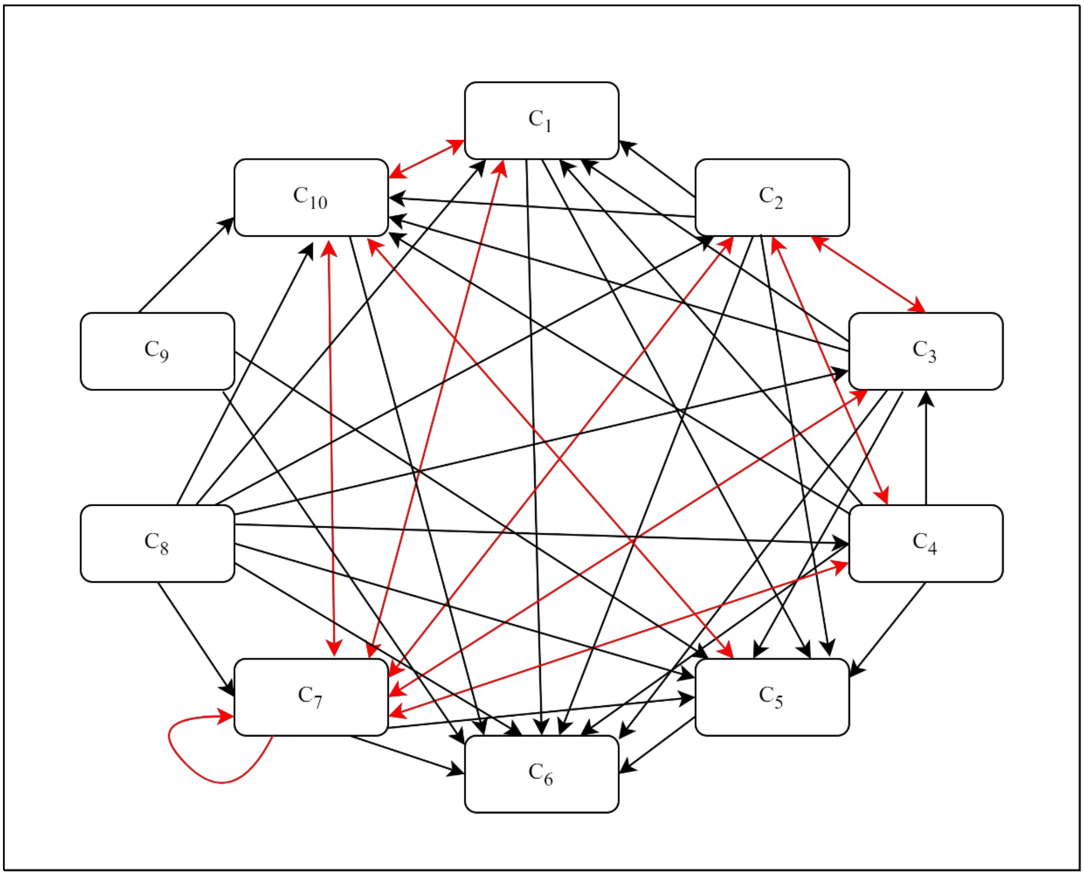

Figure 5.4 Network Relationship Map

A network relationship map is a graphical tool used to represent the connections and interactions among various elements within a system. An arrow pointing from one criterion to another signifies that the first node exerts influence over the second. In Figure 5.4, black arrows represent direct influence. For example, when organizations work together and share ideas (C_4), it supports research activities (C_5). While the red arrow represents indirect influence or mutual influence. For instance, a red arrow connecting C_2 (policy and legislation) to C_5 (research and development) indicates that policies directly impact the direction and progress of research and development activities.

Additionally, C_7 interacts extensively with almost all the criteria, both influencing and being influenced by six criteria, including itself. Therefore, the optimal strategy for SWM is to focus on improving and monitoring C_7 closely, as it has the most significant impact and interaction with other criteria. This conclusion is reinforced by the causal diagram in Figure 5.3, which shows C_7 having the most notable impact on different criteria. Enhancing C_7's efficiency will greatly benefit the overall system, highlighting its crucial role in sustainable SWM.

5.3 Summary

In conclusion, this chapter successfully integrates the proposed BNDHM operators into the SBRM and the DEMATEL-based method. The SBRM utilizes both operators separately to compare their effectiveness in ranking alternatives. Meanwhile, the DEMATEL-based method applies the latest operator, BND-IGWHM, in a real-world case study on solid waste management. Expert evaluation data were aggregated using the proposed operators and transformed into crisp values to ensure accurate analysis. The DEMATEL-based method was employed to identify the interrelationships among criteria using the NRM and to illustrate the causal relationships through a causal diagram.

CHAPTER 6
COMPARATIVE ANALYSIS OF DECISION-MAKING METHODS

6.1 Introduction

This chapter presents a comparative analysis of two decision-making methods. The sensitivity analysis and the comparison with existing methods for the score-based ranking method (SBRM) and the DEMATEL-based method are discussed in Sections 6.2 and 6.3, respectively. A summary is provided at the end of the chapter.

6.2 Score-Based Ranking Method (SBRM)

This section consists of two parts. The first part conducts a sensitivity analysis to examine the effects of parameter variations on the SBRM, specifically the BND-GWHM and BND-IGWHM calculations. The second part compares the proposed methods with existing aggregation operators.

6.2.1 Sensitivity Analysis

In this section, we compute the alternatives $\tilde{A}_i (i = 1, 2, 3, 4)$ with different operational roles for the parameter γ and the parameters p and q. The parameters γ in Dombi TN and Dombi TCN play crucial roles in controlling the shape and behavior of the aggregation process. It also provides the ability to fine-tune the behavior of these operators, making them responsive to various situations and preferences in BNS aggregation. The purpose of the analysis is to evaluate how changes in γ influence the scores and ranking orders of the alternatives.

6.2.1.1 *Sensitivity Analysis with Respect to Parameter γ for BND-GWHM and BND-IGWHM Operators*

This subsection focuses on the sensitivity analysis with respect to the parameter γ for the BND-GWHM and BND-IGWHM operators. The effect of γ on the aggregation behavior of the suggested operators is systemically investigated by changing its value while keeping the other parameters constant. The step-by-step procedure of the sensitivity analysis is presented as follows.

Step 1: The baseline parameter analysis.

To evaluate the influence of parameter γ on the model performance, a baseline analysis was first conducted. Tables 6.1 show the score values and ranking orders for different variables γ based on the BND-GWHM operator (suppose that the parameter value remains constant, $p = q = 1$).

Table 6.1
The Ranking Orders of Different γ for BND-GWHM Operator

γ	**Score value,** $s(\tilde{A}_i)$				**Ranking of alternatives**
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
2	0.4447	0.3685	0.3986	0.4113	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
3	0.5227	0.4297	0.4939	0.4946	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
7	0.6371	0.5220	0.6192	0.6082	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
10	0.6665	0.5489	0.6488	0.6364	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
20	0.7008	0.5826	0.6835	0.6690	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$

Step 2: Increase the values of parameter by 10%.

To examine the sensitivity of the model to an increase in parameter γ, each baseline value was increased by 10%. Table 6.2 provides a summary of the final rankings and scores.

