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# ANALYSIS OF IMAGE COMPRESSION USING SINGULAR VALUE DECOMPOSITION

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## ABSTRACT

*TImage processing has become a more crucial area nowadays due to the advancement of imaging technology. Although image quality is an essential criterion, in some cases, it leads to a storage bottleneck. Thus, an efficient method is needed to reduce the amount of data while maintaining quality. One preferred method is image compression using Singular Value Decomposition (SVD). In this method, the image matrix is reduced by rank, k. However, the right value of k is still debatable since it is based on the image features. Therefore, this study did some analysis to choose an ideal k. Three popular images-Afghan Girl, Eiffel Tower and Mona Lisa are selected to be experimented with, and the compression errors are recorded together with image intensity, hard-threshold and storage saved. The results show that Afghan Girl and Mona Lisa have similar singular values behavior, but not Eiffel Tower. This study suggests that the rank must be at least 30% of the total image size to have good image quality and optimal storage use.*

**Keywords***: Image Processing, Matrix Decomposition, Rank, SVD..*

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## 1. Introduction

Matrix decomposition, also known as matrix factorization, can be defined as a mathematical programming approach that simplifies a complex matrix into a much simpler matrix form, making computation a lot easier and faster (Zhang, 2015). Matrix decomposition is a fundamental issue in linear algebra and applied statistics that are important in science and engineering. Data matrices representing numerical observations, such as proximity matrices or correlation matrices, are usually enormous and challenging to identify. Hence, decomposing the data matrices into some lower-order or lower-rank forms will disclose the inherent characteristics and structure of the matrices and help interpret their definition easily. Several existing matrix decomposition methods are SVD, LU, QR, and Eigenvalue decomposition (Yang et al., 2020).

Matrix decomposition has been used in various fields, including the entertainment industry, medical domain and meteorology. The most well-known application is in the search engine. With a high mathematics algorithm, it caters to every user's personal preference, needs and recommendations. The system produces output through suggestions by analyzing the user's history, preferences, past activities, and pattern usage to enhance the user's experience. Some popular applications using the recommender system are Netflix, Amazon, and Youtube.

In the medical domain, matrix decomposition is used in analyzing medical images. The classification of this image, such as CT and MRI, can be time-consuming and may not always provide reliable results. Some parts are truncated to reduce the number of data points and storage in some circumstances. Consequently, some vital information about the studied image is eliminated, leading to misconceptions about the data (Hadi, 2018). Matrix decomposition also help in image denoising. Different level of noise in image produced different effect on the accuracy of the detected contour and hence disturbs the smoothness of image simulation (Halim et al., 2018).

As a result, using multiple techniques for automatic classification is essential. Matrix decomposition allows the image to be displayed as a smaller set of values that can retain the vital characteristics of the original image.

Another area that needs matrix decomposition is meteorology. A seasonal climate prediction is crucial in predicting unexpected precipitation to avoid a critical accident such as a flood season. However, there will inevitably be biases in every climate prediction due to external factors such as local topography. Thus, matrix decomposition effectively enhances coupled model predictions in various areas where external factors heavily impact the prediction results.

## 2. Singular Value Decomposition

The SVD is one of the important mechanisms in numerical linear algebra and the SVD of a matrix is a factorization and estimation method which effectively reduces the matrix into a smaller invertible and square matrix.

SVD is a consequence of recent breakthroughs and improvements in image protection, the importance of digital signal processing for picture identification and compression has developed. Image compression allows for a decrease in the image's storage capacity as well as its transmission bandwidth. The benefit of picture compression is that it reduces the amount of irrelevant and redundant data. This improves the transmission rate and optimizes the storage space.

Singular Value Decomposition (SVD) is one of the most successful techniques for capturing hidden latent factors in reduced dimensionality and producing high-quality recommendations. The smaller singular values can be regarded as unimportant information or random noise and can be truncated from the image information (Liu et al., 2017).

The definition of SVD is given as follows (Anton & Rorres, 2013),

**Theorem 1**. Singular Value Decomposition

If $A$ is an $m \times n$ matrix of rank $k$, then $A$ can be expressed in the form $A = U\sum V^T$ where $U$ is an $m \times m$ orthogonal matrix, $V$ is an $n \times n$ orthogonal matrix and $\sum$ has size $m \times n$ which can be expressed as,

$$\Sigma = \begin{bmatrix} D & 0_{k\times(n-k)} \\ 0_{(m-k)\times k} & 0_{(m-k)\times(n-k)} \end{bmatrix} \tag{1}$$

in which $D$ is a diagonal $k \times k$ matrix whose successive entries are the first $k$ singular values of $A$ in nonincreasing order.

