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Energy and Laplacian Energy of The Relative Squared Graph for Symmetric Group of Order Six

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ABSTRACT

The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix and the Laplacian energy is the sum of the absolute differences between the Laplacian eigenvalues and the average degree of the graph. These spectral measures have found extensive use in algebraic graph theory in the study of commutativity and other structural properties of groups, but very little has been done to the higher-order commutativity relations, like those of relative squared graphs. This paper builds the relative squared graph of the symmetric group of order six and calculates its graph energy and Laplacian energy to study the manifestation of higher- order commutativity by spectral properties. It is proposed to use the methodology as follows the relative squared graph is built with the condition $(xy)^2 = (yx)^2$ and the adjacency and Laplacian matrices are constructed to compute the eigenvalues of these matrices as a measure of the energy. The findings reveal that various subgroup structures of the symmetric group give different values of energy and Laplacian energy which represents different degrees of connectivity and irregularity of the graph. The results of these studies confirm that spectral measures like graph energy, Laplacian energy are useful in the analysis of the structural and commutative behaviour of non-abelian groups using graphs.

Keywords: Adjacency Matrices, Eigenvalues, Energy, Laplacian Energy, Laplacian Matrices, Relative Squared Graph.

INTRODUCTION

Graph theory has long been a robust structure, or framework, connecting abstract algebra and combinatorics, and providing key ways to investigate the structure of algebraic systems, including groups, rings, and semigroups. One of the most intriguing ideas related to graph theory is the energy of a graph, which was developed by Gutman et al. (2008) in the study of mathematical chemistry. It can be determined to be the sum of the absolute values of the eigenvalues of a graph's adjacency matrix and, as a consequence offers a spectral perspective of the amount, complexity, and symmetry structures of the graph of interest.

This idea was then generalized to define the Laplacian energy in 2005 by Gutman and Zhou, where they interpreted the deviation of Laplacian eigenvalues from the average degree of the graph as its energy. This refinement encodes information related to both connectivity as well as irregularity of graphs. These energy-based measures were originally introduced in chemistry, but they quickly found applications in other fields, for example, physics, biology, network science, but especially algebraic graph theory (Nikiforov, 2007).

Recent studies have explored the energy and Laplacian energy of various algebraic graphs, such as commuting graphs, non-commuting graphs, conjugacy class graphs, and relative coprime graphs, each constructed by associating group-theoretic relationships with edges in a graph (Abdollahi et al., 2006; Das & Panigrahi, 2016). These approaches have enhanced the understanding of finite groups through spectral graph theory. However, relatively small attention has been given to graphs formed specifically from the squared or cubed elements of subgroups within finite groups. These elements are not arbitrary, they satisfy additional algebraic identities that may influence the structure and spectrum of the resulting graphs. The investigation of such relative squared graphs, where edges are defined based on commutation relations involving squared elements, represents a new path with significant potential for both theoretical insight and spectral analysis.

In this study, it focuses on the symmetric group order of six, known for its rich structure and unique features, such as the existence of a non-trivial outer automorphism. This study aims to construct the relative squared graph order of six, compute its graph energy and Laplacian energy, and interpret these spectral quantities in relation to the group's algebraic properties.

PRELIMINARIES

Some preliminaries involved in this study can be found as is the following.

Definition 1 (Das, K. C. ,2024) The adjacency matrix of a graph G and denoted as $A(G)$, is a symmetric $n \times n$ matrix whose (i, j)-th entry is 1 if v_i and v_j are adjacent, 0 otherwise.

Definition 2 (Das, K. C. ,2024) The degree matrix of G is a diagonal matrix $D(G)$ whose i-th diagonal entry is the degree of the vertex v_i.