Table 6.2
The Ranking Orders of Different γ Increased by 10% for BND-GWHM Operator

γ	**Score value, $s(\tilde{A}_i)$**				**Ranking of alternatives**
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
2.2	0.4644	0.3841	0.4228	0.4327	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
3.3	0.5388	0.4423	0.5129	0.5113	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
7.7	0.6460	0.5299	0.6281	0.6167	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
11	0.6728	0.5549	0.6552	0.6424	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
22	0.7038	0.5857	0.6866	0.6719	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$

Step 3: Decrease the values of paramater by 10%.

Similarly, a sensitivity analysis was performed by decreasing each baseline value of γ by 10%. Table 6.3 presents the corresponding scores and rankings.

Table 6.3
The Ranking Orders of Different γ Decreased by 10% for BND-GWHM Operator

γ	**Score value, $s(\tilde{A}_i)$**				**Ranking of alternatives**
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
1.8	0.4220	0.3504	0.3708	0.3865	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
2.7	0.5038	0.4151	0.4712	0.4749	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
6.3	0.6265	0.5127	0.6083	0.5979	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
9	0.6588	0.5417	0.6411	0.6291	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
18	0.6970	0.5788	0.6797	0.6654	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$

Step 4: Comparative discussion with baseline.

The ranking orders of the BND-GWHM operator from Tables 6.1 to 6.3 are presented together in Table 6.4 to facilitate comparison across the baseline, increased, and decreased parameter values.

Table 6.4
The Ranking Orders with Respect to Variations in γ for BND-GWHM Operator

Baseline parameter	Increased parameter	Decreased parameter
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$

The sensitivity analysis results presented in Table 6.4 indicate that the ranking orders of the alternatives remain unchanged under both increased and decreased values of the parameter γ. Specifically, the same ranking pattern is consistently observed for the baseline case, as well as for the cases in which the parameter γ is increased and decreased by 10%. This consistency demonstrates that variations in γ within the considered range do not influence the relative ordering of the alternatives when using the BND-GWHM operator. This proved the proposed AO is robust, meaning the decision is stable and reliable. Similarly, the sensitivity analysis procedure for the BND-IGWHM operator was carried out using the same steps, and the results are presented in Table 6.5.

Table 6.5
The Ranking Orders with Respect to Variations in γ for BND-IGWHM Operator

Baseline parameter	Increased parameter	Decreased parameter
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$

Based on Table 6.5, when the value of γ for the BND-IGWHM operator changes, the ranking order remains almost consistent. Thus, the most suitable plan is $\tilde{A}_1$.

6.2.1.2 Sensitivity Analysis with Respect to Parameters p and q for BND-GWHM and BND-IGWHM Operators

This subsection presents the sensitivity analysis with respect to the parameters p and q for the BND-GWHM and BND-IGWHM operators. The analysis is performed by systematically varying the values of p and q while keeping all other parameters fixed in order to examine their influence on the aggregation results. The step-by-step procedure of the sensitivity analysis is presented as follows.

Step 1: The baseline parameter analysis.

To evaluate the influence of parameters p and q on the model performance, a baseline analysis was first conducted. The parameter values are taken from the study by Fan et al. (2019). Tables 6.6 show the score values and ranking orders for different variables p and q based on the BND-GWHM operator. In this analysis, assume the operational roles for parameter γ remain constant where $\gamma = 1$

Table 6.6
The Ranking Orders of Different p and q for BND-GWHM Operator

p,q	Score value, $s(\tilde{A}_i)$				Ranking of alternatives
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
1,0	0.2827	0.2470	0.2875	0.2961	$\tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_1 \succ \tilde{A}_2$
1,0.5	0.2715	0.2325	0.2161	0.2413	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
1,2	0.2792	0.2338	0.2073	0.2336	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
0,1	0.3108	0.2568	0.2164	0.2538	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
0.5,1	0.2792	0.2338	0.2073	0.2336	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
2,1	0.2715	0.2325	0.2161	0.2413	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
2,2	0.2736	0.2317	0.2094	0.2353	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$

Step 2: Increase the values of parameter by 10%.

To evaluate the effect of increasing the parameters p and q on the model, each baseline value was raised by 10%. A summary of the final rankings and scores is provided in Table 6.7.

Table 6.7
The Ranking Orders of Different p and q Increased by 10% for BND-GWHM Operator

p,q	Score value, $s(\tilde{A}_i)$				Ranking of alternatives
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
1.1,0	0.2827	0.2470	0.2875	0.2961	$\tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_1 \succ \tilde{A}_2$
1.1,0.55	0.2715	0.2325	0.2161	0.2413	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
1.1,2.2	0.2792	0.2338	0.2073	0.2336	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
0,1.1	0.3108	0.2568	0.2164	0.2538	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
0.55,1.1	0.2792	0.2338	0.2073	0.2336	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
2.2,1.1	0.2715	0.2325	0.2161	0.2413	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
2.2,2.2	0.2736	0.2317	0.2094	0.2353	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$

Step 3: Decrease the values of paramater by 10%.

Similarly, a sensitivity analysis was conducted by decreasing each baseline value of p and q by 10% to examine their effect on the model. The corresponding rankings and scores are presented in Table 6.8.

Table 6.8
The Ranking Orders of Different p and q Decreased by 10% for BND-GWHM Operator

p,q	**Score value, $s(\tilde{A}_i)$**				**Ranking of alternatives**
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
$0.9, 0$	0.2827	0.2470	0.2875	0.2961	$\tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_1 \succ \tilde{A}_2$
$0.9, 0.45$	0.2715	0.2325	0.2161	0.2413	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
$0.9, 1.8$	0.2792	0.2338	0.2073	0.2336	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
$0, 0.9$	0.3108	0.2568	0.2164	0.2538	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
$0.45, 0.9$	0.2792	0.2338	0.2073	0.2336	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
$1.8, 0.9$	0.2715	0.2325	0.2161	0.2413	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
$1.8, 1.8$	0.2736	0.2317	0.2094	0.2353	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$

Step 4: Comparative discussion with baseline.

The analysis from BND-GWHM operator indicates that the rankings and scores remain consistent across the baseline, increased, and decreased values of p and q, demonstrating that the model is robust with respect to variations in these parameters. Similarly, the sensitivity analysis for the BND-IGWHM operator with respect to p and q was carried out using the same steps, with the final results summarized in Table 6.9.

Table 6.9
The Ranking Orders with Respect to Different p and q for BND-IGWHM Operator

Baseline parameter	Increased parameter	Decreased parameter
$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_3 \succ \tilde{A}_4 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$

Table 6.9 shows that the ranking orders are maintained across the baseline, increased, and decreased parameter values, demonstrating the robustness of the model, where $\tilde{A}_1$ is the best option and $\tilde{A}_2$ is the lowest option.

6.2.2 Comparison with Existing Methods

In this section, the comparison analysis uses a bipolar neutrosophic environment with different types of AOs proposed by Fan et al. (2019) and Mahmood et al. (2020). The ranking results are shown in Table 6.10. Figure 6.1 portrays the ranking behavior of the four alternatives with respect to the four aggregation methods.

