A COMPARISON OF SPECTRAL CONJUGATE GRADIENT COEFFICIENTS FOR UNCONSTRAINED OPTIMIZATION

Nur Atiqah Mohd Borhan[1]**, Nur Izzah Izzati Ibrahim**[2] **and Nur Hamizah Abdul Ghani**[3*]

Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor

[1]2019207396@student.uitm.edu.my, [2]2019615292@student.uitm.edu.my,

[3*]hamizahghani@uitm.edu.my

(indicates the corresponding author)*

ABSTRACT

Conjugate gradient (CG) methods are among the most effective techniques for solving nonlinear optimization problems. Recently, a lot of research for modification and improvement of CG methods has been made. There are a few types of CG methods such as classical, spectral, hybrid, and three-term methods. The spectral CG method is shown to outperform other CG methods in past studies, where it was developed to improve the CG method's efficiency in terms of CPU time and iteration numbers. Consequently, the efficiency of three spectral CG methods—sSMARZ, SRMIL, and SCD from previous studies in solving test problems is observed in this study. The objectives of this study are to show that the selected spectral CG methods satisfy sufficient descent conditions and to determine the efficiency of the selected spectral CG methods in exact line search in solving certain standard optimization functions. The methodology of this research is conducted in four phases: literature phase, selection of three spectral CG methods, theoretical phase, and numerical phase. Conclusively, according to numerical results, the sSMARZ approach is superior and more effective than methods SRMIL and SCD.

Keywords: *Exact Line Search, Numerical Comparison, Spectral Conjugate Gradient Method, Sufficient Descent Condition, Unconstrained Optimization*

Received for review: 26-08-2022; Accepted: 09-09-2022; Published: 01-11-2022

1. Introduction

Optimization is the act of getting the best results of a situation or resource. Optimization is capable either to minimize the effort required or to maximize the desired benefits. Therefore, the final decision made gives the best results for the performance. Because of this, optimization can be defined as a process of finding conditions that provide maximum or minimal value of functionality. Optimization are divided into two types, constrained and unconstrained optimizations.

Unconstrained optimization problems are used in a variety of fields, including petroleum exploration, aerospace, transportation, and others. The amount of calculation required, on the other hand, grows exponentially with the size of the problem (Liu & Jiang, 2012). The method to solve unconstrained optimization is given by an iterative method of the form

$$x_{k+1} = x_k + \alpha_k d_k \tag{1}$$

where α_k is step size and d_k is search direction. The step size is obtained by carrying out a one-dimensional search, known as the line searches. Line searches are classified into two types: exact and inexact line searches.

The most common is the exact line search, which is,

$$f(x_k+\alpha_k d_k)=\min_{\alpha\geq 0} f(x_k+\alpha d_k) \quad (2)$$

In this study, we presume that the next development of fast computer processors will provide an advantage in employing this line search, even if many studies believe that the exact line search is relatively slow and choose to utilize the inexact line search. The advantages of the exact line search include eliminating complexity and getting the precise step size value (Liu & Jiang, 2012).

There are a few methods available to solve unconstrained optimization which are Steepest Descent, Newton, Conjugate Gradient (CG) and Quasi-Newton methods. These unconstrained optimization methods require search direction, and a different search direction choice will result in a different method. The table below shows the different of search direction for each methods.

Table 1. Methods of Unconstrained Optimization

Methods	Search Direction
Steepest Descent	$d_k=-g_k$
Newton	$d_k=-H_k^{-1}g_k$
Conjugate Gradient	$d_k=\begin{cases}-g_k, & k=0\\ -g_k+\beta_k d_{k-1}, & k\geq 1\end{cases}$
Quasi-Newton	$d_k=-H_k g_k$

According to Nocedal and Wright (2006), the CG method outperforms other methods in unconstrained optimization. Our interest in CG methods arises as they are among the most effective techniques to solve nonlinear optimization problems. This method performance is an issue for nonlinear problems and do not achieve the fast convergence rates of Newton or Quasi-Newton methods. However, CG method have the advantage of requiring only gradient evaluations and using less storage as it do not require matrices to be stored. In addition, the CG method is faster than the steepest descent method. In comparison with steepest descent direction, nonlinear CG directions are much more effective and are almost as simple to compute.

The CG method is divided into few categories which are classical, hybrid, spectral and three term CG methods. Jian et al. (2020) stated that numerous classical congujate gradient methods have been developed, each with a different choice for β_k. The most well-known CG methods are the Dai-Yuan method (DY,1999), the Polak-Ribière-Polyak (PRP,1969) method, Fletcher-Reeves (FR,1964) method, and the Hestenes-Stiefel (HS,1952) method.

The table from Shi and Guo (2008) below shows these classical CG methods along with their β_k.

