INTEGRAL ITERATIVE METHOD FOR SOLVING KORTEWEG-DE VRIES EQUATIONS

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ABSTRACT

In this paper, the Integral Iterative Method (IIM) is used to solve the Korteweg-de Vries equation. The Korteweg-de Vries equation (KdV) is a third-order nonlinear partial differential equation. The KdV equation, which describes nonlinear shallow water waves, is widely accepted in physics and engineering. Declaring the solution to the KdV problem is more difficult than with linear differential equations since it involves a nonlinear differential equation with numerous unknowns and significant nonlinearity. A notable drawback of the Adomian decomposition strategy, which is another method for solving nonlinear equations, is that it calls for the usage of higher-order differential derivatives. The main objective of this report is to analyse the accuracy and efficiency of the Korteweg-de Vries equation solution with its exact solutions. Four Korteweg-de Vries equation cases have been chosen. The equation is then implemented with IIM. The problem was analysed and solved using the Integral Iterative Method on the Korteweg-de Vries equation. The error is calculated based on the results. The results indicate that IIM is accurate, convenient, and efficient when it comes to solving nonlinear problems. As a recommendation, other researchers can use the example for non-homogeneous KdV equations with initial conditions.

Keywords*: Integral Iterative Method, Korteweg-de Vries Equation, Non-linear partial differential equation.*

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1. Introduction

Non-linear phenomena have become a beneficial tool in physics, solitary waves and applied mathematics in the last decades. The awareness of the importance of this type of equation has increased rapidly. For instance, non-linear partial differential equations have become a significant tool in modelling physical phenomena, especially in nonlinear natural sciences. In comparison to other linear equations, nonlinear partial differential equations are more complicated to solve (Yassein & Aswhad, 2019). The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation of third order. The Korteweg-de Vries equation is given by

$$u_t + au_x + u_{xxx} = f(x,t), 0 \leq x \leq 1, t > 0 \tag{1}$$

with the initial condition

$$u(x,0) = g(x) \tag{2}$$

The KdV equation is accepted broadly especially in physics and engineering for its description of nonlinear shallow-water waves. The KdV equation also explains the progression of water waves under the effect of weak nonlinearity and weak dispersion. It is highly remarkable for being an example of an exactly solvable model, that is a nonlinear partial differential equation, whose solutions are exact and precise (Xiang, 2015).

In order to solve the KdV equation, a new iterative method would be implemented, which is the Integral Iterative Method (IIM). The Integral Iterative Method (IIM) is efficient in solving a large class of differential effectively and accurately. The IIM is a simple iterative technique which depends on the integral operator explicitly. It is also convenient to solve the differentials because it reduces and simplifies the computational procedures and complexity and time (Hemeda, 2017).

2. Methodology

2.1 Introduction

This section gives an outline of the details and explanation of the methodology that is followed in the study. It includes information about the Korteweg-de Vries equation and the Integral Iterative Method and how they are used in this study to achieve the purpose of this study.

2.2 Flowchart

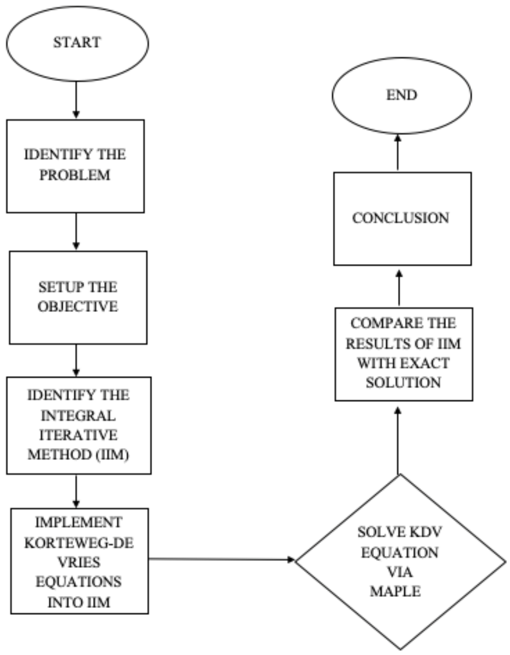

Figure 1: Flowchart of project management

Step 1:

Identify the problem of the study. The main problem of this study is to use the Integral Iterative Method (IIM) to solve the Korteweg-de Vries equation.

Step 2:

Set up the objective. Objectives are required because there is a problem to be solved.