Table 6.10
The Ranking Orders of Different p and q for BND-IGWHM Operator

Method	**Score value, $s(\tilde{A}_i)$**				**Ranking of alternatives**
	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	
BNNIGWHM operator (Fan et al., 2019) ($p = q = 1$)	0.5713	0.4818	0.4869	0.5151	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$
BNDWGA operator (Mahmood et al., 2020) ($\gamma = 1$)	0.5275	0.4518	0.3465	0.4369	$\tilde{A}_1 \succ \tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3$
BND-GWHM operator (Proposed method) $(p = q = 1, \gamma = 1)$	0.2736	0.2317	0.2094	0.2353	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_2 \succ \tilde{A}_3$
BND-GWHM operator (Proposed method) $(p = q = 1, \gamma = 1)$	0.5742	0.4836	0.4907	0.5211	$\tilde{A}_1 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_2$

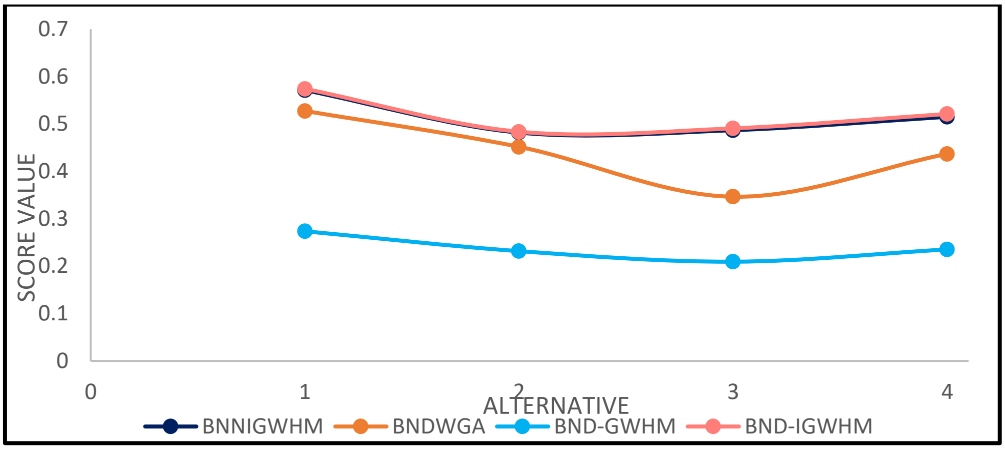

Figure 6.1 Comparison of Ranking of Alternatives

Note that the ranking order of the alternative with respect to the proposed method and existing methods (Fan et al. (2019); Mahmood et al. (2020)) is slightly different, but the optimal option is identical, with $\tilde{A}_1$ is the best option and $\tilde{A}_2$ and $\tilde{A}_3$ the least options. The slight difference in ranking order is related to the choice of AO. The existing method by Fan et al. (2019) using IGWHM, meanwhile the operator proposed by Mahmood et al. (2020) using weighted average (WA). Nevertheless, the proposed method under BNS information is more practical compared to existing methods as it can consider the interrelationship of input argument with flexible γ parameter. The characteristics of some existing aggregation operators and the proposed approach are summarized in Table 6.11.

Table 6.11
Characteristics of the Aggregation Operators

Aggregation operators	**Handle bipolarity**	**Handle indeterminacy**	**Consider the flexibility through a parameter**	**Capture interrelationship**
SVNS-Dombi	No	Yes	Yes	No
SVNS Heronian	No	Yes	No	Yes
BNS-Dombi	Yes	Yes	Yes	No
BNS-Heronian mean	Yes	Yes	No	Yes
Bipolar neutrosophic Dombi Heronian mean (proposed method)	Yes	Yes	Yes	Yes

Based on Table 6.6, the existing methods proposed by Chen and Ye (2017) and Mahmood et al. (2020) do not capture the interrelationship of the input argument since they use traditional operators which is weighted average. Furthermore, Li et al. (2016) and Fan et al. (2019) presented an approach based on the HM AO to consider the interrelationship between arguments. However, these approaches used algebraic product and algebraic sum in the operation. In comparison to these existing approaches, the proposed approach has the advantage of reflecting the interrelationships of all input arguments with flexible parameters in Dombi operation.

6.3 DEMATEL-Based Method

This section has two parts. The first conducts a sensitivity analysis to examine the effects of parameter variations on the Dombi and HM aggregation operators in BND-IGWHM calculations. The second compares the proposed BND-IGWHM-DEMATEL method with existing methods, including classical DEMATEL, SVNS-DEMATEL, and BNS-DEMATEL, which utilize arithmetic mean aggregation.

6.3.1 Sensitivity Analysis

This SWM case study uses different parameters, γ, p and q values in the aggregation process, to demonstrate their impact on the final results. Table 6.12 shows the results according to the ordering of criteria and the categorization of the cause-and-effect group for the different parameters of γ based on the bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean (BND-IGWHM) aggregation operator (Parameters p and q remain constant, at $p = q = 1$).

Table 6.12
The Ranking Orders of Criteria for Different γ Values

γ	Ranking	Category	
		Cause group	Effect group
2	$C_7 \succ C_1 \succ C_{10} \succ C_5 \succ C_2 \succ C_3 \succ C_6 \succ C_4 \succ C_8 \succ C_9$	C_2, C_3, C_4, C_8, C_9	$C_1, C_5, C_6, C_7, C_{10}$
3	$C_1 \succ C_7 \succ C_5 \succ C_{10} \succ C_2 \succ C_3 \succ C_6 \succ C_4 \succ C_9 \succ C_8$	C_2, C_3, C_4, C_8, C_9	$C_1, C_5, C_6, C_7, C_{10}$

7	$C_1 \succ C_9 \succ C_3 \succ C_5 \succ C_6 \succ C_2 \succ C_{10} \succ C_4 \succ C_7 \succ C_8$	$C_2, C_3, C_4, C_8, C_9, C_{10}$	C_1, C_5, C_6, C_7
9	$C_1 \succ C_9 \succ C_3 \succ C_5 \succ C_2 \succ C_6 \succ C_{10} \succ C_4 \succ C_7 \succ C_8$	$C_2, C_3, C_4, C_6, C_7, C_9, C_{10}$	C_1, C_5, C_8
20	$C_9 \succ C_1 \succ C_3 \succ C_2 \succ C_5 \succ C_{10} \succ C_6 \succ C_4 \succ C_7 \succ C_8$	$C_2, C_4, C_6, C_7, C_9, C_{10}$	C_1, C_3, C_5, C_8

Based on Table 6.12, it can be noticed that the ranking orders is slightly different with any value of parameter γ except for the $\gamma = 2$ and $\gamma = 20$. Even though the outcome for parameter $\gamma = 3$ shows a minor variation, the optimal and least favorable choices remain consistent with those observed for parameters $\gamma = 7$ and $\gamma = 9$. Also, the cause-and-effect group results vary dramatically depending on the parameter values employed.

Next, to illustrate the influence of parameters p and q on decision-making in this case study, we use the different values p and q using the BND-IGWHM method to rank the criteria. Assume that parameter γ is a constant where $\gamma = 1$. Hence, the ranking orders are shown in Table 6.13.