Table 2. The Classical Conjugate Gradient Methods

Author(s)	Methods

Hestenes-Stiefel (HS, 1952)	$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T(g_k - g_{k-1})}$
Fletcher-Reeves (FR, 1964)	$\beta_k^{FR} = \frac{g_k^T g_k}{g_{k-1}^T(g_{k-1})}$
Polak-Ribière-Polyak (PRP, 1969)	$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{g_{k-1}^T(g_{k-1})}$
Dai-Yuan method (DY, 1999)	$\beta_k^{DY} = \frac{g_k^T g_k}{d_{k-1}^T(g_k - g_{k-1})}$

In terms of convergence properties and numerical performance, each of these four methods has it owns benefits and drawbacks. The FR and DY methods are notable for their good convergence properties. Their numerical performance, on the other hand, is not so good. In contrast, the PRP and HS methods perform really well in practical computation. However, their convergence properties are difficult to obtain. An article by Jian et al. (2020) provided the context for the comparison of the advantages and disadvantages of the described classical CG methods. Both the FR and PRP methods have $g_{k-1}^T(g_{k-1})$ in the numerators of their parameter β_k, indicating that they have restart properties that can prevent jamming and improve performance while solving unconstrained optimization problems. Consequently, the deficiencies of these classical CGMs is resolved by modified spectral CG methods from previous studies.

2. Spectral Conjugate Gradient Method

The spectral CG method is highlighted in this study that differs from the CG and spectrum gradient methods (Liu et al., 2019). In addition, the spectral CG method is a significant generalisation of the CG method and one of the most powerful numerical methods for large-scale unconstrained optimization (Jian et al., 2020). The search direction used in spectral CG method is:

$$d_k = \begin{cases} -g_k, & k = 1 \\ -\theta_k g_k + \beta_k d_{k-1}, & k \geq 2 \end{cases} \tag{3}$$

where g_k is the gradient of $f(x)$ at the point x_k, θ_k is spectral parameter and β_k is CG coefficient (Liu & Jiang, 2012). Several studies have been conducted in order to propose a new spectral CG method that meets certain performance requirements.

Firstly, SMMAR CG method has been proposed using MMAR CG coefficient under SWP line search (Dawahdeh et al., 2020). This method satisfies the sufficient descent and global convergence condition. The MMAR CG coefficient used in the study is

$$\beta_k^{MMAR} = \begin{cases} \dfrac{\|g_k\|^2 - \dfrac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}|}{\|g_k\| + \|g_{k-1}\|^2}, & \|g_k\|^2 \geq \dfrac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}| \\ 0 & \text{, and otherwise} \end{cases} \tag{4}$$

The numerical result of iteration number indicates that the SMMAR method surpasses the MMAR CG method in most of the test problems. However, CPU time numerical result shows MMAR method surpasses the SMMAR CG method.

While Al-Bayati and Hassan (2011) proposed new spectral CG method called MMCD method has been proved to satisfy sufficient descent condition and global convergence. This method is compared with CD and MCD. The result shows MMCD method performs better than CD and MCD method.

The spectral CG method differs from other methods because it includes a spectral parameter, θ_k to compute the search direction as in (3). Therefore, Birgin and Martinez (2001) had proposed a spectral parameter as follows:

$$\theta_k = 1 - \frac{g_k^T d_{k-1}}{g_{k-1}^T d_{k-1}} \tag{5}$$

On the basis of the CD method and the spectral CG method, Liu and Jiang (2012) proposed a new nonlinear spectral conjugate descent method that satisfies the sufficient descent condition and is globally convergent under the strong Wolfe line search. The new spectral CG method, referred to as the SCD method in this paper, yields the following value of β_k:

$$\beta_k = \begin{cases} \beta_k^{CD}, & \text{if } \ g_k^T d_{k-1} \leq 0 \\ 0, & \text{else} \end{cases} \tag{6}$$

where β_k^{CD} is define as following,

$$\beta_k^{CD} = -\frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}} \tag{7}$$

The CPU time and iteration number of the proposed SCD method with the CD, FR, and PRP methods is compared on the given test problems from the CUTE test problem library using the strong Wolfe line search. The numerical results show that the SCD method performs slightly worse than the well-known PRP method in terms of performance. However, in terms of CPU time and iteration number, the SCD method outperforms the well-known CD and FR methods. As a result, the potency of the SCD method is encouraging. Under the strong Wolfe line search, the CD method will provide a descent direction, which is an important property.

