SOLUTION OF BLASIUS EQUATION USING INTEGRAL ITERATIVE METHOD (IIM)

Nurul Hafiza Majin[1], **Nor Eliza Mokhatar**[2], **Nur Athirah Wan Ismail**[3], **Mat Salim Selamat**[4*]

Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (UiTM) Cawangan Negeri Sembilan Kampus Seremban, Persiaran Seremban Tiga/1, 70300 Seremban, Negeri Sembilan.

[1]2019422586@student.uitm.edu.my, [2]2019406014@student.uitm.edu.my, [3]2019207346@student.uitm.edu.my, [4*]matsalimselamat@uitm.edu.my

(indicates the corresponding author)*

ABSTRACT

The Integral Iterative Method (IIM) was applied in this research to solve the two different forms of the Blasius equation. The Blasius equation is one of the most important equations in fluid dynamics. Blasius equation describes the velocity profile of the fluid in the boundary layer on a half infinite interval or flat plate. There were many methods have been approached to solve the problem related to the Blasius equation such as the Homotopy Perturbation Method (HPM), the Variational Iterative Method (VIM) and the Adomian Decomposition Method (ADM). However, the calculations of these methods take up too much computer memory since the data being processed is so large. Hence, this paper research presents the outcomes of the research findings with three main objectives, where we used the Integral Iterative Method (IIM) to solve the Blasius equation, examine the reliability and relevance of the IIM in solving the two forms of Blasius equation, followed by determining the precision and efficiency of the IIM compared to other existing methods used before such as Variational Iterative Method (VIM), Differential Transform Method (DTM), and Semi-Analytical Iterative Method (SAIM). Apart from that, Padé Approximant and simple Series Approximant methods are used after the IIM had been implemented into both forms of the Blasius equations. Regarding to the study's findings, it has been identified that the IIM is a simpler method in its computational procedures. The techniques used in this study introduced an easier and straightforward mathematical method to solve various differential equations.

Keywords: *Blasius Equation, Integral Iterative Method, Padé Approximant, Series Approximant.*

Received for review: 25-08-2022; Accepted: 09-09-2022; Published: 01-11-2022

1. Introduction

In numerical analysis, a numerical method is a mathematical tool that is designed to solve any numerical problems. The implementation of a numerical method with an appropriate convergence check in a programming language is known as a numerical algorithm. In previous recent years, there were a growing numbers of applying the iterative method or numerical answer strategies to solve the non-linear dynamics problems in engineering field like the Homotopy Adomian Method (HAM), the Homotopy Perturbation Method (HPM), the Variational Iterative method (VIM) and the Adomian Decomposition Method (ADM) (Liao, 2004) which have gotten a whole lot of interest in the nonlinear model fixing a number of models explored results in time-energy collection solutions, that have not in reality helped to apprehend nonlinear problems (Fernández, 2009).

Chiriţă et al. (2012) gave the example of the maximum considerable equations in fluid dynamics is the Blasius equation. On a half of an endless interval or flat plate, the Blasius equation defines the velocity profile of the fluid in the boundary layer. The Blasius equation is the prime precise solution of the Navier-Stoke problem equation, in which the Navier-Stoke equations has changed the partial differential equations into an everyday differential equation unfastened convection near a vertical waterproof surface buried in an absorbent medium and it is one of the methods regulated via the Blasius equation. This difficulty is associated with warmth approaches, which have a huge range of programs in business and geophysical fields.

Paul Richard Heinrich Blasius is the one who proposed a primary mathematical formulation for boundary-layer and introduced the flow of resistance via smooth pipes that could be represented in terms of the Reynolds number for turbulent go with the flows and laminar in early 1991. Blasius equation has been widely used in physics and fluid mechanics fields because a Blasius boundary layer is a simple function of fluid mechanics and defined because the fixed 2-D laminar boundary layer which forms on a semi-limitless plate that preserves parallel to a chronic strip line goes with the flow.

This equation is basically used to determine the laminar and turbulent flow at some objects. The examples for the laminar flow at the leading edge of solids moving relative to fluids or extremely close to solid surfaces such as the inside wall of a pipe, or in case of fluids of high viscosity which flowing slowly through small channels. Meanwhile, the common examples of turbulent flow are the flow through pumps and turbines, the flow in boat wakes, the flow around aircraft wing tips and the oil transport in pipelines.

Due to the vitality of Blasius equation in engineering and science, numerous attempts had been made to solve it numerically and also analytically at the boundary conditions of the intervals. Ertürk and Momani (2008) for example, these authors used an adjusted variation of the Differential Transform Method (DTM) to produce a convergent power series as a solution. Wang (2004) has used ADM to inspect the Blasius equation, which had been amended by another researcher, Hashim (2006). The author has used a perturbation method to decode the Blasius equation in his study.

In order to simulate the approximation solution of the Blasius equation related problem, we will use the Integral Iterative Method (IIM) in our research. Therefore, our main reasons doing this study were to solve the Blasius equation by using the IIM, examine the reliability of the IIM in solving the two forms of the Blasius equations and determine the precision and efficiency of the IIM compared to other methods used by other researchers before, such as VIM (Wazwaz, 2007), DTM (Eturk & Momani, 2008) and SAIM (Selamat et al., 2019a).

2. Methodology/Implementation

This section discusses the processes for getting the solution of Blasius equation using IIM. A clear vision on how the method works is presented with the implementation of the method. The methodology processes are shown in the flowchart of research below.