Definition 3 (Gutman & Zhou, 2005) Let G be a graph on n vertices. The Laplacian matrix $L(G)$ of G is defined as $n \times n$ matrix.

$$L(G)\begin{cases} -1, if\ v_i\ and\ v_j\ are\ adjacent\ where\ i \neq j, \\ 0, if\ v_i\ and\ v_j\ are\ not\ adjacent\ where\ i \neq j, \\ d_i, if\ i = j, \end{cases}$$

where d_i is the degree of vertex i, $i = 1, 2, 3, \ldots, n$.

Definition 4 (Bapat, 2014) The eigenvalues of a matrix are the roots of the characteristic polynomial that result from solving the characteristic equation. The eigenvalues of a $n \times n$ matrices can be complicated. The spectrum of the matrix is the entire set of eigenvalues.

Definition 5 (Gutman et al., 2008) Graph energy is the sum of the absolute values of the eigenvalues of a graph's adjacency matrix. The concept was originally derived from chemistry, modelling molecular orbital energies. For a graph with eigenvalues,
$\lambda_1, \lambda_2, \ldots, \lambda_n$ the energy is defined as:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. The graph $\Gamma_2(S_3)$ can be defined through its adjacency matrix A, where the vertices are indexed by representatives of S_3 and the edges (x, y) between them exist if and only if $(xy)^2 = (yx)^2$.

Definition 6 Gutman and Zhou (2005) extended graph energy to Laplacian energy, using the Laplacian matrix $L = D - A$, where D is the degree matrix and A the adjacency matrix. The Laplacian energy is given by:

$$LE(G) = \sum_{i=1}^{n} |\lambda_i - d|,$$

where the average degree $\bar{d} = \frac{2m}{n}$, m is the number of edges, n is the number of vertices and λ_i are the eigenvalues of L. This energy highlights structural irregularities and connectivity in graphs.

The Laplacian matrix $L = D - A$ contains two rows of information, which are the degree of each vertex, appears as a diagonal entry in the first column, while the second row contains the adjacency matrix itself. To sum up, it need to understand L to identify irregularities in $\Gamma_2(S_3)$. Laplacian eigenvalues (used for computing energy) measure degrees of unordered pairs that are not equal. The above repeated eigenvalues suggest that connectivity among vertices is symmetric, and off-diagonals in L (-1) reveal edges between transpositions or 3-cycles.

RESULTS AND DISCUSSION

This section starts by defininf a new definition as follow:

Definition 7 Let G be a non-abelian finite group and let $\tau_2(G)$ be the commuting relative squared graph of G, is defined as the simple undirected graph whose vertex set consists of all elements of G. Two distinct vertices $x, y \in G$ are adjacent if and only if their squared products commute, that is,

$$(xy)^2 = (yx)^2.$$

This graph generalizes the ordinary commuting graph by extending the concept of commutativity to a higher order. Whereas the standard commuting graph connects elements satisfying $xy = yx$, the commuting relative squared graph connects elements whose pairwise products become equal when squared. The structure of this captures second-order commutativity within G, offering a refined tool for investigating algebraic symmetry and subgroup interactions, particularly in permutation groups such as **The Commuting Relative Squared Graph of Symmetric Group of Order Six.**

The following represents the relative squared commutativity graph of symmetric group of order six.

Proposition 1 Let τ_2, (H, S_3) be the relative squared graph with a respect to subgroup H and S_3 is the symmetric group of order six.