Step 3:

Identify the method which is the Integral Iterative Method (IIM). Use a lot of articles as a reference to solve and explain a problem using the Integral Iterative Method (IIM) and find a general equation in the Integral Iterative Method (IIM).

Step 4:

The Korteweg-de Vries equation will be implemented in the Integral Iterative Method (IIM)

Step 5:

Analyse the problem and solve the problem using an application of the Integral Iterative Method (IIM) on the Korteweg-de Vries equation. There are four homogenous Korteweg-de Vries equations that will be solved using the Integral Iterative Method via Maple.

Step 6:

The result of the Integral Iterative Methods will be compared to the exact solution.

2.3 Korteweg-de Vries Equation

We consider the following general of the Korteweg-de Vries equation,

$$u_t + au_x + u_{xxx} = f(x,t), 0 \leq x \leq 1, t > 0 \tag{3}$$

with the initial condition

$$u(x,0) = g(x) \tag{4}$$

$u(x,t)$ indicates the elongation of a wave at place x and time t while the a is a constant with no great significance. Based on an article written by Yassein, and Aswhad (2019), the Korteweg-de Vries equation is solved by using the Semi-Analytical Iterative Method (SAIM). SAIM has been widely used by researchers to solve linear and non-linear differential equations. Multiple Korteweg-de Vries equations with different initial value problems are solved with the method. By the end of the research, it is found that the results are effective and efficient as well as compatible with the exact solution.

2.4 Integral Iterative Method (IIM)

The following general differential equation of arbitrary order n > 0 is considered as:

$$D_t^n u(t) = L(u) + K(u) + g(t), \qquad n \in \mathbb{N} \tag{5}$$

with the initial conditions

$$\frac{d^k}{dt^k} u(0) = h_k, \quad k = 0,1,2,\dots,n-1 \tag{6}$$

Where D_t^n is the classical different operator of order n meanwhile L and K are linear and nonlinear operators of orders less than or equal to n. $g(t)$ is a homogeneous term. To both sides of Equation (9), the classical integral operator was applied, denoted by I_t^n.

$$u(t) = \sum_{k=0}^{n-1} h_k \cdot \frac{t^k}{k!} + I_t^n[g(t)] + I_t^n[L(u) + K(u)] = f + N(u), \tag{7}$$

where

$$f = \sum_{k=0}^{n-1} h_k \cdot \frac{t^k}{k!} + I_t^n[g(t)] \; and \; N(u) = I_t^n[L(u) + K(u)]$$

The recurrence relation is used to obtain the required solution $u(t)$ for Eq. (4).

$$\begin{cases} u_0 = f, \\ u_{r+1} = u_0 + N(u_r), \quad r = 0,1,2,\dots \end{cases} \tag{8}$$

This change eliminates the requirement to calculate the integral of the given function $g(t)$ for each iteration $r, r = 1, 2, 3, \dots$

Therefore, the solution for equation (9) will be obtained by

$$u(x) = \lim_{n\to\infty} u_n(x) = \sum_{n=0}^{\infty} u_n(x) \tag{9}$$

Thus, the solution will result in a series of the following form (Al-Jawary et al.,2018):

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{10}$$

According to Odibat (2010), if the series solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$ convergent, then the series will represent the exact solution for the nonlinear problem equation (9). Moreover, if the truncated series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ is used as an approximation solution, then the maximum error $E_n(x)$ will be estimated as:

$$E_n(x) = \frac{1}{1-\eta} \eta^{n+1} \|u_0\| \tag{11}$$

The Integral Iterative Method works directly on problems with initial conditions such as Picard Method and works on problems with boundary conditions after converting these conditions to initial conditions. For solving differential and IDEs, IIM is more convenient.

2.5 Application of Integral Iterative Method (IIM) on Korteweg-de Vries Equation

We start with the Korteweg-de Vries equation (KdV) and the Initial Value Problem (IVPs):

$$u_t + au_x + u_{xxx} = 0 \tag{12}$$

$$u(x,0) = g(x) \tag{13}$$

To solve Eq. and IVPs, the general form equation as:

$$L\big(u(x,t)\big) + N\big(u(x,t)\big) + f(x,t) = 0,$$
$$with\ conditions\ C\left(u, \frac{\partial u}{\partial t}\right) = 0 \tag{14}$$

We will implement the method steps, which can be expressed as:

$$L\big(u(x,t)\big) = -(f(x,t) + au_x + u_{xxx}) \tag{15}$$

By integrating both sides, we obtain:

$$u(x,t) = \psi_0 - \int_0^t \big(f(x,t)\big)\, dt \tag{16}$$

To get $u_0(x,t)$ we solve and integrate both sides of Equation $L\big(u_0(x,t)\big) + f(x,t) = 0$ with IVPs, we obtain:

The first step:

$$u_0(x,t) = \int_0^t \big(f(x,t)\big)\, dt \tag{17}$$

The second step:

$$u_1(x,t) = u_0 - (\int_0^t (au_{0x} + u_{0xxx})\, dt) \tag{18}$$

The third step:

$$u_{n+1}(x,t) = u_0 - \left(\int_0^t (au_{nx} + u_{nxxx})\,dt\right) \tag{19}$$

Thus, this method is effective in solving and implementing the Korteweg-de Vries equation.

2.6 Validation of Integral Iterative Method (IIM)

This section illustrates how to calculate the error. The results of the Integral Iterative Method (IIM) calculation will be compared with its own exact value.

The formula of the error is as follows:
$Absolute\ Error = |V_A - V_E|$
$V_A = approximate\ (measured)\ value$
$V_E = exact\ value$

3. Results and Discussion

3.1 Example 1

We considered the homogenous Korteweg-de Vries equation from a research paper by Inc and Cherruault (2005) given by

$$u_t + u_x + u_{xxx} = 0 \tag{20}$$

with the initial condition

$$u(x,0) = g(x) = \sinh x \tag{21}$$

According to Inc and Cherruault (2005), the exact solution is given by

$$\begin{aligned} u_n(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= \sinh(x-2t) \end{aligned} \tag{22}$$

To solve equation (20), using equations (17)-(19) yields

$$u_0(x,t) = sinh\,x$$

$$u_1(x,t) = sinh\,x - 2t\,cosh\,x$$

$$u_2(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x$$

$$u_3(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4t^3\,cosh\,x}{3}$$

$$u_4(x,t) = \sinh x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3 cosh\,x + \frac{2}{3}t^4\,sinh\,x$$

$$u_5(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3cosh\;x + \frac{2}{3}t^4\,sinh\,x - \frac{4}{15}t^5\,cosh\,x$$

$$u_6(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3cosh\;x + \frac{2}{3}t^4\,sinh\,x - \frac{4}{15}t^5\,cosh\,x + \frac{4}{45}t^6\,sinh\,x$$

$$u_7(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3cosh\;x + \frac{2}{3}t^4\,sinh\,x - \frac{4}{15}t^5\,cosh\,x + \frac{4}{45}t^6\,sinh\,x - \frac{8}{315}t^7\,cosh\,x$$

$$u_8(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3cosh\;x + \frac{2}{3}t^4\,sinh\,x - \frac{4}{15}t^5\,cosh\,x + \frac{4}{45}t^6\,sinh\,x - \frac{8}{315}t^7\,cosh\,x + \frac{2}{315}t^8\,sinh\,x$$

$$u_9(x,t) = sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3cosh\;x + \frac{2}{3}t^4\,sinh\,x - \frac{4}{15}t^5\,cosh\,x + \frac{4}{45}t^6\,sinh\,x - \frac{8}{315}t^7\,cosh\,x + \frac{2}{315}t^8\,sinh\,x - \frac{4}{2835}t^9\,cosh\,x$$

$$\begin{aligned} u_{10}(x,t) = {} & sinh\,x - 2t\,cosh\,x + 2t^2\,sinh\,x - \frac{4}{3}t^3cosh\;x + \frac{2}{3}t^4\,sinh\,x \\ & - \frac{4}{15}t^5\,cosh\,x + \frac{4}{45}t^6\,sinh\,x - \frac{8}{315}t^7\,cosh\,x \\ & + \frac{2}{315}t^8\,sinh\,x - \frac{4}{2835}t^9\,cosh\,x + \frac{4}{14175}t^{10}\,sinh\,x \end{aligned} \tag{23}$$

and so on for other components. Using equation (16) leads immediately to the solution of equation (20) given in (Figure 2 and Figure 3). As a result, the IIM series solution meets Cherruault's (2005) exact solution assumption.