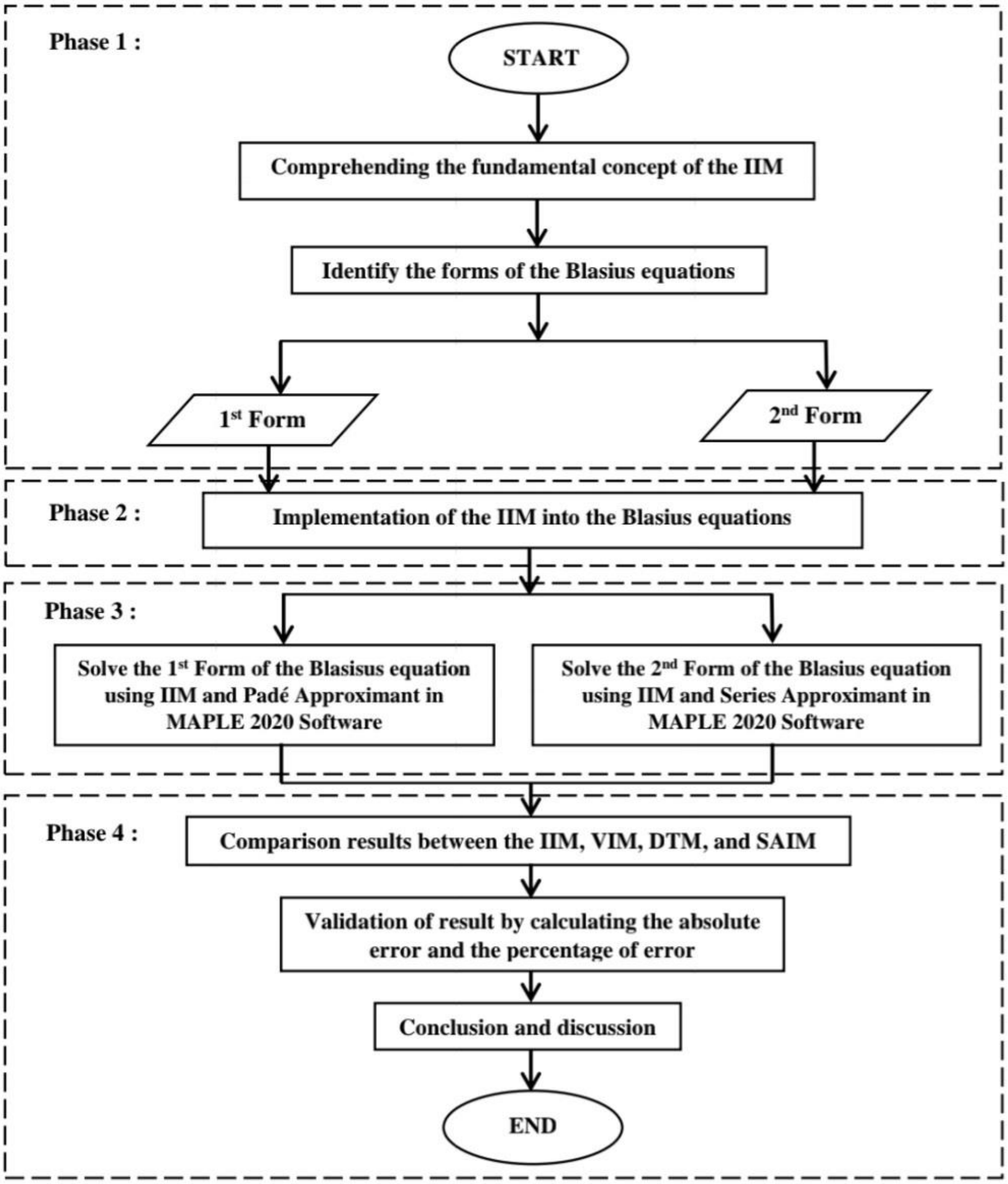

Figure 1: Flowchart of Research

Based on the figure 1 above, the methodology of this study in mainly divided into four main phases. Starting with the Phase 1, we had comprehended the fundamental concept of the IIM and had identified the two forms of the Blasius equations. In the next phase, we have done the implementation of the IIM into those two equations. Moving on to Phase 3, we solved the two forms of the Blasius equation using IIM in MAPLE 2020 software and used Padé Approximants in the first form of the Blasius equation while the Series Approximants has been used in the second form of the Blasius equation to find the values of unknown parameters. Lastly, we would compare the result obtained with VIM, DTM, and SAIM followed by validating the result via the calculations of errors, also concluded and discussed our research findings in Phase 4. Below are the detailed descriptions about each step done in this study by following the flowchart of research.

2.1 Implementation and Numerical Examples

STEP 1: Comprehending the fundamental concept of the IIM

According to Hemeda (2015), IIM is a new and simple iterative approach that relies on the integral operator which is the inverse of the differential operator was being considered. Here, the basic idea for this method is we must consider the following general differential equation of arbitrary order n greater than zero.

$$D^{n}{}_{t}u(t) = L(u) + K(u) + g(t) \quad n \in N, \tag{1}$$

with initial conditions,

$$\frac{D^k}{Dt^k}u(0) = h_k, \quad k = 0,1,2,3,\ldots n-1 \tag{2}$$

STEP 2 : Identify the forms of the Blasius equation

Blasius equation is one of the basic equations of fluid dynamics and it describes the velocity profile of the fluid in the boundary layer theory on a half-infinite interval (Wazwaz, 2007). In the mechanic theory, two forms of the Blasius equations are appeared and each of them is subjected to specific physical conditions. Moreover, it is obviously shown that the differential equations are the same but differ in initial conditions.

i) The first form of the Blasius equation:

$$u'''(x) + \frac{1}{2}u(x)u''(x) = 0\,, \tag{3}$$

(4)

$$u(0) = 0, \quad u'(0) = 1, \quad u'(\infty) = 0$$

ii) The second form of the Blasius equation:

$$u'''(x) + \frac{1}{2}u(x)u''(x) = 0\,, \tag{5}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u'(\infty) = 1 \tag{6}$$