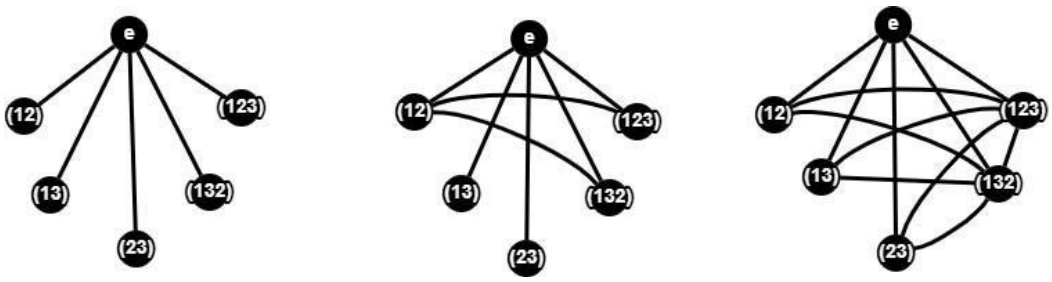

Figure 1 The commuting relative squared graph of $S3$ with respect to subgroups

$H_1 = \{(1)\}$ $\quad$ $H_2 = \{(1), (12)\}$ $\quad$ $H_5 = \{(1), (123), (132)\}$

Proof There are six elements in the symmetric group of order six, $S_3 = \{e, (12), (13), (23), (123), (132)\}$ and contains only one centre which is (1). By Definition 7, the set of commuting relative squared graph of a group are elements of S_3 and two distinct vertices $x, y \in G$ are adjacent if and only if their squared products commute, that is, $(xy)^2 = (yx)^2$.

Note: Because $H_3 = \{(1), (13)\}$ and $H_4 = \{(1), (23)\}$ share a structure with $H_2 = \{(1), (12)\}$ due to identical order, only the graph of H_2 for the subgroup of order two is shown in figure 3.

The Adjacency Matrices of the Commuting Relative Squared Graphs of Symmetric Group S_3

The following lemmas provide the adjacency matrices of the commuting relative squared graph of S_3.

Lemma 1 Let $\tau_2[H_1,S_3]$ be the commuting relative squared graph of the symmetric group S_3 with respect to subgroup H_1.The adjacency matrix is given

$$A(\tau_2, [H_1, S_3]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof Suppose u and v are two distinct vertices of $\tau_2[H_1,S_3]$. Since the identity element (1) in H_1, and the commuting property for the relative squared graph dictates that (1) commutes with every element in S_3, the vertex (1) is adjacent to all other vertices u in the graph. In this specific configuration for H_1, the subgroup structure results in only the identity element having these wide commuting squared relations across the set of order six. Since (1) is the only element in H_1 that satisfies the adjacency condition for all elements of S_3, the first row and first column of the matrix consist of 1s (excluding the diagonal), while the remaining interactions between other elements do not satisfy the commuting squared criteria. In the adjacency matrix, the connected vertices are represented as 1 and 0 otherwise.

Lemma 2 Let $\tau_2[H_2,S_3]$ be the commuting relative squared graph of the symmetric group S_3 with respect to subgroup H_2 . The adjacency matrix is defined as:

$$A(\tau_2,[H_2,S_3]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof Suppose u and v are two distinct vertices of $\tau_2[H_2,S_3]$. For example, in the case of commuting relative squared, the identity element (1) in H_2 is such that (1) commutes with the square of all elements in S_3. So the vertex (1) is neighbours to all the other 5 elements of S_3 its degree is 5. Moreover, vertex order 2 element in H_2 is commutes with the squares of vertices corresponding to order 3 elements, thus adding some more adjacencies. In the adjacency matrix, connected vertices are represented as 1 and 0 otherwise.

Lemma 3 Let $\tau_2,[H_5, S_3]$be the commuting relative squared graph of the symmetric group S_3 with respect to subgroup H_5. The adjacency matrix is stated as follows:

$$A(\tau_2,[H_5,S_3]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Proof Suppose u and v are distinct vertices. In this configuration, multiple elements in H_5 satisfy the commuting squared condition with various elements in S_3. The identity remains adjacent to all elements. Furthermore, elements in H_5 of order 2 or 3 show adjacency with other specific elements of S_3 as denoted by the 1s in the matrix. This increased connectivity results in the complex matrix structure where multiple rows and columns contain multiple 1s. In the adjacency matrix, connected vertices are represented as 1 and 0 otherwise.