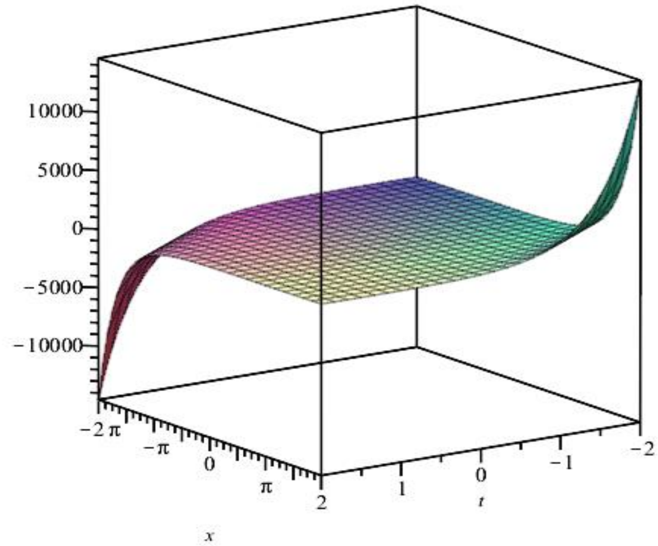

Figure 2: The surface shows the exact solution of equation (20)

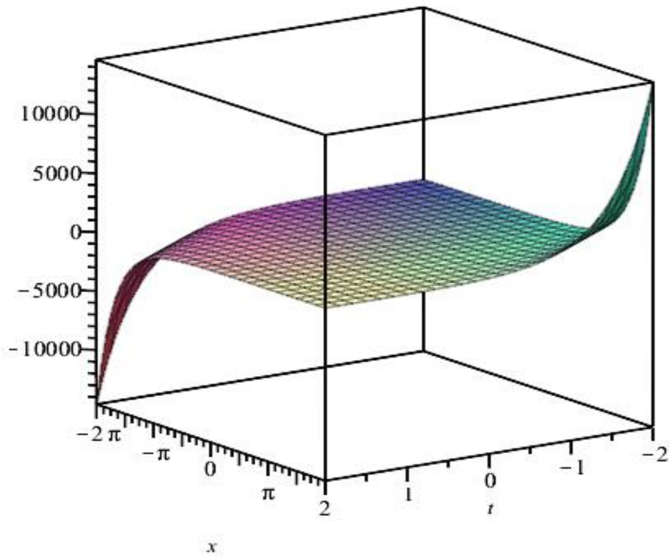

Figure 3: The surface shows the approximate solution, $u_{10}(x, t)$ of equation (20)

By substituting $x = 0$ and $t = m$ where $m = 0.1, 0.2, 0.3, ...$,we obtain absolute error as shown below.

Table 1: Absolute error for case $u_{10}(x,t)$ vs exact solution

t	Exact value, $u(x,t)$	Approximate value, $u_{10}(x,t)$	Absolute error
0.1	-0.201336003	-0.201336003	0
0.2	-0.410752326	-0.410752326	0
0.3	-0.636653582	-0.636653582	0
0.4	-0.888105982	-0.888105980	2.1×10^{-9}
0.5	-1.175201194	-1.175201169	2.5×10^{-8}
0.6	-1.509461355	-1.509461168	1.87×10^{-7}
0.7	-1.904301501	-1.904300474	1.027×10^{-6}
0.8	-2.375567953	-2.375563472	4.481×10^{-6}
0.9	-2.942174288	-2.942157848	0.000016440
1.0	-3.626860408	-3.626807760	0.000052648

The absolute errors acquired using the procedure indicated above are shown in Table 1. It is worth noting that the approximate solution was evaluated using only ten terms and from that, we were able to approximate the precise solution value to within 99.9%. As shown above, both graphs looked the same but in fact, there was a very minimal difference between them which is called error. The iteration technique was extremely effective in finding precise solutions to a wide range of problems. As a result, the numerical data showed that even when employing fewer terms to evaluate approximation solutions, a high degree of accuracy yields.

The numerical answers in Table 1 clearly showed that the error will increase as t increases. This is due to the nature of the power series which is very sensitive to the parameters inherent in the series. Accuracy will decline when the solution point moves away from the starting point. But, the larger the u term, the smaller the error produced and the more accurate the solution of the Korteweg-de Vries equation is. This shows how the Integral Iterative Method gives efficient results that are substantially closer to the true values, as expected. When the range of utility of the power series is absolutely convergent and the result of the error value is closest to zero, the iteration methodology has the advantage of being more accurate, convenient, and dependable in addressing nonlinear problems.

3.2 Example 2

Refer to the example from a research paper by Yildirim (2008), we considered the homogenous Korteweg-de Vries equation given by

$$u_t - 6uu_x + u_{xxx} = 0 \tag{24}$$

with initial condition of:

$$u(x,0) = g(x) = \frac{x}{6} \tag{25}$$

The exact solution $u_n(x,t) = \frac{x}{6(1-t)}$ was found by Yildirim (2008).