In May 2020, Mamat et al. proposed modified spectral CG which is known as sSMARZ spectral method. This modified spectral method is based on CG parameter Sulaiman et al. (2015). The spectral parameter that is used in this research (5) where this method satisfies sufficient descent condition under strong Wolfe line search

$$\beta_k^{SMARZ} = \frac{g_k^T \left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} d_{k-1} \right)}{d_{k-1}^T (d_{k-1} - g_k)} \tag{8}$$

Advantages of SMARZ method is when compared to the FR, PRP, and RAMI methods is in terms of the number of iterations and CPU time, the SMARZ approach provides the best performance. Although the FR and PRP procedures are considered among the best CG approaches, they only solved 76% and 86% of the test issues, respectively. In contrast, RAMI, a recently developed approach, solves 91% of the test functions. As a result, the SMARZ approach is thought to be superior and more effective than the other ways (Sulaiman et al., 2015).

New spectral CG method, SRMIL has been proposed by Zull et al. (2015) using β_k^{RMIL} which initially proposed by Rivaie et al. (2012) and selected spectral parameter (5) under SWP line search.

$$\beta_k^{RMIL} = \left(\frac{g_k^T (g_k - g_{k-1})}{\|d_{k-1}\|^2} \right) \tag{9}$$

β_k^{RMIL} used in this research is a simple CG coefficient that is easy to implement and when compared to other common CG approaches, numerical data reveal that this method has the best performance (Rivaie et al., 2012). SRMIL method is the best method compared to others standard CG methods for certain problems proved by numerical analysis in MatlabR2012 software.

In this study, we compared three spectral CG method that are sSMARZ, SCD and SRMIL. θ_k used in this study by Birgin and Martinez (2001) is stated below:

$$\theta_k = \left(1 - \frac{g_k^T d_{k-1}}{g_{k-1}^T d_{k-1}} \right)$$

β_k used in this study is

1. β_k^{SMARZ} proposed by Sulaiman et al. (2015) given below:

$$\beta_k^{SMARZ} = \frac{g_k^T \left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} d_{k-1} \right)}{d_{k-1}^T (d_{k-1} - g_k)}$$

2. β_k^{CD} proposed by Fletcher (1987) given below:

$$\beta_k^{CD} = -\frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}$$

3. β_k^{RMIL} proposed by Rivaie et al. (2012) given below:

$$\beta_k^{RMIL} = \left(\frac{g_k^T (g_k - g_{k-1})}{\|d_{k-1}\|^2} \right)$$

The algorithm of spectral conjugate gradient method is given as follow:

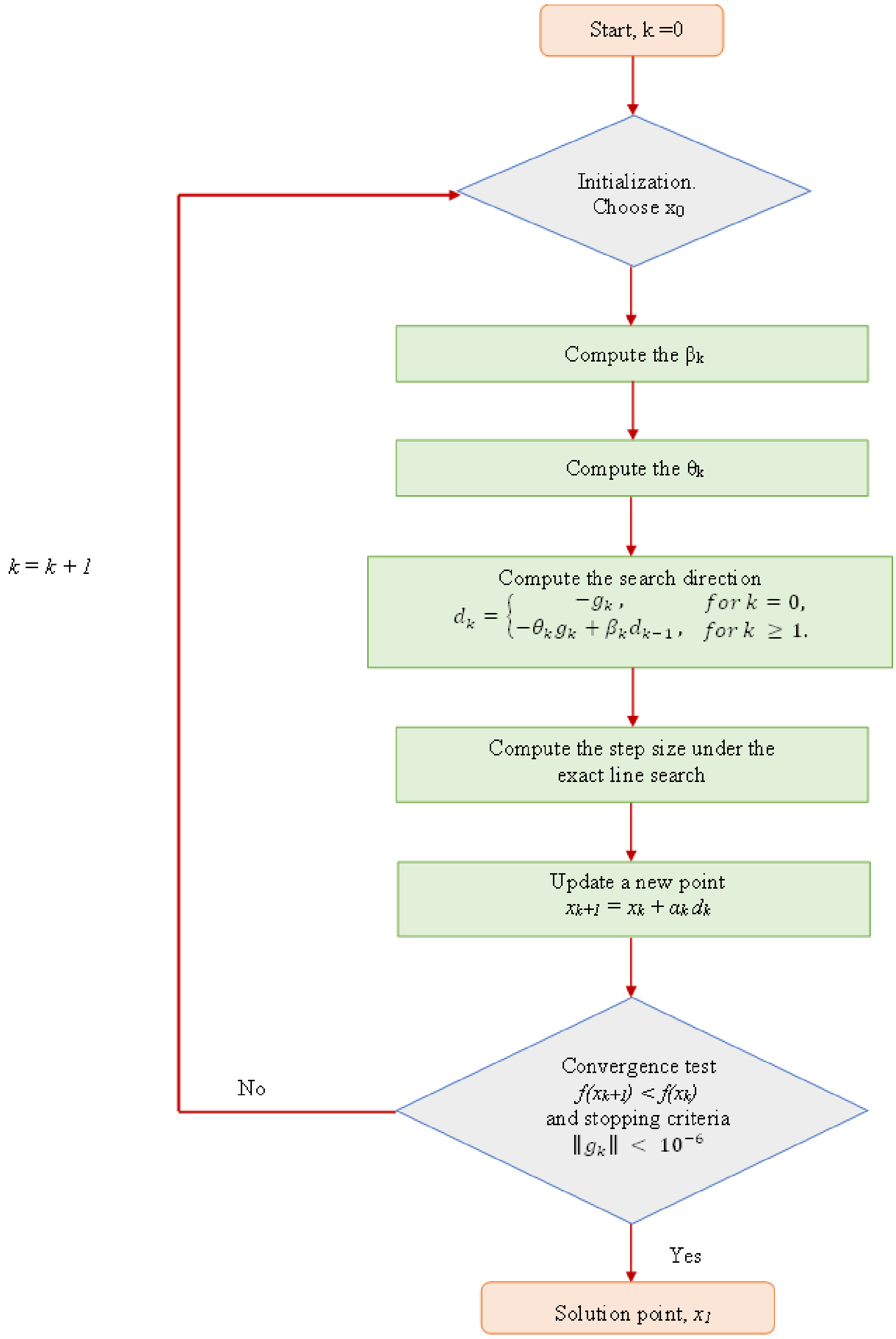

Figure 1. Spectral CG Algorithm

3. Convergence Analysis

Various new spectral CG methods that have been found by the researchers is shown to satisfy the sufficient descent condition. Hence, in the theoretical phase of this study, the spectral CG method must satisfy the sufficient descent condition.

Definition for sufficient descent condition is given below:

$$g_k^T d_k \leq -\ c\|g_k\|^2 \tag{10}$$

for all $k \geq 0$ and c is positive constant.

A sufficient descent condition, angle condition, convergence rate, and global convergence properties must all be achieved by an algorithm of proposed CG method (Zull et al., 2015) and the method's global convergence is ensured by satisfying the sufficient descent property (Faramarzi & Amini, 2018; Hamoda et al., 2016).

The following theorem shows that all three spectral CG methods with exact line search satisfy the sufficient descent condition.

Theorem 1:

Consider a spectral CG method with the search direction (3) and given as (7), (8) and (9), then condition (10) holds for all k ≥ 0.

Proof:

When k = 1,

$$g_1^T d_1 \leq -c\|g_1\|^2 \tag{11}$$

From (3), $d_1 = -g_1$ (12)

By substituting (14) in (13),

$$g_1^T(-g_1) \leq -c\|g_1\|^2$$

$$-\|g_1\|^2 \leq -c\|g_1\|^2$$

where c is any constant.

$$-\|g_1\|^2 = -c\|g_1\|^2$$

∴ Sufficient descent condition holds.

When k =2,

From (3),

$$d_k = -\theta_k\, g_k + \beta_k\, d_{k\text{-}1}$$

where θ_k as in (5)

$$d_k = -\left(1 - \frac{g_k^T\, d_{k\text{-}1}}{g_{k\text{-}1}^T\, d_{k\text{-}1}}\right) g_k + \beta_k d_{k\text{-}1}$$

Set k = k+1,

$$d_{k+1} = -\left(1 - \frac{g_{k+1}^T d_k}{g_k^T d_k}\right) g_{k+1} + \beta_{k+1} d_k$$

Multiply by g_{k+1}^T for both sides,

$$g_{k+1}^T d_{k+1} = \left(-\left(1 - \frac{g_{k+1}^T d_k}{g_k^T d_k}\right) g_{k+1} + \beta_{k+1} d_k \right) g_{k+1}^T$$

$$g_{k+1}^T d_{k+1} = -\left(1 - \frac{g_{k+1}^T d_k}{g_k^T d_k}\right) \|g_{k+1}\|^2 + \beta_{k+1} g_{k+1}^T d_k$$

By exact line search: $g_{k+1}^T d_k = 0$

$$g_{k+1}^T d_{k+1} = -\left(1 - \frac{0}{g_k^T d_k}\right) \|g_{k+1}\|^2 + \beta_{k+1} g_{k+1}^T (0)$$

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2$$

Set $k = k$-1,

$$g_k^T d_k = -\|g_k\|^2$$

∴ Sufficient descent condition holds.

Hence, we can conclude this spectral CG method satisfies sufficient descent condition by using any CG coefficients.