STEP 3 : Implementation of IIM into the Blasius equation

In this step, we implemented the IIM into both forms of the Blasius equation by using MAPLE 2020 software to obtain the solutions. Consider the following general differential equation of arbitrary order n > 0

$$\frac{\partial^n u(x)}{\partial x^n} = A(u, \partial u) + B(x), n \in N \tag{7}$$

(8)

When A is a nonlinear of function u, ∂u (partial derivatives of u with respect of x and t) and B is the source function. In view of integral operators, the initial value problem in (7) and (8) were equivalent to the following integral equation (9):

$$u(x) = \sum_{k=0}^{n-1} h_k \frac{x^k}{k!} + I_x^n B + I_x^n A = f + N(u) \tag{9}$$

where
$$f = \sum_{k=0}^{n-1} h_k\,(x) \frac{x^k}{k!} + I_x^n B, \tag{10}$$

$$N(u) = I_x^n A \tag{11}$$

and I_x^n is an $n^{th} - order\ (n - fold)$ integral operator.

The required solution of $u(x)$ for (6) which is also the solution for the equation (7) can be obtained as the limit of a sequence of function $U_{n+1}(x)$ generated by the recurrence formula:

$$u_0 = f, \tag{12}$$

$$u(x,t) = \lim_{r \to \infty} u_r\,(x,t) \tag{13}$$

$$u_{r+1} = u_0 + N(u_r), \quad r = 0,1,2,\dots \tag{14}$$

where,

The form of the Blasius equation is written as:

$$u''' + \frac{1}{2} u'' = 0 \tag{15}$$

with the initial conditions

$$u(0) = 0,\ u'(0) = 1,\ u'(\infty) = 0 \tag{16}$$

and
$$u(0) = 0, \qquad u'(0) = 0, \qquad u'(\infty) = 1 \tag{17}$$

(18)

By using IIM, the first equation from the Blasius equation is compared with the IIM :

First iterative:

$$u''' = 0$$

$$\frac{\partial^k u(x,0)}{\partial x^k} = h_x, \qquad k = 0,1,2,\dots,n-1$$

(19)

$$u_0 = \frac{Ax^2}{2} + x$$

Second iterative: (20)

$$u_2 = u_0 + \iiint \frac{u_1 u_1''}{2} dxdxdx$$

$$u_3 = u_0 + \iiint \frac{u_2 u_2''}{2} dxdxdx$$

$$u_1 = u_0 + \iiint \frac{u_0 u_0''}{2} dxdxdx \quad (21)$$

(22)

$$\vdots$$

$$u_{n+1} = u_0 + \iiint \frac{u_n u_n''}{2} dxdxdx \quad (23)$$

Where n = 0,1,2, … n. This iterative technique was straightforward to use, and it was prominent that every solution evolved from the preceding iteration as the wide variety of iterations increasing, we were able to get the answer that was convergent to the solution of equation (7).

STEP 4: Solve the first form of the Blasius equation using IIM and Padé Approximants in MAPLE 2020 software

In this step, the IIM would be implemented into the first form of the Blasius equation by using MAPLE 2020 software. To implement the IIM into the Blasius equation, we used the below commands to restart the process in getting the solutions :

```
> restart;
> u[0](x):=x+(1/2)*A*x^2;
```

First of all, we compared the equation from Blasius with IIM as shown in the equation (18). After that, we continued the procedure by pressing the below command in order to get the second iterative.

Second iterative :

```
> u[1](x):=u[0](x)-(1/2)*int(u[0](x)*diff(u[0](x),x$2),[x,x,x]);
> u[2](x):=u[0](x)-(1/2)*int(u[1](x)*diff(u[1](x),x$2),[x,x,x]);
> u[3](x):=u[0](x)-(1/2)*int(u[2](x)*diff(u[2](x),x$2),[x,x,x]);
```

From the MAPLE 2020 software, we have obtained the second iterative that contain unknown parameter *A* in the equations. Therefore, we have to use the Padé Approximant to evaluate the value of unknown parameter *A* by finding the diagonal approximant since it is the most accurate approximation method. We have calculated the Padé Approximant diagonal started with [2/2] followed by [3/3], [4/4] and [5/5] by following the steps shown below:

Step 1 :- Differentiate the $u'''(x)$

```
> series((4),x,24)
```

Step 2 :- Restart the procedure

```
> restart;
```

Step 3 :- With(numapprox)

```
> with(numapprox);
```

Step 4 :- Assign the result obtained from the step 1 as $u(x)$

```
> u(x):=1 + A*x - A^8*x^22/273145872384000 -
A^7*x^21/12415721472000 - (197*A^6*x^20)/299553914880000 -
((-83/41803776000*A^7 + 1/619315200*A^5)*x^19)/684 - ((-
1411/41803776000*A^6 + 1/520224768*A^4)*x^18)/612 +
(14057*A^5*x^17)/36480761856000 - ((5449/15328051200*A^6 -
1829/3251404800*A^4)*x^16)/480 - ((5449/1094860800*A^5 -
17/30965760*A^3)*x^15)/420 - (1147*A^4*x^14)/16907304960 -
((-10033/319334400*A^5 + 967/19353600*A^3)*x^13)/312 - ((-
10033/29030400*A^4 + 1/30720*A^2)*x^12)/264 +
(1157*A^3*x^11)/212889600 - ((-23/17920*A^2 +
25/10752*A^4)*x^10)/180 - (25*A^3*x^9)/193536 -
(43*A^2*x^8)/107520 - ((1/32*A - 11/240*A^3)*x^7)/84 +
(11*A^2*x^6)/2880 + A*x^5/160 - A^2*x^4/48 - A*x^3/12;
```

Step 5 :- Compute the Padé on $u(x)$

```
> Pade := pade(u(x),xx[2,2]);
```

Step 6 :- Lim (Padé , *x* = infinity)

```
> limit(Pade,x = infinity);
```

Step 7 :- Solve for the unknown parameter A numerically

```
> fsolve(-3*A^2+1);
```

Step 8 :- Repeat the step 2 for [3/3], [4/4], [5/5] and so on.