Table 6.13
The Ranking Orders of Criteria for Different p and q Values

p,q	**Ranking**	**Category**	
		Cause group	**Effect group**
1,0	$C_{10} \succ C_7 \succ C_2 \succ C_4 \succ C_1 \succ C_5 \succ C_3 \succ C_6 \succ C_9 \succ C_8$	$C_1, C_2, C_3, C_4, C_8, C_9$	C_5, C_6, C_7, C_{10}
1,0.5	$C_7 \succ C_{10} \succ C_2 \succ C_4 \succ C_1 \succ C_3 \succ C_5 \succ C_6 \succ C_8 \succ C_9$	C_2, C_3, C_4, C_8, C_9	$C_1, C_5, C_6, C_7, C_{10}$
1,2	$C_7 \succ C_{10} \succ C_2 \succ C_3 \succ C_4 \succ C_5 \succ C_1 \succ C_6 \succ C_8 \succ C_9$	$C_2, C_3, C_4, C_7, C_8, C_9$	C_1, C_5, C_6, C_{10}
0,1	$C_2 \succ C_7 \succ C_{10} \succ C_5 \succ C_3 \succ C_6 \succ C_1 \succ C_4 \succ C_8 \succ C_9$	$C_2, C_3, C_4, C_7, C_8, C_9$	C_1, C_5, C_6, C_{10}
0.5,1	$C_7 \succ C_{10} \succ C_2 \succ C_3 \succ C_4 \succ C_5 \succ C_1 \succ C_6 \succ C_8 \succ C_9$	$C_2, C_3, C_4, C_7, C_8, C_9$	C_1, C_5, C_6, C_{10}
2,1	$C_7 \succ C_{10} \succ C_2 \succ C_4 \succ C_1 \succ C_3 \succ C_5 \succ C_6 \succ C_8 \succ C_9$	C_2, C_3, C_4, C_8, C_9	$C_1, C_5, C_6, C_7, C_{10}$
2,2	$C_7 \succ C_{10} \succ C_2 \succ C_4 \succ C_3 \succ C_1 \succ C_5 \succ C_6 \succ C_8 \succ C_9$	$C_2, C_3, C_4, C_7, C_8, C_9$	C_1, C_5, C_6, C_{10}

The results in Table 6.13 show that the best and least criteria remain the same except for parameter $p=1, q=0$ and $p=0, q=1$ that is C_7 and C_9 respectively. The ranking orders show differences for all different parameters however for $p=1, q=0.5$ and $p=2, q=1$, obtained the similar results which are $C_7 \succ C_{10} \succ C_2 \succ C_4 \succ C_1 \succ C_3 \succ C_5 \succ C_6 \succ C_8 \succ C_9$.

6.3.2 Comparison with Existing Methods

This section presents a comparison analysis to evaluate the ranking orders derived from the proposed method against those from several existing methods: classical DEMATEL (Mavi & Standing, 2018), SVNS-DEMATEL (Liu et al., 2018), and BNS-DEMATEL (Abdullah & Rahim, 2020), all using the arithmetic mean operator. Table 6.14 below displays the ranking orders of criteria obtained from these various methods.

Table 6.14
The Ranking Orders Based on Different Methods

Method	**Ranking**	**Category**	
		Cause group	**Effect group**
DEMATEL	$C_9 \succ C_8 \succ C_3 \succ C_2 \succ C_6 \succ C_4 \succ C_5 \succ C_7 \succ C_{10} \succ C_1$	C_2, C_3, C_4, C_8, C_9	$C_1, C_5, C_6, C_7, C_{10}$
SVNS-DEMATEL	$C_9 \succ C_8 \succ C_3 \succ C_2 \succ C_4 \succ C_6 \succ C_{10} \succ C_7 \succ C_5 \succ C_1$	$C_1, C_5, C_6, C_7, C_{10}$	C_2, C_3, C_4, C_8, C_9
BNS-DEMATEL	$C_7 \succ C_2 \succ C_{10} \succ C_3 \succ C_4 \succ C_1 \succ C_5 \succ C_6 \succ C_8 \succ C_9$	$C_2, C_3, C_4, C_7, C_8, C_9$	C_1, C_5, C_6, C_{10}
BND-IGWHM-DEMATEL (Proposed method)	$C_7 \succ C_{10} \succ C_2 \succ C_4 \succ C_3 \succ C_1 \succ C_5 \succ C_6 \succ C_8 \succ C_9$	$C_2, C_3, C_4, C_7, C_8, C_9$	C_1, C_5, C_6, C_{10}

Table 6.14 demonstrates that different DEMATEL methods produce varying results for SWM. Specifically, the net impact of the SVNS-DEMATEL method differs significantly from the others, while the other DEMATEL methods show similar patterns, consistently identifying C_2, C_3, C_4, C_8, and C_9 as the cause group. Among these, the results of BNS-DEMATEL and the proposed method appear quite similar in terms of the overall ranking pattern. Both methods place criteria C_7 among the top-ranked, indicating a consistent evaluation of important criteria. . However, the proposed

BND-IGWHM-DEMATEL method goes beyond BNS-DEMATEL by offering enhanced capabilities. By incorporating BNS, the proposed method is able to handle uncertainty, indeterminacy, and bipolar information more effectively. Additionally, it employs the Dombi-based IGWHM to aggregate expert opinions with greater sensitivity and flexibility. This results in a more accurate and robust identification of the cause-and-effect groups, as seen in the classification of C_1, C_5, C_6, C_{10} as effect criteria and $C_2, C_3, C_4, C_7, C_8, C_9$ as cause criteria. Ultimately, the proposed method provides a deeper and more precise analysis of the criteria's interrelationships, making it especially valuable for complex multi-criteria decision-making problems. Some inspection of the methods is described below to explain a significant difference in the comparative results:

i) The DEMATEL method evaluates all criteria using crisp numbers. However, utilising DEMATEL in SWM problems is inappropriate because of hesitancy and uncertainty.

ii) The SVNS method captures truth, indeterminate, and false membership degrees but mainly focuses on positive membership degrees. In contrast, the BNS method effectively handles positive and negative membership degrees. As a result, SVNS proves less suitable for solid waste management, where accurately capturing a full range of membership degrees is essential.

iii) The BNS-DEMATEL method aggregates decision-maker judgments by calculating the arithmetic mean. This approach averages the input arguments without considering their interrelationships.

Regarding the above, the proposed BND-IGWHM-DEMATEL is validated as a more rational and practical solution for the SWM problem. This innovative approach excels in scenarios where experts face uncertainty and bipolarity, adeptly handling these challenges. It considers the interrelationships among input arguments and integrates variable operational parameters, effectively aggregating information within the bipolar neutrosophic framework.

6.4 Summary

In conclusion, this chapter has explored a case study on SWM, addressing all three primary objectives of the study. First, it aggregated expert evaluation data using the BND-IGWHM operator, transforming these values into crisp numbers. Next, the DEMATEL method was employed to elucidate the interrelationships between criteria through the NRM and to visualize causal relationships by creating a causal diagram. The findings highlight that the 'collection and transportation system' is the most influential criterion for decision-making problem especially in the case study of SWM. Additionally, the analysis shows that the parameters have a minimal impact on the criteria ranking within the SWM system. Applying these methods to an SWM case can assist institutions and organizations in developing strategic approaches to achieve sustainable SWM.

CHAPTER 7
CONCLUSION

7.1 Introduction

Chapter 7 concludes the thesis by summarizing the study's objectives and outcomes. In Section 7.2 presents the summary of research objectives and outcomes of the study. Next, Section 7.3 discusses the research contributions and highlights its limitations, followed by recommendations for future research.

7.2 Research Objectives and Outcomes

The complexity of real-world decision-making problems has increased due to multiple conflicting criteria, numerous available alternatives, and the involvement of various experts. Addressing these issues can be challenging, as it requires managing different clusters, assessing weights, and evaluating the interrelationships among input arguments. To tackle these challenges, this study achieves its first objective by integrating the bipolar neutrosophic Dombi Heronian mean (HM) aggregation operator, incorporating Dombi operations into the HM operator. As a result, two novel aggregation operators are developed: the bipolar neutrosophic Dombi-based generalized weighted Heronian mean (BND-GWHM) and the bipolar neutrosophic Dombi-based improved generalized weighted Heronian mean (BND-IGWHM). The properties of each operator are thoroughly analyzed such as idempotency, monotonicity, and boundedness property. The study also includes illustrative examples and a comparative analysis to evaluate the applicability of the proposed methods. These proposed AOs offer effective method for capturing the complex interrelationships among input arguments and provide flexibility through adjustable operational parameters.