4. Numerical Result and Discussion

This section discussed numerical result of sSMARZ, SCD, and SRMIL method. All 20 unconstrained optimization test problems are solved using MATLAB software. The list of test problem (Andrei, 2008; Mishra, 2006; Molga & Smutnicki, 2005) and its formula and initial points are shown in Table 3 and Table 4 respectively. The numerical performance is tested by 20 distinct test problems with four different initial points and different variables. All codes were executed using an Intel® Core™ i3-6006U processor running at 2 GHz, 12 GB of RAM, and Windows 10 Home as the operating system. The stopping criteria are set as $\|g_k\| \leq 10^{-6}$(Liu & Jiang, 2012; Rivaie et al., 2012; Zull et al., 2015). In some cases, the result of CPU time and iteration number is considered a failure if the test problem does not satisfy the convergence test and stopping criteria. The iteration number greater than 10,000 (Salihu et al., 2020) is also considered a failure. Numerical results of all three spectral CG are compared based on the iteration numbers and CPU time.

Table 3. List of test problems formula

No	**Problems**	**Formula**
1	Six Hump	$f(x_1, x_2) = \left(4 - 2.1x_1^2 + \frac{x_1^4}{3}\right)x_1^2 + x_1x_2 + (-4 + 4x_2^2)x_2^2$
2	Three Hump	$f(x_1, x_2) = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{6} + x_1x_2 + x_2^3$
3	Extended White & Holst	$f(x) = \sum_{i=1}^{n/2} 100(x_{2i} - x_{2i-1}^3)^2 + (1 - x_{2i-1})^2$
4	Extended Rosenbrock	$f(x) = \sum_{i=1}^{n/2} 100(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2$
5	Extended Freudenstein & Roth	$f(x) = \sum_{i=1}^{n/2} (-13 + x_{2i-1} + ((5 - x_{2i})x_{2i} - 2)x_{2i})^2 + (-29 + x_{2i-1} + ((x_{2i} + 1)x_{2i} - 14)x_{2i})^2$
6	Extended Beale	$f(x) = \sum_{i=1}^{n/2} \left(1.5 - x_{2i-1}(1 - x_{2i})\right)^2 + \left(2.25 - x_{2i-1}(1 - x_{2i}^2)\right)^2 + \left(2.625 - x_{2i-1}(1 - x_{2i}^3)\right)^2$
7	Generalized Quartic	$f(x) = \sum_{i=1}^{n-1} x_i^2 + (x_{i+1} + x_i^2)^2$

8	Raydan 1	$f(x)=\sum_{i=1}^{n}\frac{i}{10}(\exp(x_i)-x_i)$
9	Extended Tridiagonal 1	$f(x)=\sum_{i=1}^{n/2}(x_{2i-1}+x_{2i}-3)^2+(x_{2i-1}-x_{2i}+1)^4$
10	Diagonal 4	$f(x)=\sum_{i=1}^{n/2}\frac{1}{2}(x_{2i-1}^2+100x_{2i}^2)$
11	Extended Himmelblau	$f(x)=\sum_{i=1}^{n/2}(x_{2i-1}^2+x_{2i}-11)^2+(x_{2i-1}+x_{2i}^2-7)^2$
12	Generalized Tridiagonal 1	$f(x)=\sum_{i=1}^{n-1}(x_i+x_{i+1}-3)^2+(x_i-x_{i+1}+1)^4$
13	FLETCHCR	$f(x)=\sum_{i=1}^{n-1}100(x_{i+1}-x_i+1-x_i^2)$
14	Generalized Tridiagonal 2	$f(x)=\left((5-3x_1-x_1^2)x_1-3x_2+1\right)^2+\sum_{i=1}^{n-1}\left((5-3x_i-x_i^2)x_i-x_{i-1}-3x_{i+1}+1\right)^2+\left((5-3x_n-x_n^2)x_n-x_{n-1}+1\right)^2$
15	Extended Maratos	$f(x)=\sum_{i=1}^{n/2}x_{2i-1}+100(x_{2i-1}^2+x_{2i}^2-1)^2$
16	Extended DENSCHNB	$f(x)=\sum_{i=1}^{n/2}\left((2(x_{2i-1}+x_{2i})^2+(x_{2i-1}-x_{2i})^2-8)\right)^2+(5x_{2i-1}^2+(x_{2i}-3)^2-9)^2$
17	Hager	$f(x)=\sum_{i=1}^{n}(\exp(x_i)-\sqrt{i}x_i)$
18	Quadratic QF1	$f(x)=\frac{1}{2}\sum_{i=1}^{n}ix_i^2-x_n$
19	Quadratic QF2	$f(x)=\frac{1}{2}\sum_{i=1}^{n}i(x_i^2-1)^2-x_n$
20	Extended Penalty	$f(x)=\sum_{i=1}^{n-1}(x_i-1)^2+\left(\sum_{j=1}^{n}x_j^2-0.25\right)^2$