To solve equation (24), using equation (17)-(19) the solution reads

$$u_0(x,t) = \frac{x}{6}$$

$$u_1(x,t) = \frac{x}{6} + \frac{1}{6}xt$$

$$u_2(x,t) = \frac{x}{6} + \frac{1}{6}xt + \frac{1}{6}xt^2 + small\ terms$$

$$u_3(x,t) = \frac{x}{6} + \frac{1}{6}xt + \frac{1}{6}xt^2 + \frac{1}{6}xt^3 + small\ terms$$

$$u_4(x,t) = \frac{x}{6} + \frac{1}{6}xt + \frac{1}{6}xt^2 + \frac{1}{6}xt^3 + \frac{1}{6}xt^4 + small\ terms \tag{26}$$

and so on for other components.

This in turn yields the components

$$u_1(x,t) = \frac{x}{6} + \frac{1}{6}xt$$

$$u_2(x,t) = \frac{x}{6} + \frac{1}{6}xt + \frac{1}{6}xt^2 + small\ terms, \tag{27}$$

where $\frac{1}{6}xt^3$ is considered as a small term appearing between the components $u_1\ and\ u_2$. According to Rangkuti and Noraini (2012), we can disregard the small term and use the remaining terms of u1 and u2 to validate the convergent series in numerical terms. We then cancelled these small terms between the components and justify the remaining terms of u_2, i.e. $u_2(x,t) = \frac{x}{6} + \frac{1}{6}xt + \frac{1}{6}xt^2$ to get a convergent series. It is repeated until *n* tends to infinity. As a result, the series solution using IIM meets with the assumption of the exact solution of the problem found by Yildirim (2008).

By substituting $x = 0$ and $t = m$ where $m = 0.1, 0.2, 0.3, ...$, we get as below.

Table 2: Absolute error for case $u_4(x,t)$ vs exact solution

t	Exact value, $u(x,t)$	Approximate value, $u_4(x,t)$	Absolute error
0.1	0.1851851852	0.1851848972	2.880×10^{-7}
0.2	0.2083333333	0.2083211164	0.0000122169
0.3	0.2380952381	0.2379691010	0.0001261372
0.4	0.2777777778	0.2770320839	0.0007456939
0.5	0.3333333333	0.3300127221	0.0033206112
0.6	0.4166666667	0.4039634666	0.0127032001
0.7	0.5555555556	0.5100149801	0.0455405754
0.8	0.8333333333	0.6658233150	0.1675100184
0.9	1.666666667	0.8994616578	0.7672050092
1.0	Float(infinity)	1.255556256	Float(infinity)

Table 2 illustrates the absolute errors obtained using the procedure indicated above. It is worth noting that the approximate solution was evaluated using only four terms, and we were able to approximate the precise solution value to within 99.9 percent of the exact solution value using that.

The numerical results in Table 2 clearly demonstrate that the error increases as t increases. This is due to the nature of the power series, which is extremely sensitive to the series' inherent parameters. When the solution point moves away from the starting point, the accuracy decreases.

3.3 Example 3

We considered the homogenous Korteweg-de Vries equation from a research paper by Yassein and Aswhad (2019), given by

$$u_t - 6uu_x + u_{xxx} = 0 \tag{28}$$

with initial condition

(29)

$$u(x,0) = g(x) = \frac{1}{6}(x-1)$$

Yassein and Aswhad (2019) obtained the exact solution $u_n(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right)$

Proceeding as before, using equations (17)-(19) we find

$$u_0(x,t) = \frac{x}{6} - \frac{1}{6} \tag{30}$$

$$u_1(x,t) = \frac{x}{6} - \frac{1}{6} + \left(\frac{x}{6} - \frac{1}{6}\right)t$$

$$u_2(x,t) = \frac{x}{6} - \frac{1}{6} + \left(\frac{x}{6} - \frac{1}{6}\right)t + \left(\frac{x}{6} - \frac{1}{6}\right)t^2 + small\ terms$$

$$u_3(x,t) = \frac{x}{6} - \frac{1}{6} + \left(\frac{x}{6} - \frac{1}{6}\right)t + \left(\frac{x}{6} - \frac{1}{6}\right)t^2 + \left(\frac{x}{6} - \frac{1}{6}\right)t^3 + small\ terms$$

$$u_4(x,t) = \frac{x}{6} - \frac{1}{6} + \left(\frac{x}{6} - \frac{1}{6}\right)t + \left(\frac{x}{6} - \frac{1}{6}\right)t^2 + \left(\frac{x}{6} - \frac{1}{6}\right)t^3 + \left(\frac{x}{6} - \frac{1}{6}\right)t^4 + small\ terms$$

and so on for other components.