Starting from the step 2 to 7, the steps will be repeated in order to obtain the result of Padé Approximants of [3/3], [4/4], and [5/5].

STEP 5: Solve the second form of the Blasius equation using IIM and Series Approximant in MAPLE 2020 software

In this step, we would also implement the IIM into the second form of the Blasius equation and solved the equation by using MAPLE 2020 software. We used the below commands to restart the process in obtaining the solutions:

```
> restart;
> y[0](x):=(1/2)*x^2
```

Then, from the boundary condition 0 to 2, we have to find the series solution up to (x^{23}) using the below commands:

```
> for m from 0 to 2 do
> y[m+1](x):=y[0](x)-
(1/2)*int(y[m](x)*diff(y[m](x),x$2),[x,x,x]);
> end do;
```

The next step was to find the $y_3{}'(x)$, where we differentiated the $y_3(x)$ that we obtained from the previous step using below command:

```
> y := diff(y[3](x),x);
```

In order to find the value of parameter *B*, we have to use the Series Approximants where the values of used are 2.0, 2.2, 2.4, 2.6, and 2.8 that would be substituted in the equation of $y_3{}'(x)$. Then, we would obtain the value of the parameter B directly after using the formula $\boldsymbol{B} = \sqrt{y'(\infty)}$.

Step 1 :- Substitute *x* = 2.0 into $y_3{}'(x)$

We substituted the value of $x = 2.0$ into the equation of $y_3{}'(x)$ that have been obtained from the previous step by using the below command:

```
> evalf(subs(x = 2.0, y))
```

Step 2 :- Simplify the value obtained from the previous step using the square root

Since $B = \sqrt{y'(\infty)}$ and graph of $y'(x)$ is levelled off for $x > 2$ then, we have to square root the answer obtained for the unknown parameter *B*.

```
> sqrt(value obtained)
```

Step 3 :- Repeat step 1 and step 2 for *x* = 2.2, 2.4, 2.6 and 2.8

Step 1 and step 2 will be repeated by substituting *x* = 2.2, 2.4, 2.6 and 2.8 into $y_3{}'(x)$ then square rooted the value obtained for each value of substituted *x.*

STEP 6: Comparison result between IIM and other existing methods

We would compare the result obtained for the Padé Approximant and Series Approximant with the other existing methods that have been used by the other researchers such as VIM (Wazwaz, 2007), DTM (Erturk & Momani, 2008) and SAIM (Selamat et al., 2019a) in the first form of Blasius equation whereas we only selected VIM (Wazwaz, 2007) to be compared with our proposed method, IIM in the second form of the Blasius equation.

STEP 7: Validation by calculating the absolute error and the percentage of error

In this section, the calculation of error is illustrated. In this study, we would use the VIM, the method that has been proposed by (Wazwaz, 2007) as the exact value whereas the IIM as our approximated value to find the absolute error. The errors of the outcomes have been calculated using the formulas as shown below:

$$\text{Absolute Error} \quad = |\, V_A - V_E \,| \tag{24}$$

$$\text{Percentage Error} \quad = \frac{V_A - V_E}{V_E} \times 100\% \tag{25}$$

V_A = Approximated value
V_E = Exact value

STEP 8: Conclusion and discussion

Lastly, we would make a conclusion based on the comparison result between IIM and other existing methods. We have also discussed whether all of the objectives had been achieved and satisfied in this research study.

3. Results and Discussion

3.1 Solutions for the First Form of Blasius equation

The followings are the results obtained from MAPLE 2020 Software for the first form of Blasius equation with the implementation of IIM. The approximations were obtained by using IIM and initial conditions given for U_0, then we selected $U_0(x) = \frac{1}{2}Ax^2 + x$ as an initial solution, where $U_{n'}(0) = A$.

Solution:

$$u_0(x) := x + \frac{1}{2}Ax^2$$

$$u_1(x) := x + \frac{A x^2}{2} - \frac{A\left(\frac{1}{24}x^4 + \frac{1}{120}A x^5\right)}{2}$$

$$
-\frac{\left(\frac{5449}{1094860800}A^5-\frac{17}{30965760}A^3\right)x^{16}}{6720}-\frac{1147A^4x^{15}}{253609574400}
-\frac{\left(-\frac{10033}{319334400}A^5+\frac{967}{19353600}A^3\right)x^{14}}{4368}-\frac{\left(-\frac{10033}{29030400}A^4+\frac{1}{30720}A^2\right)x^{13}}{3432}
+\frac{1157A^3x^{12}}{2554675200}-\frac{\left(-\frac{23}{17920}A^2+\frac{25}{10752}A^4\right)x^{11}}{1980}-\frac{5A^3x^{10}}{387072}-\frac{43A^2x^9}{967680}
-\frac{\left(\frac{1}{32}A-\frac{11}{240}A^3\right)x^8}{672}+\frac{11A^2x^7}{20160}+\frac{Ax^6}{960}-\frac{A^2x^5}{240}-\frac{Ax^4}{48}
$$

3.2 Result of the Padé Approximants

The Padé Approximant has the advantage of manipulating the polynomial approximation into a rational function to gain more information about *u(x)*. It is well-known that Padé Approximant would converge on the entire real axis if *u(x)* is free of singularities on the real axis. The diagonal approximation is also known as the most accurate approximant method. Therefore, we constructed only the diagonal approximations in the following discussions. By using the boundary condition, $u'(\infty)=0$, the diagonal approximation [*n/n*] vanishes if the coefficient of *x* with the highest power in the numerator vanishes. The diagonal approximations would be determined for *u(x)*.

