MERGING LANES: WHERE E-LEARNING DIVERSITY MEETS FUTURE TRENDS

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EVALUATING STUDENTS' SPATIAL REASONING IN SETTING UP TRIPLE INTEGRALS

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ABSTRACT

Multivariable calculus is a critical component of STEM education, demanding sophisticated spatial reasoning and abstract geometric thinking to solve advanced problems related to multiple integrals – specifically double and triple integrals. However, the transition from single-variable to multivariable calculus presents significant cognitive challenges for students. This study investigates students' understanding and the underlying cognitive obstacles they face when setting up triple integrals to determine the volume of three-dimensional (3D) solids. A qualitative case study was conducted with undergraduate engineering students, whose written solutions were evaluated across four distinct phases: visualization, projection, coordinate selection, and algebraic setup. Utilizing the Action-Process-Object-Schema (APOS) theoretical framework as a diagnostic lens, the findings reveal that the primary learning obstacle is not a lack of procedural competence, but rather a fundamental deficit in 3D spatial visualization. While many students successfully executed dimension reduction and coordinate selection, they frequently failed to translate textual descriptions into accurate geometric boundaries, resorting instead to mechanical calculations characteristic of the APOS Action stage. To bridge this cognitive gap, the study recommends a pedagogical shift away from rote algorithmic drills toward instructional strategies that build spatial schema, specifically highlighting the integration of dynamic geometry software like GeoGebra and established pedagogical strategies, such as employing spatial metaphors to conceptualize the bounds of a solid.

Keywords: Calculus, multiple integrals, triple integrals, visualization spatial reasoning, APOS

Introduction

Multivariable calculus is a compulsory subject in Science, Technology, Engineering, and Mathematics (STEM) degree programs, equipping students with the mathematical tools required to model complex, multidimensional physical phenomena. Within this domain, the conceptualization and application of multiple integrals, specifically double and triple integrals, are critical for solving advanced problems related to volume, mass, moments of inertia, and surface areas of 3D objects (Milenković & Vučićević, 2024).

The transition from single-variable calculus to multivariable calculus represents a formidable cognitive leap, contrary to what most educators believe is a seamless generalization of previously learned calculus concepts (Martínez-Planell & Trigueros, 2021; Padayachee & Craig, 2020). While single-variable integration often relies on straightforward two-dimensional (2D) graphical interpretations, multivariable integration demands a sophisticated synthesis of 3D spatial reasoning, advanced algebraic manipulation, and abstract geometric thinking.

The first part of solving multiple integrals involves the process of defining the integration area and setting boundaries for the variables. In multiple integral problems, students are given information about mathematical objects that define the integral function. To start calculating multiple integrals, they must utilize both the graphical and algebraic representations of those objects. They are also required to solve specific equations and inequalities to determine the boundaries for the multiple integral. Understanding the properties of different surfaces in space is a critical skill that must be acquired to successfully determine the boundaries of these integrals (Milenkovic et al., 2023).

Much work has been done on student understanding of multivariable functions over the past twenty years (Martínez-Planell & Trigueros, 2021). However, studies concentrated on students' understanding of multiple integrals are still lacking. A few of these studies, such as Gemechu et al. (2021), Padayachee (2020), and Martínez-Planell et al. (2025), used the Action-Process-Object-Schema (APOS) theory to investigate the levels of students' conception of double and triple integrals. This theory conceptualizes mathematical learning as a progression beginning with "Actions," which are externally driven, step-by-step algorithmic transformations of mathematical entities. Through repetition and cognitive reflection, students internalize these rigid procedures into a continuous "Process," thereby gaining autonomous internal control over the mathematical operations. When students are subsequently able to perform new actions upon these internalized processes, they encapsulate them into unified cognitive "Objects." Ultimately, these actions, processes, and objects are integrated into a broader cognitive framework called a "Schema," which students deploy to navigate and resolve complex mathematical problem situations.