Furthermore, this study has addressed the shortcomings of the aggregation process in the DEMATEL method, enhancing its effectiveness compared to traditional approaches, such as the arithmetic mean. An aggregation operator takes essential steps to obtain accurate results. To solve this, we integrate the bipolar neutrosophic Dombi-based IGWHM with the DEMATEL method. Our primary goal is to enhance the

DEMATEL method by strengthening the interrelationships between criteria and improving its effectiveness in solving MCDM problems. We present an illustrative example and comparative results to validate the proposed approach. The findings show that the BND-IGWHM-DEMATEL method significantly impacts the outcomes. Additionally, the results reveal slight variations in the importance ranking compared to other methods, demonstrating that our approach enables a more comprehensive analysis of causal relationships among the identified factors.

The developed operator, BND-IGWHM, with the DEMATEL method, was applied in a real case study focusing on Solid Waste Management (SWM). Through a literature review, we selected SWM as the case study for group decision-making, utilizing secondary data from Abdullah et al. (2019). Based on this data, a survey was conducted with five experts who assessed the influential criteria for SWM in Malaysia. The results indicated that the collection and transportation system (C_7) is the most critical criterion in SWM. Furthermore, the collection and transportation system were identified as the causal group that impacts the other criteria. In summary, the collection and transportation system significantly enhance the other criteria related to SWM. The validated causal diagram and NRM of BND-IGWHM-DEMATEL offer valuable insights into improving the SWM challenges.

7.3 Contribution of the Research

This study makes several important contributions to the field of multi-criteria decision-making (MCDM), particularly in addressing uncertainty and complex interrelationships among criteria. The first contribution is conceptual. The study develops two novel aggregation operators, BND-GWHM and BND-IGWHM, which extend the Heronian mean (HM) under bipolar neutrosophic set (BNS) theory by incorporating Dombi operations. These operators overcome key limitations of traditional HM, such as inadequate weight handling and lack of idempotency, while providing flexible mechanisms to aggregate uncertain and interdependent information. This advancement enriches the literature on aggregation operators and provides a solid foundation for future research in MCDM under uncertainty.

The second contribution is methodological. By integrating the BND-IGWHM operator into both the score-based ranking method (SBRM) and the DEMATEL-based method, the study enhances existing MCDM methods for analyzing complex systems.

This integration enables more accurate evaluation of alternatives and a more comprehensive understanding of causal relationships among criteria, improving the robustness and reliability of decision-making under uncertainty. Comparative and sensitivity analyses further demonstrate the effectiveness and adaptability of the proposed framework relative to existing aggregation methods, highlighting its value for complex decision contexts.

The third contribution is practical. The use of the BND-IGWHM-DEMATEL framework to solve a real-world solid waste management (SWM) problem shows that it can help people make decisions based on facts. The study identifies the most critical and causal criteria affecting SWM, providing actionable insights for policymakers and practitioners. Moreover, the generalizable structure of the framework allows it to be applied to other MCDM problems characterized by uncertainty and interdependent criteria, extending its usefulness beyond the specific case study.

7.4 Research Limitations and Suggestions for Future Research

This study has made significant contributions, but there are also some limitations, which are listed below:

i) The use of BNS is most effective in scenarios where it is crucial to capture the nuances of truth, indeterminacy, and falsity to represent bipolarity information accurately. However, this method may not be suitable for all decision-making problems, particularly those that do not require and are limited in such a complex representation of these attributes or where a more straightforward approach suffices.

ii) Also, this study faces limitations due to the reliance on secondary data. The relevance of this data may not align perfectly with the research objectives, resulting in gaps or misalignment with the study's focus. Furthermore, the accuracy and quality of secondary data hinge on the source, which can differ significantly; challenges such as outdated information, biased collection methods, or incomplete data sets may emerge. Researchers also lack control over how the data was gathered, processed, or classified, restricting their ability to analyze or interpret it effectively.

The findings of this study provide several recommendations for future research. First, the aggregation procedure can be integrated with other alternative ranking methods within the BNS framework, such as BNS-TOPSIS, BNS-VIKOR, and BNS-AHP. Additionally, the proposed aggregation operator can be extended using other operators, such as Archimedean t-norms and t-conorms or power averages. The proposed methods also have potential for application across various fields of MCDM, particularly in science and technology domains, including pattern recognition and ambiguous programming. Furthermore, the framework can be tested and adopted in other real-world MCDM problems, which would demonstrate its broader practical applicability. Finally, the proposed methods can be adapted for use in other neutrosophic and fuzzy environments, including Pythagorean neutrosophic sets (PNS), Fermatean neutrosophic sets, and emerging information theories such as basic uncertain information.

REFERENCES

Abdel-Basset, M., Manogaran, G., Gamal, A., & Smarandache, F. (2018). A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. *Design Automation for Embedded Systems*, 22(3), 257–278.

Abdullah, L., & Rahim, S. N. (2020). Bipolar neutrosophic DEMATEL for urban sustainable development. *Journal of Intelligent & Fuzzy Systems, 39*(5), 6109–6119.

Abdullah, L., Zulkifli, N., Liao, H., Herrera-Viedma, E., & Al-Barakati, A. (2019). An interval-valued intuitionistic fuzzy DEMATEL method combined with Choquet integral for sustainable solid waste management. *Engineering Applications of Artificial Intelligence, 82*, 207–215.

Abdullah, F. M., Al-Ahmari, A. M., & Anwar, S. (2023). An integrated fuzzy DEMATEL and fuzzy TOPSIS method for analyzing smart manufacturing technologies. *Processes*, *11*(3), 906.

Abushammala, M. F., Basri, N. E. A., Basri, H., El-Shafie, A. H., & Kadhum, A. A. H. (2011). Regional landfills methane emission inventory in Malaysia. *Waste Management & Research, 29*(8), 863–873.

Alballa, T., Alamer, A., Nasir, K., Yousaf, A., Alhabeeb, S. A., & Khalifa, H. A. E. W. (2024). A multiple attribute group decision making model based on 2-tuple linguistic Pythagorean fuzzy Dombi aggregation operators for optimal selection of potential global suppliers. *Heliyon, 10*(14), e34570.

Ali, Z., Emam, W., Mahmood, T., & Wang, H. (2024). Archimedean Heronian mean operators based on complex intuitionistic fuzzy sets and their applications in decision-making problems. *Heliyon, 10*(3).

Altuntas, S. & Dereli, T. (2015). A novel approach based on DEMATEL method and patent citationanalysis for prioritizing a portfolio of investment projects. *Expert Systems with Applications.* 42, 1003-101

Anusha, V., & Sireesha, V. (2022). Einstein Heronian mean aggregation operator and its application in decision making problems. *Computational and Applied Mathematics*, *41*(2), 69.

Asadi, S., Nilashi, M., Iranmanesh, M., Ghobakhloo, M., Samad, S., Alghamdi, A., Almulihi, A., & Mohd, S. (2022). Drivers and barriers of electric vehicle usage in Malaysia: A DEMATEL approach. *Resources, Conservation and Recycling*, 177, 105965.

Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 20*(1), 87–96.

Atanassov, K., & Gargov, G. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 31*(3), 343–349.

Awang, A., Ab Ghani, A. T., Abdullah, L., & Ahmad, M. F. (2018). The Shapley weighting vector-based neutrosophic aggregation operator in DEMATEL. *Journal of Physics: Conference Series*, 1132, 012059.

Awang, A., Aizam, N. A. H., & Abdullah, L. (2019). An integrated decision-making method based on neutrosophic numbers for investigating factors of coastal erosion. *Symmetry, 11*(3), 328.

Beliakov, G., Pradera, A., & Calvo, T. (2007). *Aggregation functions: A guide for practitioners*. Springer.

Bonferroni, C. (1950). Sulle medie multiple di potenze. *Bollettino dell'Unione Matematica Italiana, 5*(3–4), 267–270.

Bosc, P., & Pivert, O. (2013). On a fuzzy bipolar relational algebra. *Information Sciences, 219*, 1–16.

Chang, B., Chang, C. W., & Wu, C. H. (2011). Fuzzy DEMATEL method for developing supplier selection criteria. *Expert Systems with Applications, 38*(3), 1850–1858.

Chen, J. Q., & Ye, J. (2017). Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision-making. *Symmetry, 9*(6), 82.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2002). The ordered weighted geometric operator: Properties and application in MCDM problems. *Technologies for Constructing Intelligent Systems 2: Tools*, 173-183.

Choquet, G. (1953). Theory of capacities. *Annales de l'Institut Fourier, 5*, 131–295.

Deli, I., Ali, M., & Smarandache, F. (2015, August). Bipolar neutrosophic sets and their application based on multi-criteria decision-making problems. *International Conference on Advanced Mechatronic Systems*, 249–254.

Deepu, T. S., & Ravi, V. (2021). Supply chain digitalization: An integrated MCDM approach for inter-organizational information systems selection in an electronic supply chain. *International Journal of Information Management Data Insights*, *1*(2), 100038.

Dombi, J. (1982). A general-class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets and Systems, 8*, 149–163.

Ebrahimi, S. H. (2023). Introducing the λ-Scale DEMATEL method: Developing the classical DEMATEL technique. *Politeknik Dergisi*, *26*(2), 751-758.

Fan, C., Ye, J., Feng, S., Fan, E., & Hu, K. (2019). Multi-criteria decision-making method using Heronian mean operators under a bipolar neutrosophic environment. *Mathematics, 7*(1), 97.

Fan, J., Jia, X., & Wu, M. (2019). Green supplier selection based on Dombi prioritized Bonferroni mean operator with single-valued triangular neutrosophic sets. *International Journal of Computational Intelligence Systems, 12*(2), 1091–1101.

Gabus, A., & Fontela, E. (1973). Perceptions of the world problematique: Communication procedure, communicating with those bearing collective responsibility (pp. 11–18).

Ghadami, L., Masoudi, I., Hessam, S., & Modiri, M. (2019). Developing hospital accreditation standards, applying fuzzy DEMATEL. *International Journal of Healthcare Management, 14*(3), 847–855.

Hashim, H., Garg, H., Al-Quran, A., Awang, N. A., & Abdullah, L. (2022). Heronian mean operators considering Shapley fuzzy measure under interval neutrosophic vague environment for an investment decision. *International Journal of Fuzzy Systems, 24*(4), 2068–2091.

Hu, K. H. (2023). An exploration of the key determinants for the application of AI-enabled higher education based on a hybrid Soft-computing technique and a DEMATEL approach. *Expert Systems with Applications*, *212*, 118762.

Hu, X., Yang, S., & Zhu, Y. R. (2022). Multiple-attribute decision making based on interval-valued intuitionistic fuzzy generalized weighted heronian mean. *Information*, *13*(3), 138.

Jana, C., Muhiuddin, G., & Pal, M. (2021). Multi-criteria decision-making approach based on SVTrN Dombi aggregation functions. *Artificial Intelligence Review, 54*(5), 3685–3723.

Kakati, P., Senapati, T., Moslem, S., & Pilla, F. (2024). Fermatean fuzzy Archimedean Heronian mean-based model for estimating sustainable urban transport solutions. *Engineering Applications of Artificial Intelligence, 127*, 107349.

Khan, M., & Gulistan, M. (2023). Unification of some generalized aggregation operators in neutrosophic cubic environment and its applications in multi-expert decision-making analysis. *Neutrosophic Sets and Systems, 56*(1), 8.

Khatun, M., Wagner, F., Jung, R., & Glaß, M. (2023). An application of DEMATEL and fuzzy DEMATEL to evaluate the interaction of safety management system and cybersecurity management system in automated vehicles. *Engineering Applications of Artificial Intelligence*, *124*, 106566.

Kiani Mavi, R., & Standing, C. (2018). Cause and effect analysis of business intelligence (BI) benefits with fuzzy DEMATEL. *Knowledge Management Research & Practice, 16*(2), 245–257.

Kilic, H. S., Yurdaer, P., & Aglan, C. (2021). A leanness assessment methodology based on neutrosophic DEMATEL. *Journal of Manufacturing Systems, 59*, 320–344.

Koushki, P. A., Al-Humoud, J., & Al-Duaij, U. S. A. M. A. (2004). Municipal solid waste in Kuwait: Trends and attitudes on collection, separation, and willingness to pay. *Kuwait Journal of Science and Engineering, 31*(2), 173.

Leake, C. (2000). GIS and multicriteria decision analysis. *Journal of the Operational Research Society, 51*(2), 247.

Li, B., & Yang, L. (2021). Hamacher Heronian mean operators for multi-criteria decision-making under multi-valued picture fuzzy uncertain linguistic environment. *Journal of Intelligent & Fuzzy Systems, 41*(1), 1531–1552.

Li, Y., Liu, P., & Chen, Y. (2016). Some single valued neutrosophic number Heronian mean operators and their application in multiple attribute group decision making. *Informatica, 27*(1), 85–110.

Lin, M., Li, X., Chen, R., Fujita, H., & Lin, J. (2022). Picture fuzzy interactional partitioned Heronian mean aggregation operators: an application to MADM process. *Artificial Intelligence Review*, *55*(2), 1171-1208.

Liu, F., Aiwu, G., Lukovac, V., & Vukic, M. (2018,a). A multicriteria model for the selection of the transport service provider: A single valued neutrosophic DEMATEL multicriteria model. *Decision Making: Applications in Management and Engineering, 1*(2), 121–130.

Liu, H. Z., & Pei, D. W. (2012). HOWA operator and its application to multi-attribute decision making. *Journal of Zhejiang Sci-Tech University, 25*, 138-142.

Liu, P., & Shi, L. (2017). Some neutrosophic uncertain linguistic number Heronian mean operators and their application to multi-attribute group decision making. *Neural Computing and Applications, 28*, 1079–1093.

Liu, P., & You, X. (2018). Some linguistic intuitionistic fuzzy Heronian mean operators based on Einstein T-norm and T-conorm and their application to decision-making. *Journal of Intelligent & Fuzzy Systems, 35*(2), 2433–2445.

Liu, P., Khan, Q., Mahmood, T., Smarandache, F., & Li, Y. (2019). Multiple attribute group decision making based on 2-tuple linguistic neutrosophic Dombi power Heronian mean operators. *IEEE Access, 7*, 100205–100230.