$$
u_2(x):=x+\frac{1}{2}Ax^2-\frac{1}{5702400}A^4x^{11}-\frac{1}{518400}A^3x^{10}-\frac{1}{193536}A^2x^9
+\frac{11}{161280}A^3x^8+\frac{11}{20160}A^2x^7+\frac{1}{960}Ax^6-\frac{1}{240}A^2x^5-\frac{1}{48}Ax^4
$$

$$
u_3(x):=x+\frac{Ax^2}{2}-\frac{A^8x^{23}}{6282355064832000}-\frac{A^7x^{22}}{273145872384000}-\frac{197A^6x^{21}}{6290632212480000}
-\frac{\left(-\frac{83}{41803776000}A^7+\frac{1}{619315200}A^5\right)x^{20}}{13680}
-\frac{\left(-\frac{1411}{41803776000}A^6+\frac{1}{520224768}A^4\right)x^{19}}{11628}+\frac{14057A^5x^{18}}{656653713408000}
-\frac{\left(\frac{5449}{15328051200}A^6-\frac{1829}{3251404800}A^4\right)x^{17}}{8160}
$$

Solution:

Table1: The comparison values of *A* between VIM, DTM, SAIM, and IIM

Padé Approximants	**VIM (Wazwaz, 2007)**	**DTM (Erturk & Momani, 2008)**	**SAIM (Selamat et al., 2019a)**	**Current Method (IIM)**
[2/2]	0.5773502693	0.5773502692	0.5773502692	0.5773502692
[3/3]	0.5163977793	0.5163977795	0.5163977795	0.5163977795
[4/4]	0.5227030796	0.5227030798	0.5227030798	0.5227030798
[5/5]	Complex Numbers	Complex Numbers	Complex Numbers	0.5107780670

Table 1 above shows the numerical values of $u''(0) = A$ that were obtained by using the diagonal Padé Approximants and the results in IIM were in proper and decent agreement with the results obtained for VIM, DTM and SAIM. Regarding to these proofs, it is clearly shown that IIM is one of the efficient methods that can solve the Blasius equation efficiently and has high accuracy in getting the exact solutions.

As presented in the table above, the values of Padé Approximants obtained in IIM have

$$
\begin{aligned}
u := x \mapsto\ & 1 + A\cdot x - \frac{A^8\cdot x^{22}}{273145872384000} - \frac{A^7\cdot x^{21}}{12415721472000} - \frac{197\cdot A^6\cdot x^{20}}{299553914880000} \\
& - \frac{\left(-\frac{83}{41803776000}\cdot A^7 + \frac{1}{619315200}\cdot A^5\right)\cdot x^{19}}{684} \\
& - \frac{\left(-\frac{1411}{41803776000}\cdot A^6 + \frac{1}{520224768}\cdot A^4\right)\cdot x^{18}}{612} + \frac{14057\cdot A^5\cdot x^{17}}{36480761856000} \\
& - \frac{\left(\frac{5449}{15328051200}\cdot A^6 - \frac{1829}{3251404800}\cdot A^4\right)\cdot x^{16}}{480} \\
& - \frac{\left(\frac{5449}{1094860800}\cdot A^5 - \frac{17}{30965760}\cdot A^3\right)\cdot x^{15}}{420} - \frac{1147\cdot A^4\cdot x^{14}}{16907304960} \\
& - \frac{\left(-\frac{10033}{319334400}\cdot A^5 + \frac{967}{19353600}\cdot A^3\right)\cdot x^{13}}{312} \\
& - \frac{\left(-\frac{10033}{29030400}\cdot A^4 + \frac{1}{30720}\cdot A^2\right)\cdot x^{12}}{264} + \frac{1157\cdot A^3\cdot x^{11}}{212889600} \\
& - \frac{\left(-\frac{23}{17920}\cdot A^2 + \frac{25}{10752}\cdot A^4\right)\cdot x^{10}}{180} - \frac{25\cdot A^3\cdot x^9}{193536} - \frac{43\cdot A^2\cdot x^8}{107520} \\
& - \frac{\left(\frac{1}{32}\cdot A - \frac{11}{240}\cdot A^3\right)\cdot x^7}{84} + \frac{11\cdot A^2\cdot x^6}{2880} + \frac{A\cdot x^5}{160} - \frac{A^2\cdot x^4}{48} - \frac{A\cdot x^3}{12}
\end{aligned}
$$

$$y'''(x) + \frac{1}{2}y(x)y''(x) = 0$$

similar values with some methods compared. For instance, the Padé Approximant of [2/2] in IIM was 0.5773502692 which was similar to the results for SAIM and DTM which both of this method are also getting the same value as IIM while for VIM, there was a slight difference in its value compared to IIM. The same went to the Padé Approximants of [3/3] and Padé Approximant of [4/4] for IIM, DTM and SAIM, these three methods have the same values for *A* which were 0.5163977795 and 0.5227030798 respectively. Meanwhile, for the result of Padé Approximants of [5/5] for each method, VIM, DTM and SAIM have obtained complex numbers whereas the IIM has obtained 0.5107780670. According to Table 1, we selected the approximation of value *A* = *u''*(0) = 0.5227030798.