The Laplacian Matrices of the Commuting Relative Squared Graphs of Symmetric Group S_3

The adjacency matrices of the commuting relative squared graph of S_3 are given by the following lemmas.

Lemma 4 The Laplacian matrix $\tau_2[H_1, S_3]$ for the commuting relative squared graph of S_3 with respect to H_1 is given by:

$$L(\tau_2, [H_1, S_3]) = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof Suppose u and v are two distinct vertices of $\tau_2[H_1, S_3]$. Since (1) in H_1 and is satisfied in the squared context, then the vertex (1) is adjacent to all elements u. The remaining vertices are adjacent only to the identity vertex and have no edges between them. Thus, each of these non-identity vertices has a degree of 1. In the Laplacian matrix, the a_{ii} diagonal entry represents the degree of vertex i. Furthermore, for $a_{ij} = -1$ if vertex i and j are adjacent, and $a_{ij} = 0$ if they are not adjacent.

Lemma 5 The Laplacian matrix $\tau_2[H_2, S_3]$ for the commuting relative squared graph of S_3 with respect to H_2 is given by:

$$L(\tau_2, [H_2, S_3]) = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & 0 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Proof Let u and v be distinct vertices of $\tau_2[H_2, S_3]$. In H_2, the identity (1) commutes with the square of each element of S_3. So the vertex (1) touches all the other 5 elements of S_3, hence the degree of vertex (1) = 5.

In addition, since vertex 2 (for the order 2 element in H_2) satisfies the relative squares property with respect to the vertices (order 3 elements). In the matrix, it has 5 and 6 as elements of order 3 (and the other vertices 2,3,4 as elements of order 2). As a result, vertex 2 has degree 3, as it is adjacent to the identity vertex together with the two vertices of order 3. Thus vertices 5 and 6, which represent elements of order 3, each have degree 2 since they are both connected to the identity vertex and the particular element of order 2.

In the Laplacian matrix, the a_{ii} diagonal entry represents the degree of vertex i. Furthermore, for $a_{ij} = -1$ if vertex i and j are adjacent, and $a_{ij} = 0$ if they are not adjacent.

Lemma 6 The Laplacian matrix $\tau_2[H_5, S_3]$ for the commuting relative squared graph of S_3 with respect to H_5 is given by:

$$L(\tau_2,[H_5,S_3]) = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & 0 & 0 & -1 & -1 \\ -1 & 0 & 3 & 0 & -1 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}$$

Proof Suppose u and v are two distinct vertices of $\tau_2[H_5,S_3]$. Since (1) in H_5 and is satisfied in the squared context, making the identity vertex adjacent to all other 5 vertices. Consequently, the degree of the identity vertex (1) is 5. In this subgroup structure, other elements of H_5 also satisfy the combined squared condition along with the various vertices in S_3. In fact, vertices order three are neighbours to all other vertices of the graph, thus each has degree 5. The vertices order two, and vertices order three are adjacent to each other, giving vertex degrees of 3 respectively.

The Energy of the Commuting Relative Squared Graphs of Symmetric Group S_3

In this section, the energies of commuting relative squared graph of a symmetric group of order six are evaluated.

Proposition 2 Let G be the symmetric group of order six, S_3 and $\tau_2(H_1,G)$ be the commuting relative squared graph of G with respect to a subgroup H_1. Then, the energy of the commuting relative squared graph of G:

$$E(\tau_2[H_1,G]) = 4.4721$$

Proof Based on Lemma 1, the eigenvalues of the adjacency matrix are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_5 = \sqrt{5}, \lambda_6 = -\sqrt{5}$. Then, by the definition of graph energy $E(\tau_2[H_1,G]) = 4|0| + |\sqrt{5}| + |-\sqrt{5}| = 4.4721$.