The process flow of the methodology involved in this study is shown in Figure 1. The process begins with data collection. The input image is converted to grayscale before the SVD is applied. Then, the compressed image is produced. The last process is to calculate the compression error.

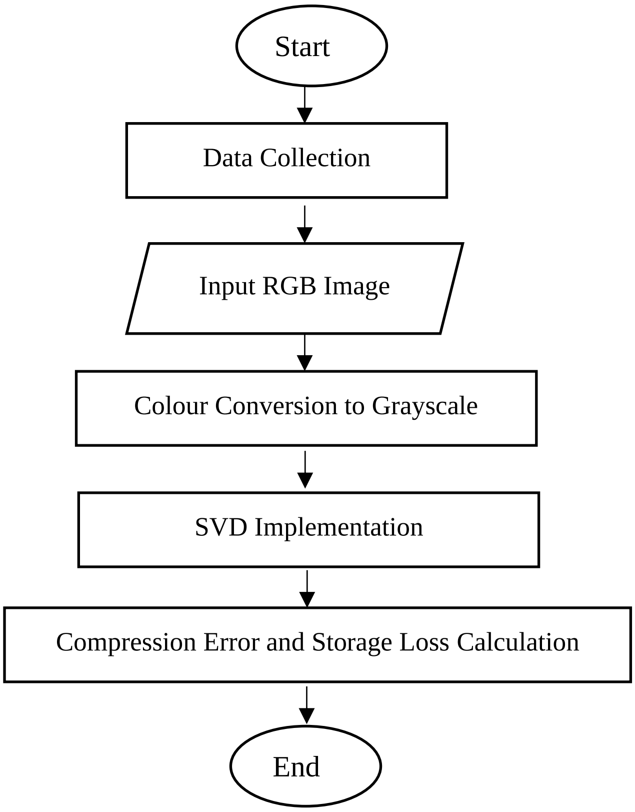



Figure 1. Process Flow of Research.

## 2.1 Data Collection

This study applies three different images for the experiment. The images are well-known images and have different properties in size and intensity, which are Afghan Girl (AG), Eiffel Tower (ET) and Mona Lisa (ML). Figure 2 shows the selected images. The images originally in RGB colour.

(a) Afghan Girl (b) Eiffel Tower (c) Mona Lisa

Figure 2. Three Samples of Coloured Images.

Figure 2(a) shows a 1984 photographic portrait of Sharbat Gula, a 12 years old girl with a red scarf draped loosely over her head, and her eyes were staring directly into the camera. This image was taken by National Geographic Society photographer Steve McCurry entitled Afghan Girl (Callwood, n.d.). It had appeared on the June 1984 cover of National Geographic and had been called "The First World's Third World Mona Lisa". She was a Pashtun child living in the Nasir Bagh refugee camp in Pakistan during the Soviet occupation of Afghanistan. The original size for this image is $658 \times 691$ pixels.

Figure 2(b) shows the Eiffel Tower (ET), a historical landmark located on the Champ de Mars in Paris, France (Alexander, 2021). The name of the Eiffel Tower is derived from the engineer Gustave Eiffel, whose company designed and constructed the tower. The tower was built from 1887 until 1889 as a part of the World's Fair 1889. Nowadays, the ET has become the historical landmark of France and one of the largest famous structures worldwide. The ET is the tallest building in Paris, with 324 metres (1063 ft) tall. The original size of this image is $487 \times 730$ pixels.

Figure 2(c) shows the Mona Lisa (ML), one of the world's the most expensive paintings (Alicja Zelazko, n.d.). It holds the Guinness World Record for the highest reported art policy value, with an insurance amount of US$100 million in 1962. There is no doubting that the ML is a masterpiece. The original image size is $1000 \times 671$ pixels.

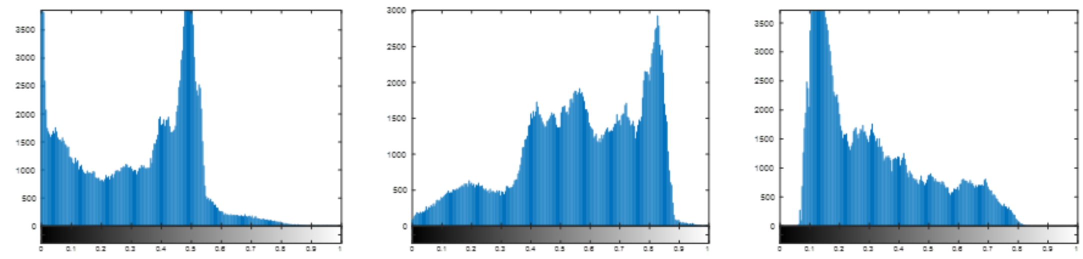
### 2.2 Colour Conversion to Grayscale

All three figures are converted to a grayscale image and resized to $500 \times 500$ to assist the comparison process, as shown in Figure 3.