To address these existing gaps in the literature, the primary aim of this study is to investigate the levels of students' understanding of multiple integrals. Specifically, this research focuses on their cognitive processes in defining the geometric integration area and setting up the precise limits of integration for triple integrals. Because triple integrals involve the integration of a function of three variables over a defined 3D solid region, they serve as a rigorous context to expose students' spatial and symbolic reasoning. Students' work will be evaluated across four distinct phases of setting up triple integrals: the visualization of the 3D solid region, the projection of the solid onto an appropriate coordinate plane, the selection of a coordinate system, and the algebraic setup of the integral. Furthermore, APOS theory is utilized as a diagnostic lens to pinpoint the specific stage of students' understanding. Conducting an in-depth case study analysis regarding student understanding in the precise formulation and setup of triple integrals will provide essential insights into these learning obstacles, ultimately informing more effective, spatially grounded teaching strategies in multivariable calculus.

Methodology

The purpose of this study was to analyze students' understanding of setting up triple integrals to determine the volume of a solid region. The participants were selected from a single class enrolled in the "Further Calculus for Engineers" course during the October 2025 to February 2026 academic semester at Universiti Teknologi MARA (UiTM) Pulau Pinang. The students were divided into eight small groups consisting of two to three members. Each group was randomly assigned one of eight different tasks requiring them to set up a triple integral across various coordinate systems to calculate the volume of a specific solid region. The groups were given 30 minutes to complete the task, and the problem-solving sessions were conducted in person under the direct observation of the lecturer. During this time, students were permitted to use the dynamic geometry software, GeoGebra, to plot surface graphs and assist in their integral setup.

To evaluate student performance, their written solutions were analyzed using a four-phase criteria framework. Table 1 outlines the rubric used to assess these four distinct phases: Visualization, Projection, Coordinate Selection, and Setup. Additionally, the APOS theory was utilized specifically as a diagnostic lens to classify the students' level of understanding. Based on the types of procedural and conceptual errors made during the integral setup, students' cognitive progression was categorized into the stages of APOS theory (Action, Process, Object or Schema stages) to better diagnose their learning obstacles.

Table 1: Assessment Rubric for Visualization, Projection, Coordinate Selection, and Setup

Phase criteria	**Description**
Visualization	The ability to translate text descriptions into a 3D geometric representation and identify all bounding surfaces
Projection	The ability to dimensionally reduce the 3D solid to a 2D region of integration on the appropriate coordinate plane with correct dimensions
Coordinate	The ability to choose the optimal coordinate system (Cartesian, cylindrical and spherical coordinates) and convert variables or differentials correctly
Setup	The ability to define precise integration limits to find the volume of a solid

Results and Discussion

The overall evaluation of the eight student groups, utilizing the four-phase assessment rubric, revealed a wide variance in understanding levels. While the majority of the cohorts demonstrated a moderate to high proficiency across the setup process, only one group demonstrated complete conceptual and procedural mastery across all four phases. Conversely, two groups struggled significantly, failing to successfully initiate the basic visualization and projection requirements.

To better understand the cognitive mechanisms underlying these varying levels of success, three groups representing high, moderate, and low understanding were selected as case studies. Each case is discussed as follows:

Case 1: High-understanding cohort

Question: Setup the triple integral to find the volume of the solid that is bounded above by $z = 8 - x^2 - y^2$ and below by $z = x^2 + y^2$. Use suitable coordinates to setup the integral.

Correct answer: $\int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r \, dzdrd\theta$

Figure 1 illustrates the work of the cohort demonstrating a high level of understanding. This group successfully navigated the critical first step of geometric-to-algebraic translation by accurately identifying the solid's bounding surfaces: a lower paraboloid ($z = x^2 + y^2$) and an upper paraboloid ($z = 8 - x^2 - y^2$). Furthermore, they excelled in Phase 2 (Projection). Rather than guessing the bounds, they implemented an algebraic strategy, equating the two boundary surfaces to determine the intersection curve. This allowed them to accurately define the 2D region of integration as a circle with radius 2.