Table 4. List of test problems with number of variable and initial point

No	Problems	Number of Variable, N	Initial Point
1	Six Hump	2	8, -8, 10, -10
2	Three Hump	2	13, 17, 21, 23
3	Extended White & Holst	2, 4, 100, 500, 1000	-3, 6, 10, 12
4	Extended Rosenbrock	2, 4, 100, 500, 1000	2, 3, 5, 8
5	Extended Freudenstein & Roth	2, 4, 100, 500, 1000	3, 13, 25, 36
6	Extended Beale	2, 4, 100, 500, 1000	2, 3, 5, 100
7	Generalized Quartic	2, 4, 100, 500, 1000	10, 50, 100, 200
8	Raydan 1	2, 4, 100, 500, 1000	1, 5, 10, 20
9	Extended Tridiagonal 1	2, 4, 100, 500, 1000	12, 17, 20, 30
10	Diagonal 4	2, 4, 100, 500, 1000	10, 15, 20, 25
11	Extended Himmelblau	2, 4, 100, 500, 1000	2, 6, 10, 12
12	Generalized Tridiagonal 1	2, 4, 100, 500, 1000	3, 21, 45, 90
13	FLETCHCR	2, 4, 100, 500, 1000	3, 8, 14, 18
14	Generalized Tridiagonal 2	2, 4, 100, 500, 1000	15, 30, 100, 150
15	Extended Maratos	2, 4, 100, 500, 1000	10, 50, 100, 200
16	Extended DENSCHNB	2, 4, 100, 500, 1000	2, 4, 8, 11
17	Hager	2, 4, 100, 500, 1000	2, 4, 6, 10
18	Quadratic QF1	2, 4, 100, 500, 1000	3, 5, 10, 13
19	Quadratic QF2	2, 4, 100, 500, 1000	4, 20, 40, 80
20	Extended Penalty	2, 4, 100, 500, 1000	100, 105, 135, 200

Numerical results in terms of iteration number and CPU time will be collected and analyzed using performance profiles initiated by Dolan and Moré (2002). They introduced the concept of a means to evaluate and compare the performance of the set solvers S on a test set P in this performance profile. Assuming there are n_s solvers and n_p problem, the definitions for each problem p and solver s are given below:

$t_{p,s}$ = computing time needed for solver s to solve problem p

In this case, computation time is specified by the number of iterations or CPU time. The performance ratio is required in the performance profile as a benchmark for evaluating problem p by solver s, which is shown below:

$$t_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}} \tag{13}$$

Parameter $r_m \geq r_{p,s}$ for all p,s is selected, and $r_{p,s} = r_m$ if a solver s fails to solve a problem p. The performance ratio is used to determine performance solver s for any given problem which given below:

$$p_s(t) = \frac{1}{n_p}\text{size}\{p \in P : r_{p,s} \leq t\} \tag{14}$$

where $p_s(t)$ is the probability for solver $s \in S$ that a performance ratio $r_{p,s}$ within a factor $t \in R$ of the best possible ratio while p_s is the cumulative distribution function for the performance ratio. The solver performance profile is $p_s : R \rightarrow [0,1]$. Hence, $p_s(1)$ is the highest probability that the solver will select as the best solver among the other solvers.

Generally, the highest value of $p(\tau)$ indicates the best solver. In the performance profile graph, the top right of the graph indicates the best solver. The fastest method performance on certain unconstrained optimization test problem can be identify on the left side of the graph while the percentage of spectral CG method successfully solve the test problems can be identify on the right side of the graph.

The performance results are shown in Figures 2 and 3, respectively, using a performance profile introduced by Dolan and Moré (2002). The graph of performance profile is observed on both left and right side. The left side of the graph interpret the rate of method performance on solving test problems. The higher the line graph, the faster the method performs. While the right side of the graph represent the percentage of test problem solved by the spectral CG methods. The greater the percentage of test problems solved, the lower the failure rate of the method in solving the test problems.