(a) Afghan Girl (b) Eiffel Tower (c) Mona Lisa

Figure 3. Three Samples of the Grayscale Images.

To illustrate the difference in intensity between images, the histogram is plotted as in Figure 4.



(a) Afghan Girl (b) Eiffel Tower (c) Mona Lisa

Figure 4. Histogram of the Grayscale Images.

The histogram indicates that all three images have different peak values. ET consists more of lighter pixels, while ML consists more of darker pixels. This is due to the original image of the ET (refer to Figure 2(b)) consists various RGB, which contributes to the low intensity in grayscale. Meanwhile, the colour range for ML is relatively small, which is more to brownish and yellowish (refer to Figure 2(c)). Interestingly, AG has two different peaks values: the darkest (0) and the medium darkness (0.5).

**2.3 Image Compression using SVD**

MATLAB software computes the SVD for image compression and reconstructs the image by approximating a smaller rank. Different ranks of the SVD truncation have been applied, such as $k = 450, 350, 250, 150, 50$. The $k$ values are obtained by reducing the image quality by $10\%, 30\%, 50\%, 70\%, 90\%$, respectively.

**2.4 Compression Error and Storage Loss Calculation**

The compression error, $e$ which is also known as Frobineus-norm (Sadek, 2012) is calculated as,

$$e = \sqrt{\sum^{m}\sum^{m}(A - B)^2} \tag{2}$$

where $A$ is the image matrix in grayscale and $B$ is the image matrix after compression. As stated in Theorem 1, $A$ can be decomposed into three different matrices, $A = U\Sigma V^T$. Thus, $B$ can be represented as,

$$B = U_k \Sigma_k V_k^T \tag{3}$$

where $U_k, \Sigma_k$ and $V_k^T$ are the reduced matrices based on rank $k$.

The effect of compression can also be measured by using storage loss (Mounika et al., 2015). An $m \times n$ matrix $A$ has been decomposed into three matrices, $U, \Sigma, A$, with dimensions $m \times m, m \times n$, and $n \times n$, respectively. For $\Sigma$, the dimension can be reduced to only $m$ since all elements in the matrix are zero, except the diagonal. Thus, the use storage can be calculated as,

$$D_{ori} = m \times m + m + n \times n \tag{4}$$

.

Since this study has converted the image into $500 \times 500$, thus the used storage is 500,500. However, when each sub-matrix is reduced to a rank $k$, the dimension in (4) becomes,

$$D_{new} = m \times r + r + r \times n = r(m + n + 1) \tag{5}$$

.

## 3. Results and Analysis

Figure 5 to 7 shows the compressed images of the grayscale image as in Figure 3. For each image, the rank $k$ and the compression error $e$ are stated.