When we look at the components $u_2, u_3\ and\ u_4$ in equations (30), we can see that they contain recurring terms in the series, which we labelled as small terms. By truncating the small terms, the exact solution of the equation is obtained. As a result, the IIM series solution meets the assumption of exact solution of the problem discovered by Yassein and Aswhad (2019).

Table 3: Absolute error for case $u_4(x,t)$ vs exact solution

t	Exact value, $u(x,t)$	Approximate value, $u_4(x,t)$	Absolute error
0.1	-0.1851851852	-0.1851848972	2.880×10^{-7}
0.2	-0.2083333333	-0.2083211164	0.0000122169
0.3	-0.2380952381	-0.2379691010	0.0001261372
0.4	-0.2777777778	-0.2770320839	0.0007456939
0.5	-0.3333333333	-0.3300127221	0.0033206112
0.6	-0.4166666667	-0.4039634666	0.0127032001
0.7	-0.5555555556	-0.5100149801	0.0455405754

0.8	-0.8333333333	-0.6658233150	0.1675100184
0.9	-1.666666667	-0.8994616578	0.7672050092
1.0	Float(undefined)	-1.255556256	Float(undefined)

Table 3 shows the absolute errors obtained using the above-mentioned procedure. It is worth noting that the approximate solution was evaluated using only four terms, and we were able to use that to approximate the precise solution value to within 99.9% of the exact solution value.

Table 3's numerical results clearly show that the error increases as t increases. This is because the power series is extremely sensitive to the series' inherent parameters. The accuracy decreases as the solution point moves away from the starting point.

Finally, it is clear that using the integral iterative methodology and paying attention to the small terms for the homogeneous situation results in a very quick convergence. This approach was applied to a variety of scenarios and demonstrated superiority to existing techniques, particularly in terms of determining the exact solution and reducing the number of computations and work.

3.4 Example 4

Based on a research paper by Yassein and Aswhad (2019), we considered the homogenous Korteweg-de Vries equation given by

$$u_t - 6uu_x + u_{xxx} = 0 \tag{31}$$

where

$$\alpha = -6$$

with initial condition

$$u(x,0) = g(x) = \frac{2}{(x-3)^2} \tag{32}$$

After implementing the Integral Iterative Method, the solution that we get

$$u_0(x,t) = \frac{2}{(x-3)^2}$$

$$u_1(x,t) = \frac{2}{(x-3)^2}$$

$$u_2(x,t) = \frac{2}{(x-3)^2}$$

$$u_3(x,t) = \frac{2}{(x-3)^2}$$

(33)

$$u_4(x,t) = \frac{2}{(x-3)^2}$$

$$u_5(x,t) = \frac{2}{(x-3)^2}$$

and so on for other components.

After we observe the u_0 until u_5, we can find that the exact solution was

$$u_n(x,t) = \lim_{n\to\infty} u_n(x,t)$$

$$= \frac{2}{(x-3)^2} \quad (34)$$

The example was solved easier by using the Integral Iterative Method and the results are a good indicator of the exact solution.

4. Conclusion and Recommendations

This study is about a nonlinear differential equation known as the Korteweg-de Vries (KdV) equation and an integral iterative method. Four examples of the KdV equations were taken from four different articles to be solved using the integral iterative method. The Integral Iterative Method (IIM) was implemented to solve homogeneous examples of KdV equations with initial conditions. Therefore, the first objective was achieved. Then, the results obtained were compared to the exact solutions given by each of the examples. The comparison of the results of all methods shows that IIM provides the most accurate results. This was because the results were slightly closer to the true values. When the power series' utility range is absolutely convergent, and the resulting error value is close to zero. It is possible to conclude that the effectiveness and accuracy of IIM were proven when applied to KdV equations from all examples. Hence, the second objective was achieved.

It is recommended for researchers to use the example for non-homogeneous KdV equations with initial conditions. Researchers can use IIM because it gives more advantages than other methods. When compared to the classical methods, the Adomian Decomposition Method, Modified Adomian Decomposition Method, and Variation Iteration Method, which require longer times, this method is characterised by significant analytical work that significantly lowers the amount of computational work. The results of IIM have also been proven to be more accurate.

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