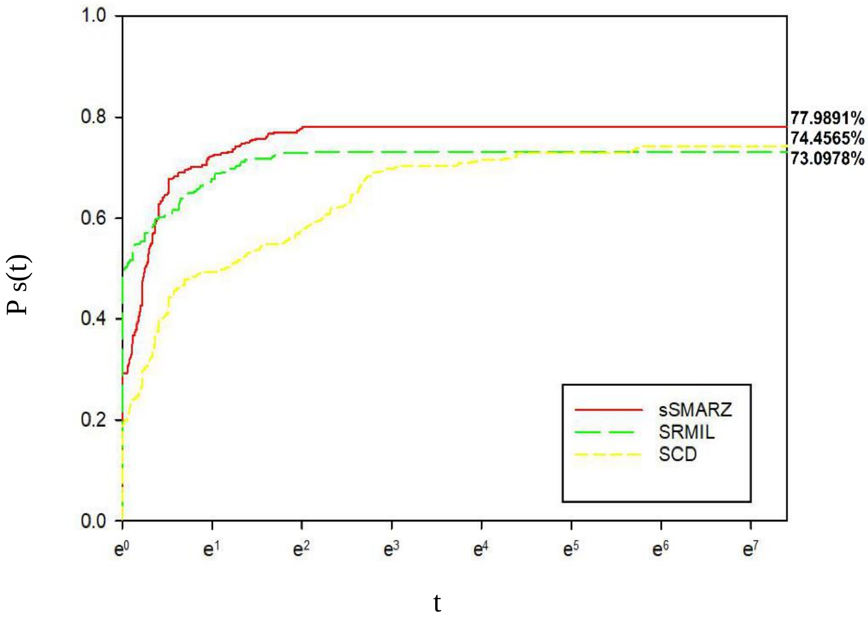

Figure 2. Performance Profile based on Iteration Number

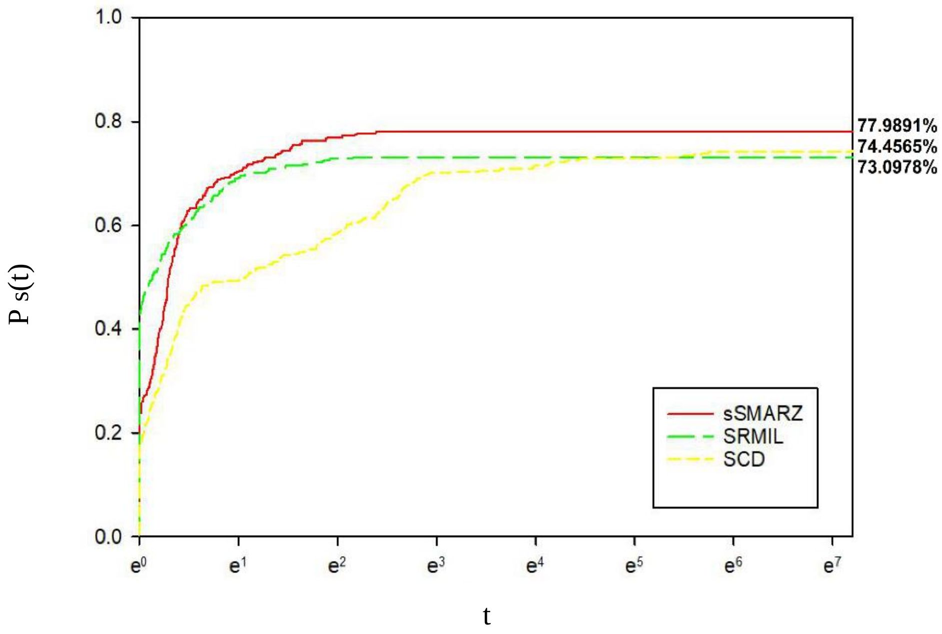

Figure 3. Performance Profile based on CPU time

Table 5 below indicate sSMARZ has the best performance on both iteration number and CPU time because it can solve more test problems and achieve approximately 78%.

Table 5. Percentage of test problem solved using the following methods

Method	Percentage of method successfully solved the test problem
sSMARZ	77.9891%
SCD	74.4565%
SRMIL	73.0978%

In contrast to SRMIL and SCD, the other two spectral CG methods, which can only resolve approximately 73% and 74% of the test problems, respectively. In terms of the performance, SRMIL has better performance at the beginning compared to the other two methods. SCD, in comparison, is the slowest method available. SRMIL performs better early in the graph, while later on, sSMARZ outperforms other methods. Since sSMARZ has the highest performance and the greatest number of test problems solved compared to the other methods, we can say that sSMARZ is the best and most effective method. The initial points that were chosen at random from the various past studies that have been gathered and the new points that have been randomly added will help to support our method's convergence proof. As a result, our test result using the first four initial points is both valid and significant.

5. Conclusion and Recommendations

At the end of this study, all objectives are achieved. CG methods have undergone multiple past studies, leading to new variations on these methods. As a result, the performance of each of these new spectral CG methods varies when used to solve test problems. Additionally, the CG coefficients of the spectral CG method chosen are unique and may be more or less complicated than the others. The first objective is achieved as we proved that these three spectral CG methods satisfy the sufficient descent condition under the exact line search. At the same time, another objective is achieved as our numerical findings indicate that, when compared to the SRMIL and SCD methods, the sSMARZ spectral CG method performs the best. However, there are limitations in this study where this study only focused on three coefficients of spectral CG method on 20 test problems using exact line search. The recommendation suggested for next study are to determine the efficiency on more CG method coefficients that not included in this study on both exact and inexact line search.

Acknowledgement

The author would like to thank to Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA for supporting this project.

References

Al-Bayati, A. Y., & Hassan, B. A. (2011). A spectral conjugate gradient method with inexact line searches. Iraqi Journal of Statistical Sciences, 11(20),155–163. https://doi.org/10.33899/iqjoss.2011.27895

Andrei, N. (2008). An unconstrained optimization test functions collection. Adv. Model. Optim, 10(1), 147-161.