(a) $k = 450, e = 0.07244$

(b) $k = 350, e = 0.4970$

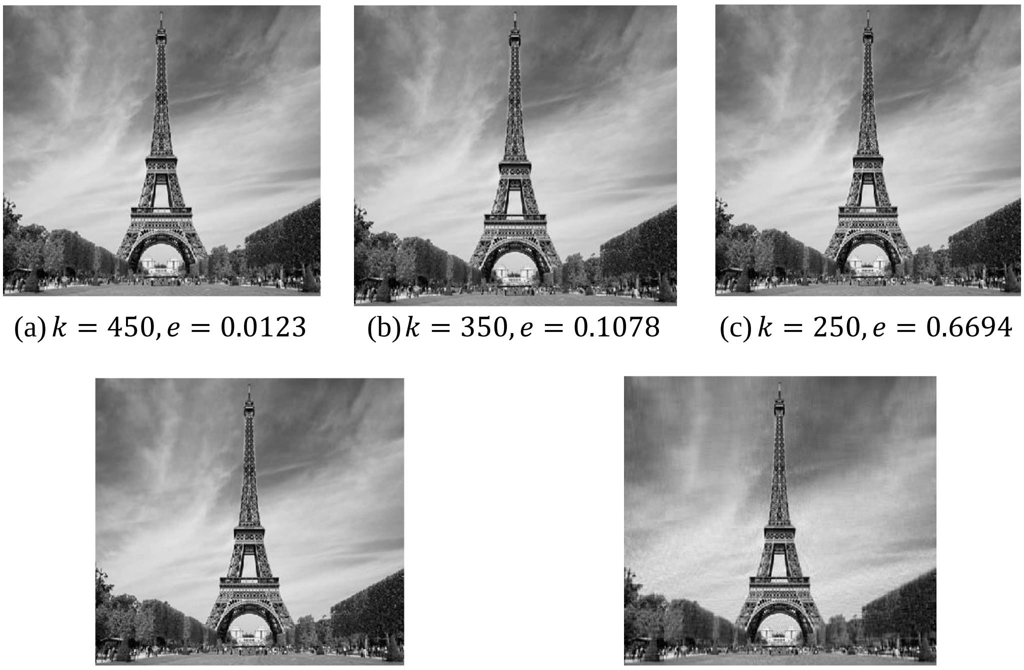
(c) $k = 250, e = 1.4289$

(d) $k = 150, e = 3.48090$

(e) $k = 50, e = 10.0065$

Figure 5. Afghan Girl (AG) with Different $k$ and $e$.

Figure 5 shows the results for AG. For Eiffel Tower, the results are in Figure 6.

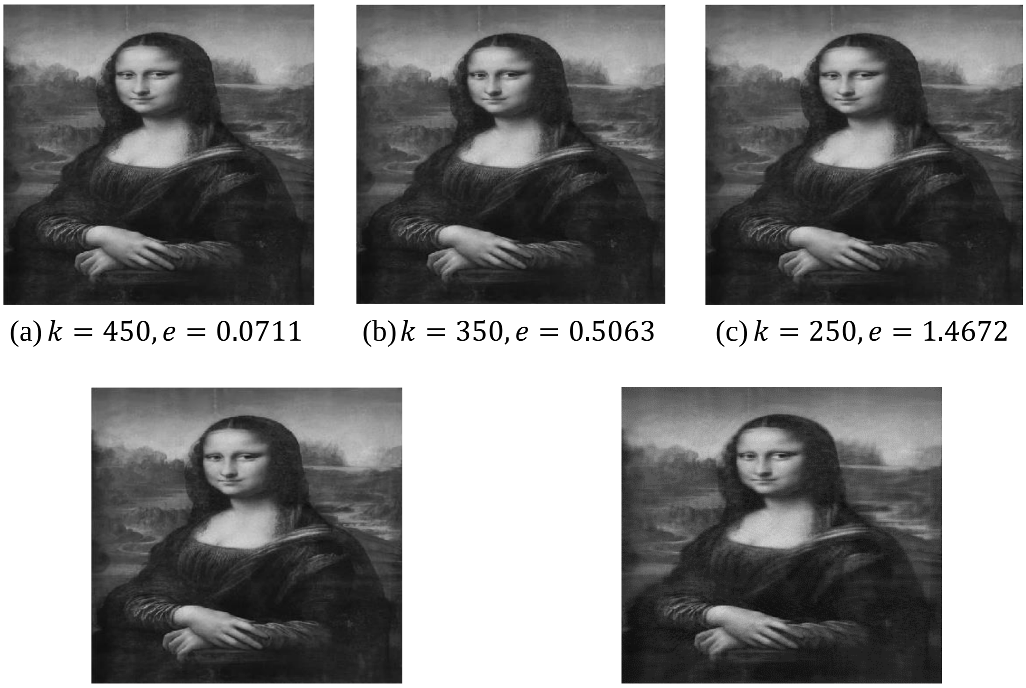


(a) $k = 450, e = 0.0123$ (b) $k = 350, e = 0.1078$ (c) $k = 250, e = 0.6694$

(d) $k = 150, e = 3.2208$ (e) $k = 50, e = 14.8923$

Figure 6. Eiffel Tower with Different $k$ and $e$.

Figure 7 shows the results of compressed images of Mona Lisa.



(a) $k = 450, e = 0.0711$ (b) $k = 350, e = 0.5063$ (c) $k = 250, e = 1.4672$

(d) $k = 150, e = 3.3317$ (e) $k = 50, e = 8.2963$

Figure 7. Mona Lisa with Different $k$ and .$e$

Figure 8 shows the compression error of the three images for each rank as compared with the original image. In the general, it can be observed that the image quality is increasing as the rank is increase because the error is reduced.