Demonstrating high coordinate fluency, the group seamlessly translated these Cartesian parameters into cylindrical coordinates, formulating the final triple integral without error. This case study highlights that when students possess strong foundational spatial visualization (Phase 1), the subsequent symbolic setup of the integral limits (Phase 4) naturally follows. Viewed through the lens of APOS theory, this cohort demonstrates cognitive behaviors consistent with the Process and Object stages; they have successfully interiorized the geometric transformations and can treat the bounded region as a unified mathematical object to construct an accurate integral.

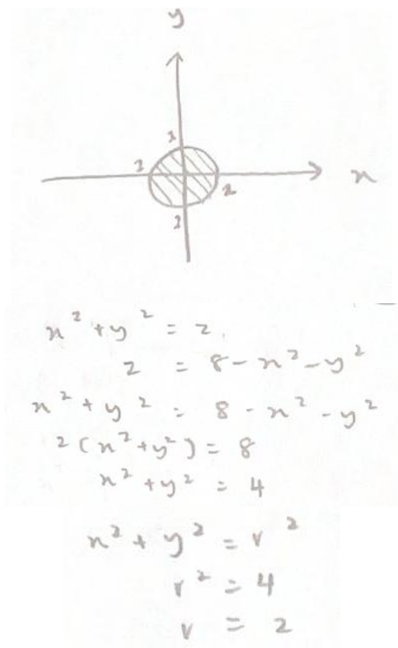
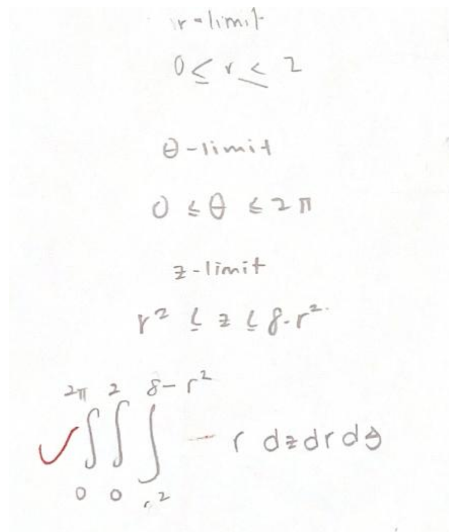
Student work: *Projection of solid onto xy-plane* 	*Setup the triple integral in cylindrical coordinates* 

Figure 1: Written Solution from the High-Understanding Cohort

Case 2: Moderate-understanding cohort

Question: Setup the triple integral to find the volume of the solid that is bound by the cylinder $x^2 + y^2 = 1$, the cone $z = \sqrt{x^2 + y^2}$ and the xy-plane. Use suitable coordinates to setup the integral.

Correct answer: $\int_0^{2\pi} \int_0^1 \int_0^r r\, dzdrd\theta$

Figure 2 illustrates the work of a cohort demonstrating a moderate level of understanding. This group was tasked with setting up a triple integral to determine the volume of a solid bounded below by the xy-plane, above by a cone ($z = \sqrt{x^2 + y^2}$), and laterally by a cylinder ($x^2 + y^2 = 1$). As evidenced by their written solution, the students experienced a significant geometric visualization failure. They misinterpreted the constraints of the solid, erroneously placing the cone as the lower bound and inventing a flat upper boundary at $z = 1$ - a plane likely derived from the intersection of the cone and the cylinder.

Despite this profound spatial misconception, the group demonstrated resilience in their procedural execution. They successfully projected the solid onto the xy-plane, accurately identifying the 2D region of integration as a circle of radius 1 based on the cylinder's base. Additionally, the cohort correctly selected cylindrical coordinates as the optimal system and properly applied the volume formula. However, the students failed to convert the Cartesian expression $\sqrt{x^2 + y^2}$ into its cylindrical equivalent within the inner integration limits. This case exemplifies the 'moderate' understanding profile: students who possess algorithmic fluency in dimension reduction (Phase 2) and coordinate selection (Phase 3) but lack the robust spatial-to-symbolic mapping skills required to correctly visualize the solid (Phase 1) and formalize its boundaries (Phase 4).