3.3 Solution for the Second Form of Blasius Equation

The results obtained from MAPLE 2020 software for the second form of the Blasius equation with the implementation of IIM are presented in this section. In the second form of the Blasius equation, it consists of unknown parameter *B* that needed to be evaluated then compared with other existing methods same as in the first form of the Blasius equation before. Here, we would use the Series Approximants in order to get the value of parameter *B* and also for the value of *u''*(0).

$$y_0(x) := \frac{x^2}{2}$$

$$y_1(x) := \frac{1}{2}x^2 - \frac{1}{240}x^5$$

$$y_2(x) := \frac{1}{2}x^2 - \frac{1}{5702400}x^{11} + \frac{11}{161280}x^8 - \frac{1}{240}x^5 \qquad (26)$$

Solution :

The following is the second equation of the Blasius equation and would be used as

$$u'''(x) + \frac{1}{2}u(x)u''(x) = 0\,, \qquad (27)$$

where *B* is an unknown parameter that needed to be solved and evaluated. An equation would be equivalent with *u* which is

(28)

with boundary conditions given as

$$y(0) = 0, \qquad y'(0) = 0, \qquad y'(\infty) = 1 \quad = \quad V^2 f^2(bx) \qquad as\ x \longrightarrow \infty$$

After that, the approximations were obtained by using the IIM and the boundary conditions given for y_0 been used hence, $y_0 = \frac{1}{2}x^2$ selected as the initial value.

Solution :

After we obtained the third iteration, we have the $y_3(x)$ then y was considered as y_3. From that, we would obtain the $y'(x)$ as shown below.

Solution :

$$y'(x) =$$

3.4 Result of the Series Approximants

By referring to the research by Wazwaz (2007) who used the VIM, *y′(x)* has a levelled off for *x* > 2. Thus, we could use $B = \sqrt{y'(\infty)}$ to estimate the value of *B* by using the several values of *x*. In this section, the example of calculations in getting the value of *B* using *x* = 2.0 is shown below.

$$y_3(x) := \frac{1}{2}x^2 - \frac{1}{6282355064832000}x^{23} + \frac{83}{571875655680000}x^{20} - \frac{5449}{125076897792000}x^{17} + \frac{10033}{1394852659200}x^{14} - \frac{5}{4257792}x^{11} + \frac{11}{161280}x^8 - \frac{1}{240}x^5$$

Series Approximant (2.0)

After getting the $y'(x)$ in the previous step, we substituted *x* = 2.0 in the equation obtained. We would like to highlight here that $y'(x) = B^2$. Therefore, to obtain the value of *B* only, we have to square root the B^2 then substituted the *x* = 2.0 into the equation.

Solution:

$$y'(2.0) = x - \frac{1}{273145872384000}x^{22} + \frac{83}{28593782784000}x^{19} - \frac{5449}{7357464576000}x^{16} + \frac{10033}{99632332800}x^{13} - \frac{5}{387072}x^{10} + \frac{11}{20160}x^7 - \frac{1}{48}x^4$$

$$B^2 = 1.7$$

$$B = \sqrt{1.7}$$

$$B = 1.313034017$$

For the second form of the Blasius equation, we have to find the value of $u''(0)$. Therefore, by using the formula $u''(0) = \frac{1}{B^3}$, we would substitute the value of *B* obtained before into the formula given.

Solution:

$$u''(0) = \frac{1}{B^3}$$

$$u''(0) = \frac{1}{(1.313034017)^3}$$
$$u''(0) = 0.4417454320$$

Table 2: The comparison values of *B* and $u''(0) = \frac{1}{B^3}$ between VIM and IIM

Series Approximants, *x*	VIM (Wazwaz, 2007)		Current Method (IIM)	
	B	$u''(0)$	***B***	$u''(0)$
2.0	1.313034017	0.4417454320	1.313034017	0.4417454320
2.2	1.347736192	0.4084936660	1.347736192	0.4084936660
2.4	1.373000106	0.3863565574	1.373000106	0.3863565574
2.6	1.387743095	0.3741732832	1.387743095	0.3741732833
2.8	1.388836100	0.3732905625	1.388836099	0.3732905632

Table 2 above shows the comparison values of *B* and $u''(0) = 1/B^3$ between VIM and IIM that have been obtained by using the simple Series Approximants method. According to the result shown in the table, the values of Series Approximants when x = 2.0 obtained in IIM have similar values with VIM, where *B* was 1.313034017 and the value of $u''(0)$ was 0.4417454320. The same went to the Series Approximants when x = 2.2 and x = 2.4 for IIM and VIM, both of these methods have similar values where the value of *B* obtained were 1.347736192 and 1.373000106 respectively whereas the value of $u''(0)$ obtained were 0.4084936660 and 0.3863565574 respectively.

Moving on to the result of Series Approximants, when x = 2.6, the value of *B* obtained for both methods was similar where *B* = 1.387743095, but there was a slightly difference in the value compared of $u''(0)$. The value of $u''(0)$ obtained for both methods were 0.3741732832 and 0.3741732833 respectively. Lastly, for x = 2.8, the result of Series Approximants obtained for IIM was 1.388836099 while for $u''(0)$ was 0.3732905632 which were different with the results of *B* and $u''(0)$ in VIM where the value of *B* for VIM was 1.388836100 and for its value of $u''(0)$ was 0.3732905625. Hence, the selected approximated value for *B* is 1.388836099 while the approximated value for $u''(0)$ is 0.3732905632.

3.5 Result of the Validation by Calculating the Errors

In this section, we presented the calculations of the error between the exact value and our selected approximated value by using the absolute error formula and evaluated the percentage of error. We have calculated the error occurs for both forms of the Blasius equation then selected the results of the VIM as our exact values whereas the results obtained by the IIM as our approximated values.