Liu, P., Liu, J., & Merigó, J. M. (2018,b). Partitioned Heronian means based on linguistic intuitionistic fuzzy numbers for dealing with multi-attribute group decision making. *Applied Soft Computing, 62*, 395–422.

Mahmood, M. K., Zeng, S., Gulfam, M., Ali, S., & Jin, Y. (2020). Bipolar neutrosophic Dombi aggregation operators with application in multi-attribute decision making problems. *IEEE Access, 8*, 156600–156614.

Mahmoudi, S., Jalali, A., Ahmadi, M., Abasi, P., & Salari, N. (2019). Identifying critical success factors in heart failure self-care using fuzzy DEMATEL method. *Applied Soft Computing, 84*, 105729.

Mamites, I., Almerino, P., Sitoy, R., Atibing, N. M., Almerino, J. G., Cebe, D., Ybañez, R., et al. (2022). Factors influencing teaching quality in universities: Analyzing causal relationships based on neutrosophic DEMATEL. *Education Research International, 2022*(1), 9475254.

Mishra, A. R., Rani, P., Deveci, M., Gokasar, I., Pamucar, D., & Govindan, K. (2023). Interval-valued Fermatean fuzzy heronian mean operator-based decision-making method for urban climate change policy for transportation activities. *Engineering Applications of Artificial Intelligence*, *124*, 106603.

Moh, Y., & Abd Manaf, L. (2017). Solid waste management transformation and future challenges of source separation and recycling practice in Malaysia. *Resources, Conservation and Recycling, 116*, 1–14.

Osman, M. S., Nasir, M. H., & Mujeebu, M. A. (2009). Assessment of municipal solid waste generation and recyclable materials potential in Kuala Lumpur, Malaysia. *Waste Management, 29*(7), 2209–2213.

Park, J. H., & Park, E. J. (2013). Generalized fuzzy Bonferroni harmonic mean operators and their applications in group decision making. *Journal of Applied Mathematics*, 1–14.

Peng, H. G., Wang, J. Q., & Cheng, P. F. (2018). A linguistic intuitionistic multi-criteria decision-making method based on the Frank Heronian mean operator and its application in evaluating coal mine safety. *International Journal of Machine Learning and Cybernetics, 9*, 1053–1068.

Peng, J., Wang, J.-Q., Hu, J.-H., Tian, C., Juan-Juan, P., Jian-Qiang, W., & Jun-Hua, H. (2018). Multi-criteria decision-making approach based on single-valued neutrosophic hesitant fuzzy geometric weighted Choquet integral Heronian mean operator. *Journal of Intelligent & Fuzzy Systems, 35*, 3661–3674.

Petrović, G., Mihajlović, J., Marković, D., Hashemkhani Zolfani, S., & Madić, M. (2023). Comparison of aggregation operators in the group decision-making process: A real case study of location selection problem. *Sustainability, 15*(10), 8229.

Rahim, S. N. (2022). Bipolar neutrosophic DEMATEL-MABAC for sustainable energy selection (Unpublished Master's thesis). Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu.

Rao, F., & Xiao, M. (2023). A novel MADM algorithm for physical education teaching quality evaluation based on 2-tuple linguistic neutrosophic numbers power Heronian mean operators. *PLOS ONE, 18*(2), e0279534.

Riaz, M., Farid, H. M. A., Pamucar, D., & Tanveer, S. (2022). Spherical fuzzy information aggregation based on Aczel–Alsina operations and data analysis for supply chain. *Mathematical Problems in Engineering, 2022*(1), 9657703.

Saleh, W. A., Abdelkader, S. M., Rashada, H., & Abdelgawad, A. (2023). Analyzing critical success factors of IoT-enabled green supply chain management using bipolar neutrosophic-DEMATEL. *Neutrosophic Sets and Systems, 55*(1), 8.

Salehi, R., Asaadi, M. A., Rahimi, M. H., & Mehrabi, A. (2021). The information technology barriers in supply chain of sugarcane in Khuzestan province, Iran: A combined ANP-DEMATEL approach. *Information Processing in Agriculture, 8*(3), 458–468.

Sathyan, R., Parthiban, P., Dhanalakshmi, R., & Sachin, M. S. (2023). An integrated Fuzzy MCDM approach for modelling and prioritising the enablers of responsiveness in automotive supply chain using Fuzzy DEMATEL, Fuzzy AHP and Fuzzy TOPSIS. *Soft computing,* 27(1), 257-277.

Si, S. L., You, X. Y., Liu, H. C., & Zhang, P. (2018). DEMATEL technique: A systematic review of the state-of-the-art literature on methodologies and applications. *Mathematical Problems in Engineering, 2018*(1), 3696457.

Smarandache, F. (1998). Neutrosophy: Neutrosophic probability, set, and logic: Analytic synthesis & synthetic analysis. American Research Press.

Soltani, A., Hewage, K., Reza, B., & Sadiq, R. (2015). Multiple stakeholders in multi-criteria decision making in the context of municipal solid waste management. *Waste Management, 35*, 318–328.

Sýkora, S. (2009). Mathematical means and averages: Generalized Heronian means. Stan's Library.

Tchobanoglous, G., Theisen, H., & Vigil, S. (1993). Integrated solid waste management: Engineering principles and management issues. McGraw-Hill.

Torkayesh, A. E., Tavana, M., & Santos-Arteaga, F. J. (2022). A multi-distance interval-valued neutrosophic approach for social failure detection in sustainable municipal waste management. *Journal of Cleaner Production, 336*, 130409.

Torra, V., Narukawa, Y., Aguiló, I., & González-Hidalgo, M. (Eds.). (2018). Modeling decisions for artificial intelligence: 15th International Conference, MDAI 2018, Mallorca, Spain, October 15–18, 2018, Proceedings (Vol. 11144). Springer.

Tuğrul, F., Çitil, M., Karasolak, B., & Dağlı, M. (2020). Interpretation of physical conditions of schools with fuzzy multi criteria decision making. *Journal of Universal Mathematics, 3*(1), 46–52.

Wang, H., Madiraju, P., Zhang, Y., & Sunderraman, R. (2004). Interval neutrosophic sets. *Mathematics, 1*, 274–277.

Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single valued neutrosophic sets. *Infinite Study*, 410.

Wu, L., Wei, G., Wu, J., & Wei, C. (2020). Some interval-valued intuitionistic fuzzy Dombi Heronian mean operators and their application for evaluating the ecological value of forest ecological tourism demonstration areas. *International Journal of Environmental Research and Public Health, 17*(3), 829.

Yager, R. R. (1988). Fusion of ordinal information using weighted median aggregation. *International Journal of Approximate Reasoning, 18*(1–2), 35–52.

Yager, R. R. (2001). The power average operator. IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans, 31(6), 724–731.

Yalcin, A. S., Kilic, H. S., & Delen, D. (2022). The use of multi-criteria decision-making methods in business analytics: A comprehensive literature review. *Technological forecasting and social change*, *174*, 121193.

Yu, D.J., & Wu, Y.Y. (2012) Interval-valued intuitionistic fuzzy Heronian mean operators and their application in multi-criteria decision making. Afr. J. Bus. Manage. 6(11), 4158–4168.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*, 338–353.

Zhang, W. R. (1994). Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis. In *Proceedings of IEEE Conference* (pp. 305–309).

Zhu, Y., Niu, X., Gu, C., Yang, D., Sun, Q., & Rodriguez, E. F. (2020). Using the DEMATEL-VIKOR method in dam failure path identification. *International Journal of Environmental Research and Public Health*, *17*(5), 1480.