Analyzed through the lens of APOS theory, this profile strongly indicates an Action level conception. The students' approach relies heavily on substituting given equations into a memorized integral template as a step-by-step external procedure. The failure to unify the variables in the integrand, coupled with the algebraic invention of the $z = 1$ boundary, demonstrates that they are executing disconnected algebraic actions rather than interiorizing the coordinate transformation and geometric constraints as a cohesive mathematical process.

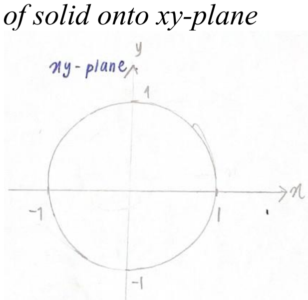
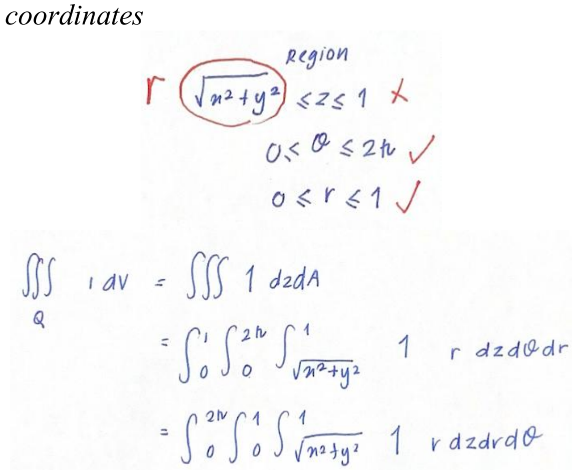
Student work: *Projection of solid onto xy-plane* 	*Setup the triple integral in cylindrical coordinates* 

Figure 2: Written Solution from the Moderate-Understanding Cohort

Case 3: Low-understanding cohort

Question: Setup the triple integral to find the volume of the solid that is bounded above the sphere $x^2 + y^2 + z^2 = 2$ and below the paraboloid $z = x^2 + y^2$. (Use cylindrical coordinates).

Correct answer: $\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\, dzdrd\theta$

Figure 3 illustrates the work of a cohort demonstrating a low level of understanding. The students were tasked with finding the volume of a solid bounded below by a paraboloid ($z = x^2 + y^2$) and above by a sphere ($x^2 + y^2 + z^2 = 2$). Their written solution shows major struggles with both visualizing the shape and setting up the mathematics. When defining the upper spherical boundary, they made a sign error, writing $\sqrt{x^2 + y^2 - 2}$ instead of $z = \sqrt{2 - x^2 - y^2}$. Furthermore, they misidentified the lower boundary by using a constant limit of $z = 1$ instead of the paraboloid equation, showing they could not correctly identify the geometric base of the solid. Although their sketch showed a correct 2D circular region with a radius of 1, they could not prove this dimension through calculation. They failed to find the intersection of the two surfaces algebraically, requiring the lecturer to step in and show the correct steps.

During the final Phase 4 setup, they experienced further issues writing the integral. Even though they explicitly listed the formulas for cylindrical coordinates, they did not use them, leaving Cartesian variables inside their integration limits. Most importantly, they misunderstood the volume formula itself. Instead of integrating the function 1, they inserted the sphere's equation directly into the integral. This highlights a serious confusion between finding a volume and integrating a specific function, marking a complete failure in setting up the problem. Viewed through the framework of APOS theory, these students exhibit pre-Action conception; they attempt rote memorization but are entirely unable to coordinate external actions to initiate the problem-solving process.