Solution:

First form of Blasius equation, value *A*

i) Absolute Error

$$= |\, V_A - V_E \,|$$
$$= |\, 0.5227030798 - 0.5227030796 \,|$$
$$= 2 \times 10^{-10}$$

ii) Percentage Error

$$= \frac{V_A - V_E}{V_E} \times 100\%$$
$$= \frac{0.5227030798 - 0.5227030796}{0.5227030796} \times 100\%$$
$$= 3.826 \times 10^{-8}\,\%$$

In order to calculate the absolute error for the value of parameter *A*, we selected the Padé Approximants of [4/4] for both methods where the exact value in VIM was 0.5227030796 would be subtracted with the approximated value obtained in IIM where the value was 0.5227030798. We could see from the calculation above, the absolute error obtained was 2×10^{-10} with percentage of error 3.826×10^{-8} %. It was clearly proved that the IIM has a high accuracy in getting the solutions approaching to the exact solutions because of the very small error occurred.

b) Second form of the Blasius equation, value *B*

i) Absolute Error

$$= |\, V_A - V_E \,|$$
$$= |\, 1.388836099 - 1.38883610 \,|$$
$$= 1 \times 10^{-9}$$

ii) Percentage Error

$$= \frac{V_A - V_E}{V_E} \times 100\%$$
$$= \frac{1.388836099 - 1.388836100}{1.388836100} \times 100\%$$
$$= 7.2003 \times 10^{-8}\,\%$$

Secondly, to calculate the absolute error for the value *B*, we selected the Series Approximants when $x = 2.8$ for both methods where the exact value of *B* in VIM was 1.388836100 would be subtracted with the approximated value of *B* obtained in IIM where the value was 1.388836099. We could see from the calculation above, the absolute error obtained was 1×10^{-9} while the percentage of error obtained was 7.2003×10^{-8} %. It was clearly proved that the IIM has a high accuracy in getting the solutions approaching to the exact solutions because of the very small error occurred.

c) Second form of the Blasius equation, $u'' = 1/B^3$

i) Absolute Error

$$= |\, V_A - V_E \,|$$

$$= |\, 0.3732905632 - 0.3732905625 \,|$$

$$= 7 \times 10^{-10}$$

ii) Percentage Error

$$= \frac{V_A - V_E}{V_E} \times 100\%$$

$$= \frac{0.3732905632 - 0.3732905625}{0.3732905625} \times 100\%$$

$$= 1.8752 \times 10^{-7}\,\%$$

Finally, to calculate the absolute error for the value of $u''(0) = 1/B^3$, we also selected the Series Approximants when x = 2.8 for both methods where the exact value of $u''(0)$ in VIM was 0.3732905625 would be subtracted with the approximated value of $u''(0)$ obtained in IIM where the value was 0.3732905632. We could see from the calculation above, the absolute error obtained was 7×10^{-10} with percentage of error 1.8752×10^{-7} %. It was clearly proved that the IIM has a high accuracy in getting the solutions approaching to the exact solutions because of the very small error occurred.

4. Conclusion and Recommendation

4.1 Conclusion

Ultimately, this study has managed to solve both forms of the Blasius equation by using the IIM. The IIM has been implemented into both forms of the Blasius equation and has been programmed in the MAPLE 2020 software to facilitate obtaining the calculations. Next, we had successfully examined the reliability and relevance of the IIM in solving the two forms of Blasius equation. In this section, we have presented the table comparison between IIM and other existing methods. As for the first form of the Blasius equation, the IIM has been compared with three different methods known as VIM, DTM and SAIM whereas for the second form of the Blasius equation, we only compared our proposed method with the VIM, a method that was proposed by Wazwaz (2007).

Besides that, we also managed to determine the precision and efficiency of the IIM compared to other methods used by the other researchers before and validated the results obtained from the calculations shown in the MAPLE 2020 software. The first form of Blasius equation was combined with the Padé Approximants because we needed to find the value of unknown parameter A. We had used the diagonal approximation [*n/n*] of the Padé Approximants [2/2], [3/3], [4/4], and [5/5]. As a result, we could see that the values of Padé Approximants obtained in our proposed method, IIM have similar values with some of the methods.

For example, the Padé Approximant of [2/2] in IIM which was 0.5773502692 was similar to the results for SAIM and DTM where both of these methods were also getting the same value as IIM while for VIM, there was a slight difference in its value compared to IIM. The same went with Padé Approximants of [3/3] and Padé Approximant of [4/4] for IIM, DTM and SAIM have same values which were 0.5163977795 and 0.5227030798 respectively. Plus, for the Padé Approximants of [5/5] the VIM, DTM and SAIM, these methods obtained complex numbers, but the IIM obtained 0.5107780670. Therefore, the most obvious finding from this study is that the IIM is one of the reliable methods that could solve the Blasius equation due to its high accuracy in getting the results same as the exact solutions.

In addition, this study has conducted a few more calculations that involved coding programming in the MAPLE 2020 software by referring to the research done by Wazwaz (2007) who using VIM in his research. By taking the results obtained when solving the second form of the Blasius equation in VIM to be compared with our proposed method, we have identified the comparison values of B and $u''(0) = 1/B^3$ between VIM and IIM by using the simple Series Approximants method. Regarding the result shown in Table 2 before, the result of Series Approximants for $x = 2.8$ obtained for IIM was 1.388836099 while for $u''(0)$ was 0.3732905632 which were different with the results of B and $u''(0)$ in VIM where the value of B for VIM was 1.388836100 and for its value of $u''(0)$ was 0.3732905625. Thus, the absolute errors for the selected approximated value of parameter B and $u''(0)$ are 1.388836099 and 0.3732905632 respectively.

To tide up, all of the this study objectives have been achieved. In this study, we managed to solve the Blasius equation by using IIM with the help of a very useful mathematical application known as MAPLE 2020 software. Other than that, we have also examined the reliability and relevance of the IIM in solving the two forms of Blasius equations. Finally, we have determined the precision and efficiency of the IIM compared to other existing methods used in solving the Blasius equation such as VIM, DTM, and SAIM by calculating the absolute errors and percentage errors for VIM and IIM in both forms of the equations.