Birgin, E. G., & Martinez, J. M. (2001). A spectral conjugate gradient method for unconstrained optimization. Applied Mathematics and Optimization, 43(2), 117–128. https://doi.org/10.1007/s00245-001-0003-0

Dawahdeh, M. (2020). A new spectral conjugate gradient method with strong Wolfe-Powell Line Search. International Journal of Emerging Trends in Engineering Research, 8(2), 391–397. https://doi.org/10.30534/ijeter/2020/25822020

Dolan, E. D., & Moré, J. J. (2002). Benchmarking optimization software with performance profiles. Mathematical Programming, 91(2), 201–213. https://doi.org/10.1007/s101070100263

Faramarzi, P., & Amini, K. (2019). A modified spectral conjugate gradient method with global convergence. Journal of Optimization Theory and Applications, 182(2), 667–690. https://doi.org/10.1007/s10957-019-01527-6

Fletcher, R. (1987). Practical methods of optimization (Vol. 1). Wiley.

Hamoda, M., Mamat, M., Rivaie, M., & Salleh, Z. (2016). A conjugate gradient method with strong Wolfe-Powell Line Search for unconstrained optimization. Applied Mathematical Sciences, 10, 721–734. https://doi.org/10.12988/ams.2016.56449

Ibrahim, S. M., Yakubu, U. A., & Mamat, M. (2020). Application of spectral conjugate gradient methods for solving unconstrained optimization problems. An International Journal of Optimization and Control: Theories & Applications (IJOCTA), 10(2), 198-205

Jian, J., Yang, L., Jiang, X., Liu, P., & Liu, M. (2020). A spectral conjugate gradient method with descent property. Mathematics, 8(2), 280. https://doi.org/10.3390/math8020280

Liu, J. K., Feng, Y. M., & Zou, L. M. (2019). A spectral conjugate gradient method for solving large-scale unconstrained optimization. Computers & Mathematics with Applications, 77(3), 731–739. https://doi.org/10.1016/j.camwa.2018.10.002

Liu, J., & Jiang, Y. (2012). Global convergence of a spectral conjugate gradient method for unconstrained optimization. Abstract and Applied Analysis, 2012, 1–12. https://doi.org/10.1155/2012/758287

Malik, M., Mamat, M., Abas, S. S., & Sulaiman, I. M. (2021). Performance Analysis of New Spectral and Hybrid Conjugate Gradient Methods for Solving Unconstrained Optimization Problems. IAENG International Journal of Computer Science, 48(1).

Mamat, M., Sulaiman, M. I., Maulana, M., Sukono, & Zakaria, Z. A. (2020). An Efficient Spectral Conjugate Gradient Parameter with Descent Condition for Unconstrained Optimization. Journal of Advanced Research in Dynamical and Control Systems, 12(02), 2487–2493. https://doi.org/10.5373/JARDCS/V12I2/S20201296

Mishra, S. K. (2006). Some new test functions for global optimization and performance of Repulsive Particle Swarm method. SSRN Electronic Journal. https://doi.org/10.2139/ssrn.926132

Molga, M., & Smutnicki, C. (2005). Test functions for optimization needs. new.zsd.iiar.pwr.wroc.pl/files/docs/functions.pdf

Nocedal, J., & Wright, S. J. (2006). Numerical optimization. https://link.springer.com/book/10.1007/978-0-387-40065-5

Rivaie, M., Mamat, M., June, L. W., & Mohd, I. (2012). A new class of nonlinear conjugate gradient coefficients with global convergence properties. Applied Mathematics and Computation, 218(22), 11323–11332. https://doi.org/10.1016/j.amc.2012.05.030

Salihu, N., Halilu, A. S., Waziri, M. Y., & Odekunle, M. R. (2020). A New Hybrid Conjugate Gradient Method Based on Secant Equation for Solving Large Scale Unconstrained Optimization Problems. Iranian Journal of Optimization, 12(1), 33–44.

Shi, Z.-J., & Guo, J. (2008). A new algorithm of nonlinear conjugate gradient method with strong convergence. Computational & Applied Mathematics, 27(1). https://doi.org/10.1590/s0101-82052008000100006

Sulaiman, M. I., Mamat, M., Abashar, A., Rivaie, M., & Salleh, Z. (2015). A modified nonlinear conjugate gradient method for unconstrained optimization. Applied Mathematical Sciences, 9, 2671–2682. https://doi.org/10.12988/ams.2015.5141

Zull, N., Rivaie, M., Mamat, M., Salleh, Z., & Amani, Z. (2015). Global convergence of a new spectral conjugate gradient by using strong Wolfe Line Search. Applied Mathematical Sciences, 9(63), 3105–3117. https://doi.org/10.12988/ams.2015.53240