Önden, İ., Pamucar, D., Deveci, M., As, Y., Birol, B., & Yıldız, F. Ş. (2024). Prioritization of transfer centers using GIS and fuzzy Dombi Bonferroni weighted Assessment (DOBAS) model. *Expert Systems with Applications, 238*, 121827.

APPENDICES

APPENDIX 1

Evaluation Values by Experts

E1	C1						C2						C3					
C1	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C2	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00
C4	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C7	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C8	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C9	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C10	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70

E1	C4						C5						C6					
C1	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C2	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C4	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00
C7	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C8	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C9	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C10	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70

E1	C7						C8						C9					
C1	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C2	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C4	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C7	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C8	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70
C9	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00
C10	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70

E1	C1					
C1	0.35	0.60	0.70	-0.35	-0.60	-0.70
C2	0.35	0.60	0.70	-0.35	-0.60	-0.70
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70
C4	0.35	0.60	0.70	-0.35	-0.60	-0.70
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70
C7	0.35	0.60	0.70	-0.35	-0.60	-0.70
C8	0.35	0.60	0.70	-0.35	-0.60	-0.70
C9	0.35	0.60	0.70	-0.35	-0.60	-0.70
C10	0.00	0.00	0.00	0.00	0.00	0.00

E2	C1						C2						C3					
C1	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C4	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C7	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E2	C4						C5						C6					
C1	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C7	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E2	C7						C8						C9					
C1	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C7	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E2	C1					
C1	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15
C7	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.00	0.00	0.00	0.00	0.00	0.00

E3	C1						C2						C3					
C1	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C4	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C7	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E3	C4						C5						C6					
C1	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.50	0.40	0.45	-0.50	-0.40	-0.45
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C4	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.00	0.00	0.00	0.00	0.00	0.00
C7	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C8	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.35	0.60	0.70	-0.35	-0.60	-0.70
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70

E3	C7						C8						C9					
C1	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C6	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C7	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C10	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E3	C1					
C1	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.50	0.40	0.45	-0.50	-0.40	-0.45
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.50	0.40	0.45	-0.50	-0.40	-0.45
C5	0.50	0.40	0.45	-0.50	-0.40	-0.45
C6	0.50	0.40	0.45	-0.50	-0.40	-0.45
C7	0.50	0.40	0.45	-0.50	-0.40	-0.45
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.00	0.00	0.00	0.00	0.00	0.00

E4	C1						C2						C3					
C1	0.00	0.00	0.00	0.00	0.00	0.00	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C2	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.90	0.10	0.10	-0.90	-0.10	-0.10
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C4	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.90	0.10	0.10	-0.90	-0.10	-0.10
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.80	0.20	0.15	-0.80	-0.20	-0.15
C7	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.90	0.10	0.10	-0.90	-0.10	-0.10
C10	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.90	0.10	0.10	-0.90	-0.10	-0.10

E4	C4						C5						C6					
C1	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.35	0.60	0.70	-0.35	-0.60	-0.70
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.00	0.00	0.00	0.00	0.00	0.00	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.90	0.10	0.10	-0.90	-0.10	-0.10
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.00	0.00	0.00	0.00	0.00	0.00
C7	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.90	0.10	0.10	-0.90	-0.10	-0.10
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E4	C7						C8						C9					
C1	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.50	0.40	0.45	-0.50	-0.40	-0.45
C2	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C3	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C4	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.90	0.10	0.10	-0.90	-0.10	-0.10
C7	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45
C9	0.90	0.10	0.10	-0.90	-0.10	-0.10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15

E4	C1					
C1	0.90	0.10	0.10	-0.90	-0.10	-0.10
C2	0.90	0.10	0.10	-0.90	-0.10	-0.10
C3	0.90	0.10	0.10	-0.90	-0.10	-0.10
C4	0.50	0.40	0.45	-0.50	-0.40	-0.45
C5	0.50	0.40	0.45	-0.50	-0.40	-0.45
C6	0.50	0.40	0.45	-0.50	-0.40	-0.45
C7	0.50	0.40	0.45	-0.50	-0.40	-0.45
C8	0.50	0.40	0.45	-0.50	-0.40	-0.45
C9	0.50	0.40	0.45	-0.50	-0.40	-0.45
C10	0.00	0.00	0.00	0.00	0.00	0.00

E5	C1						C2						C3					
C1	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45
C3	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.00	0.00	0.00	0.00	0.00	0.00
C4	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C5	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C7	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45

E5	C4						C5						C6					
C1	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45
C3	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C4	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45
C5	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.00	0.00	0.00	0.00	0.00	0.00
C7	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45

E5	C7						C8						C9					
C1	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C2	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C3	0.35	0.60	0.70	-0.35	-0.60	-0.70	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C4	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45
C5	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15
C6	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15
C7	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45
C8	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.40	0.45	-0.50	-0.40	-0.45
C9	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.00	0.00	0.00	0.00	0.00	0.00
C10	0.80	0.20	0.15	-0.80	-0.20	-0.15	0.50	0.40	0.45	-0.50	-0.40	-0.45	0.80	0.20	0.15	-0.80	-0.20	-0.15

E5	C1					
C1	0.50	0.40	0.45	-0.50	-0.40	-0.45
C2	0.50	0.40	0.45	-0.50	-0.40	-0.45
C3	0.50	0.40	0.45	-0.50	-0.40	-0.45
C4	0.80	0.20	0.15	-0.80	-0.20	-0.15
C5	0.35	0.60	0.70	-0.35	-0.60	-0.70
C6	0.35	0.60	0.70	-0.35	-0.60	-0.70
C7	0.80	0.20	0.15	-0.80	-0.20	-0.15
C8	0.80	0.20	0.15	-0.80	-0.20	-0.15
C9	0.50	0.40	0.45	-0.50	-0.40	-0.45
C10	0.00	0.00	0.00	0.00	0.00	0.00

AUTHOR'S PROFILE

Siti Nurhidayah binti Yaacob obtained Diploma in Mathematical Science in 2020 and Bachelor of Science in Mathematics (Hons.) in 2022 from Universiti Teknologi MARA (UiTM). Currently she pursues master's degree in mathematics at College of Computing, Informatics and Mathematics, Universiti Teknologi Mara (UiTM) Shah Alam branch, Selangor. Her research area includes the neutrosophic set theory and its application in decision-making problems.

LIST OF PUBLICATION

Yaacob, S. N., Hashim, H., Halim, N. Q. A., Awang, N. A., Abdullah, L., & Al-Quran, A. (2025). Bipolar Neutrosophic Dombi Shapley Weighted Average Aggregation Operator and Its Application in Decision-Making Problem. Journal of Advanced Research in Applied Sciences and Engineering Technology, 46(2), 239-250.

Yaacob, S. N., Hashim, H., Awang, N. A., Sulaiman, N. H., Al-Quran, A., & Abdullah, L. (2024). Bipolar Neutrosophic Dombi-Based Heronian Mean Operators and Their Application in Multi-criteria Decision-Making Problems. International Journal of Computational Intelligence Systems, 17(1), 181.

Yaacob, S. N., Hashim, H., Sulaiman, N. H., Awang, N. A., Al-Quran, A., & Abdullah, L. (2025). An Integrated DEMATEL with Bipolar neutrsophic Dombi-based Heronian Mean Operator and Its Applications in Decision-making Problem. International Journal of Neutrosophic Science, (1), 08-8.