Proposition 3 Let G be the symmetric group of order six, S_3 and $\tau_2(H_2,G)$ be the commuting relative squared graph of G with respect to a subgroup H_2. Then, the energy of the commuting relative squared graph of G:

$$E(\tau_2[H_2,G]) = 6.6859$$

Proof Based on Lemma 2, the eigenvalues of the adjacency matrix are $\lambda_1 = \lambda_2 = 0, \lambda_3 = -2, \lambda_4 = 2.8137, \lambda_5 = -1.3429, \lambda_6 = 0.5293$. Then, by the definition of graph energy $E(\tau_2[H_2,G]) = 2|0| + |-2| + |2.8137| + |-1.3429| + |0.5293| = 6.6859$.

Proposition 4 Let G be the symmetric group of order six, S_3 and $\tau_2(H_5,G)$ be the commuting relative squared graph of G with respect to a subgroup H_5. Then, the energy of the commuting relative squared graph of G:

$$E(\tau_2[H_5,G]) = 8.3245$$

Proof Based on Lemma 3, the eigenvalues of the adjacency matrix are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1 +$

$\sqrt{10}, \lambda_4 = 1 - \sqrt{10}, \lambda_5 = \lambda_6 = -1$. Then, by the definition of graph energy $E(\tau_2[H_5, G]) = 2|0| + |1 + \sqrt{10}| + |1 - \sqrt{10}| + 2|-1| = 8.3245$.

The Laplacian Energy of the Commuting Relative Squared Graphs of Symmetric Group S_3

In this section, the Laplacian energies of commuting relative squared graph of a symmetric group of order six are evaluated.

Proposition 5 Let G be the symmetric group of order six, $S3$ and $\tau 2(H1,G)$ be the commuting relative squared graph of G with respect to a subgroup $H1$. Then, the Laplacian energy of the commuting relative squared graph of G:

$$LE(\tau[H_1, G]) = \frac{26}{3}$$

Proof Based on Lemma 4, the eigenvalues of the Laplacian matrix are $\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1$. The average degree for subgroup order one is $\frac{5}{3}$. Then, by the definition of Laplacian energy $LE(\tau_2[H_1, G]) = \left|0 - \frac{5}{3}\right| + \left|6 - \frac{5}{3}\right| + 4\left|1 - \frac{5}{3}\right| = \frac{26}{3}$.

Proposition 6 Let G be the symmetric group of order six, $S3$ and $\Gamma 2(H2, G)$ be the commuting relative squared graph of G with respect to a subgroup $H2$. Then, the Laplacian energy of the commuting relative squared graph of G:

$$LE(\tau_2[H_2, G]) = \frac{32}{3}$$

Proof Based on Lemma 5, the eigenvalues of the Laplacian matrix are $\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 2, \lambda_4 = 4, \lambda_5 = \lambda_6 = 1$. The average degree for subgroup order one is $\frac{5}{3}$. Then, by the definition of Laplacian energy $LE(\tau_2[H_2, G]) = \left|0 - \frac{5}{3}\right| + \left|6 - \frac{5}{3}\right| + \left|2 - \frac{5}{3}\right| + \left|4 - \frac{5}{3}\right| + 2\left|1 - \frac{5}{3}\right| = \frac{32}{3}$.

Proposition 7 Let G be the symmetric group of order six, $S3$ and $\Gamma 2(H1, G)$ be the commuting relative squared graph of G with respect to a subgroup $H1$. Then, the Laplacian energy of the commuting relative squared graph of G:

$$LE(\tau_2[H_5, G]) = 12$$

Proof Based on Lemma 6, the eigenvalues of the Laplacian matrix are are $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 3, \lambda_4 = 6, \lambda_5 = 6, \lambda_6 = 6$. The average degree for subgroup order three is 4. Then, by the definition of Laplacian energy $LE(\tau_2[H_2, G]) = |0 - 4| + 2|3 - 4| + 3|6 - 4| = 12$.