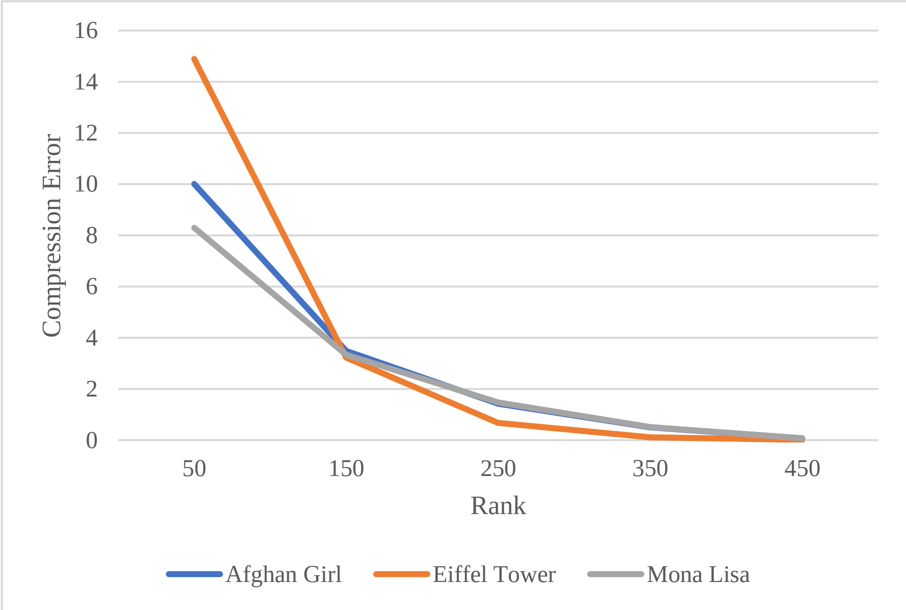



Figure 8. Rank versus the Compression Error.

From Figure 8, it can be observed that 150 which is 30% of the original rank, acts as the turning point for all images. The ET approaches good quality faster when the rank is increased from 150 to 450, while AG and ML show similar behaviour in error reduction for the same range. On the other hand, ET shows the most drastic error increment, followed by AG and ML when the rank is really low.

This phenomenon happens because singular values that form rank for ET have a more significant role than AG and ML. Equation (6) shows the threshold to determine the significant singular value, $\sigma$ (Gavish & Donoho, 2014).

$$Threshold, \tau = \frac{4}{\sqrt{3}}\sqrt{n}p \tag{6}$$

The threshold $\tau$ in (4) is called the hard threshold for image matrix $n \times n$ and white noise, $p$. In this study, we set $p = 0.01, 0.005, 0.001$. The threshold and the number of significant $\sigma$ for each image are displayed in the following table.

Table 1. The number of significant singular value, $\sigma$ for a certain threshold, $\tau$

| **Image** | $\boldsymbol{N(\sigma) > \tau}$ | | |
|---|---|---|---|
| | $\boldsymbol{p = 0.005}$ | $\boldsymbol{p = 0.003}$ | $\boldsymbol{p = 0.001}$ |
| Afghan Girl (AG) | 217 | 275 | 389 |
| Eiffel Tower (ET) | <u>202</u> | <u>236</u> | <u>302</u> |
| Mona Lisa (ML) | 213 | 280 | 390 |

In Table 1, the most less number of significant $\sigma$ for each $p$ is underlined. It can be seen that ET has the most less significant $\sigma$, while AG and ML have a similar number of significant $\sigma$ for smaller $p$. This explains the similar curve behaviour in Figure 8 for AG and ML for $k > 150$.

The ET has the least significant s, reflecting it needs a smaller rank than the other two images. Therefore, a higher rank gives better image quality (less error) for ET, but ET does not have sufficient features if the rank is too small, which leads to a greater error. The $D_{new}$ for each compression is given in Table 2.

Table 2. The storage used and loss for each compression

| $\boldsymbol{r}$ | **50** | **150** | **250** | **350** | **450** | **500** |
|---|---|---|---|---|---|---|
| **Storage used ($\boldsymbol{D_{new}}$)** | 50,050 | 150,150 | 250,250 | 350,350 | 450,450 | 500,500 |
| **Storage loss (%)** | 90 | 70 | 50 | 30 | 10 | 0 |

Table 2 shows that the percentage of loss storage is the compression percentage itself. Since the suggested rank must be not less than 150, SVD can save up to 70% storage.

## 4. Conclusion and Recommendations

Matrix decomposition is a mathematical approach that simplifies a complicated matrix into a simpler form. This study utilised the SVD method for three different images for compression. These images have different colour ranges and intensity distributions. Each image has been compressed based on five different ranks, and the compression errors and storage saved are recorded. The results suggest that the rank must be greater than 30% of the image's original size to have a good quality image. Additionally, the result also shows that the SVD can save up to 77% of the storage.

For future direction, SVD can be compared with other decomposition methods such as QR and EVD. More types of images can be considered to enhance the analysis. Different techniques, such as the soft and Frobenius threshold, can be considered for the threshold.

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