4.2 Recommendation

We would recommend that the IIM is used by Physics and Engineering students or other researchers in solving any problems related to Blasius equation or perhaps enhance this research in the future. IIM is not only required less iterations, but it also provides shorter time to compute the coding programming to obtain the solutions especially when using any useful mathematical application such as MAPLE software. Besides that, IIM has a high accuracy in getting the solutions and only small error occurred in certain iteration calculated. Hence, the IIM is one of the best alternative methods that can solve the Blasius equation efficiently and accurately.

Acknowledgement

The authors would like to thank to Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA for supporting this project.

5. References

Chiriţă, A., Ene, H., & Nicolescu, B. N. (2012). On some application of the Blasius problems. *Bulletin of the Transilvania University of Brasov. Mathematics, Informatics, Physics. Series III, 5, 93-101.*

Ertürk, V. S., & Momani, S. (2008). Numerical solutions of two forms of Blasius equation on a half-infinite domain. *Journal of Algorithms & Computational Technology*, 2(3), 359–370. https://doi.org/10.1260/174830108785302850

Fazio, R. (2013). Blasius problem and Falkner-Skan model: Töpfer's algorithm and its extension. *Computers and Fluids*, 73, 202–209. https://doi.org/10.1016/j.compfluid.2012.12.012

Fazio, R. (2016). A non-iterative transformation method for Blasius equation with moving wall or surface gasification. *International Journal of Non-Linear Mechanics*, 78, 156–159. https://doi.org/10.1016/j.ijnonlinmec.2015.01.004

Fernández, F. M. (2009). On some approximate methods for nonlinear models. *Applied Mathematics and Computation*, 215(1), 168-174.

Hashim, I. (2006). Comments on "A new algorithm for solving classical Blasius equation" by L. Wang. *Applied Mathematics and Computation*, 176(2), 700-703. https://doi.org/10.1016/j.amc.2005.10.016

Hemeda, A. A. (2015). An integral iterative method for solving fractional physical differential equations. *International Information Institute (Tokyo)*. Information, 18(2), 365.

Hemeda, A. A. (2018). A friendly iterative technique for solving nonlinear integro-differential and systems of nonlinear integro-differential equations. *International Journal of Computational Methods*, 15(3), 1–15. https://doi.org/10.1142/S0219876218500160

Hemeda, A. A., & Eladdad, E. E. (2018). New iterative methods for solving Fokker-Planck equation. *Mathematical Problems in Engineering*, 2018(2). https://doi.org/10.1155/2018/6462174

Kalateh Bojdi, Z., Ahmadi-Asl, S., & Aminataei, A. (2013). A new extended Padé approximation and its application. *Advances in Numerical Analysis*, *2013*, 1–8. https://doi.org/10.1155/2013/263467

Khandelwal, Y., Umar, B. A., & Kumawat, P. (2018). Solution of the Blasius equation by using adomain mahgoub transform. *International Journal of Mathematics Trends and Technology*, 56(5), 303–306. https://doi.org/10.14445/22315373/ijmtt-v56p541

Kim, K. M., Choe, S. H., Ryu, J. M., & Choi, H. (2020). Computation of analytical zoom locus using Padé approximation. *Mathematics*, 8(4), 1–19. https://doi.org/10.3390/math8040581

Lee, Z. Y. (2006). Method of bilaterally bounded to solution Blasius equation using particle swarm optimization. *Applied Mathematics and Computation*, 179(2), 779-786. https://doi.org/10.1016/j.amc.2005.11.118

Liao, S. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, 147(2), 499-513.

Marinca, V., & Herişanu, N. (2014). The optimal homotopy asymptotic method for solving Blasius equation. *Applied Mathematics and Computation*, 231(1), 134–139. https://doi.org/10.1016/j.amc.2013.12.121

Parand, K., & Taghavi, A. (2009). Rational scaled generalized Laguerre function collocation method for solving the Blasius equation. *Journal of Computational and Applied Mathematics*, 233(4), 980–989. https://doi.org/10.1016/j.cam.2009.08.106

Rahman, M. M. (2019). The comparison between analytical and numerical solutions of Blasius Equation on boundary layer parallel flow over a flat plate. *Journal of Scientific and Engeneering Research*, 6(2), 1–12.

Santatra, F. R., & Solofoniaina, J. (2018). Solutions of the boundary layer of a viscous fluid in forced convection on a flate plate by the variational iteration method. *HAL, 2018*.

Selamat, S., Halmi, N. A., & Ayob, N. A. (2019a). A semi analytic iterative method for solving two forms of Blasius equation. *Journal of Academia,* 7(2), 76–85.

Selamat, M., Khusairi, N., Abdul Gaffar, N., & Syed Huzaini, S. (2019b). The integral iterative method for approximate solution of Newell-Whitehead-Segel equation*. Mathematical Sciences And Informatics Journal,* 3(1), 1-10.

Tran, D. X. (2015). Padé approximants for finite time ruin probabilities. *Applied Journal of Computational and Applied Mathematics*, 278, 130-137. https://doi.org/10.1016/j.cam.2014.09.109

Wang, L. (2004). A new algorithm for solving classical Blasius equation. *Applied Mathematics and Computation*, *157*(1), 1-9. https://doi.org/10.1016/j.amc.2003.06.011

Wazwaz, A. M. (2007). The variational iteration method for solving two forms of Blasius equation on a half-infinite domain. *Applied Mathematics and Computation,* 188(1), 485–491. https://doi.org/10.1016/j.amc.2006.10.009

Zhou, S. (2018). Padé approximant for hard sphere + square well and hard sphere + square well + square shoulder model fluids. *Physica A: Statistical Mechanics and its Applications*, 512, 1260-1277.