Energy-based analysis for the Commuting Relative Squared Graphs of Symmetric Group S_3

The graph energy $E(G)$ is a numerical measure of the connectivity and structural pattern in the graphs obtained from commuting relative squared operator. In the group of three subgroups, H_5 has the highest vertex distribution complexity with graph energy value to 8.3246. This high value means that the commuter squared relations in H_5 are among the largest, leading to a highly connected and dense graph. H_2, on the other hand is a problem of moderate size with energy 6.6859. Lastly, H_1 has the lowest complexity with a graph energy of 4.4721. Its lower energy is due to less connected structure and just a simple graph. This is the energy level associated with its star graph, where the identity element serves as a center and all elements are linked to it, but no extra links associate non- identity elements.

Laplacian Energy-based analysis for the Commuting Relative Squared Graphs of Symmetric Group S_3

The Laplacian energy $LE(G)$ is a parameter which quantifies the strength of graph, but not only diffusion process on it depended upon degrees of vertices. Subgroup H_5 has the maximum Laplacian energy of 12 among all the subgroups that illustrate its best strength. This is because the nodes are highly connected to one another on average ($\bar{d} = 4$) and there is central preferred vertex with a high degree which is 5. On the other hand, H_2 is a graph with a considerable Laplacian energy of about 10.6667. The smallest Laplacian energy is 8.6667 from H_1. Being a star graph itself, it keeps Laplacian energy and makes the identity vector connected with all vertices (presenting maximum degree 5 only in a node).

Conclusion

The results indicate subgroup H_5 has the highest graph energy and Laplacian energy, suggesting a greater level of commutativity in its commuting relative squared graph when compared to both H_1 and H_2. On the other end, H_1 to yield the weakest energies of them all.

CONCLUSION AND RECOMMENDATIONS

In conclucion, spectral analysis of the relative squared graph for the symmetric group of order six, S_3 , indicates that this research successfully achieved its main goals. Extending the notion of classical commuting graphs to a second order relation $(xy)^2 = (yx)^2$, the research led to a unique spectral description of this non-abelian group. The graphs $\tau_2(H, S_3)$ corresponding to the subgroups H_1, H_2 and H_5 show that the internal algebraic structure of the graphs (in particular the presence of transpositions and 3-cycles) plays a big role in the connectivity and therefore matrices of the graph.

Its spectral analysis highlighted the amount of the group's activity and structure irregularities. The graph energy was computed to be 4.4721, and the Laplacian energy was 8.6667 for the identity subgroup H_1 and it is lowest from the other subgroup. On the other hand, taking the subgroup H_2 of order two gave a moderate graph energy of 6.6859 and a Laplacian energy of 10.6667. The highest work gotten by the subgroup H_5 of order three, yielded to be the most complicated structure with the most elevated graph energy of 8.3245 and a Laplacian energy of 12. These results suggest that high order commutativity relations in conjunction with particular subgroup structures are a better closer examination of the symmetries and connectivity behaviours contained within S_3, which are not directly obvious from basic algebraic algorithms.

Along the lines of this study, few research directions are suggested that are worth considering in conjunction with algebraic graph theory and spectral analysis. The first recommendation is for future researchers to apply this framework to larger symmetric groups S_4 and S_5 or to other families of finite groups such as Dihedral or Quaternion groups. S_3 scale would help determine if the spectral signatures observed in S_3 are indeed scalable to larger groups or if new structural phenomena arise with increasing group order.

Second, future work can investigate higher-power relations for example $(xy)^3 = (yx)^3$ and relations arising from commutation to see how further developing the local structure of sequent transformations affects the global algebraic and graphical properties of the group. Also, we would expect to be able to use other graph invariants for example, the Sombor energy or the Estrada index to enhance the characterization of such algebraic structures. Lastly, a comparison of relative squared graphs to other types of graphs, such as non-commuting (or conjugacy class) graphs, may provide more nuanced perspectives on groups that are relatively close to abelian with respect to different spectral measures.

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