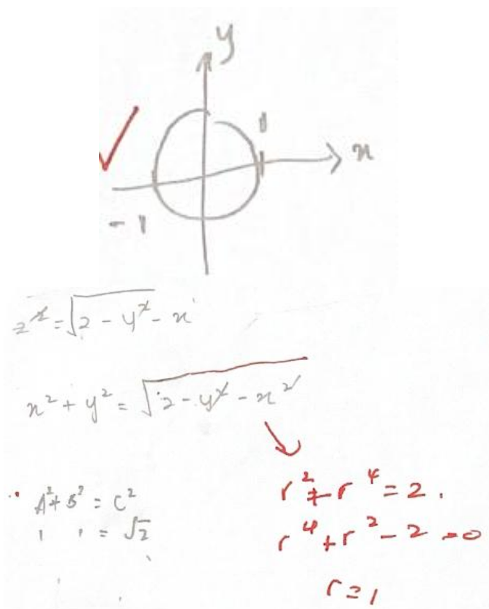
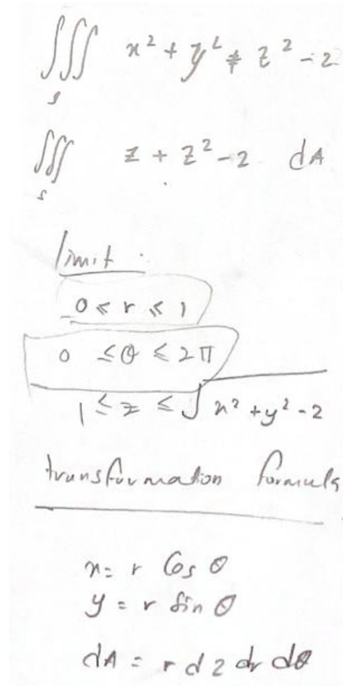
Student work: *Projection of solid onto xy-plane*	*Setup the triple integral in cylindrical coordinates*
	

Figure 3: Written Solution from the Low-Understanding Cohort

The findings indicate that the majority of students experienced significant difficulty in visualizing 3D solids. They struggled to translate textual descriptions into 3D geometric representations and to identify the correct bounding surfaces, highlighting a fundamental weakness in spatial reasoning. Viewed through the lens of APOS theory, this poor spatial reasoning prevented students from progressing to the Process stage. Instead, they remained at the Action stage, tending to memorize procedures from prior examples and rigidly applying them to new situations (Gemechu et al., 2021). This aligns closely with the observations of Gemechu et al. (2021), who noted that most of their respondents struggled with the algebraic representation of integration regions. Similarly, Padayachee and Craig (2020) highlighted that students frequently fail to recognize the region of integration due to a lack of geometric understanding. Ultimately, these procedural and setup failures are deeply rooted in inadequate spatial reasoning abilities.

To address these cognitive obstacles, instructional strategies must adapt. Martínez-Planell and Trigueros Gaisman (2012) observed that students develop a much better understanding of multivariable functions when they are explicitly exposed to their graphical representations. Building on this, to assist students in correctly setting up the limits of integration, Martínez-Planell et al. (2025) suggest employing spatial metaphors, such as identifying the "floor" and "ceiling" of a solid, which specifically helps students conceptualize the bounding surfaces required for the inner limits of a triple integral. Therefore, teaching should place a stronger emphasis on connecting the algebraic equations of surfaces with their visual graphs, particularly focusing on the intersections between different surfaces. This geometric understanding can then be significantly enhanced through the use of dynamic geometric software applications like GeoGebra. Supporting this approach, Lepellere (2025) reported that integrating GeoGebra into university-level multivariable calculus courses led to a marked improvement in the overall delivery and effectiveness of the teaching.

Conclusion

In conclusion, this study reveals that the primary learning obstacle in setting up triple integrals is visualizing the 3D solid, rather than a lack of procedural competence. While students can often project solids onto 2D planes and select appropriate coordinate systems, they fundamentally struggle to translate textual descriptions into accurate 3D geometries, frequently resulting in invented boundaries or inappropriate constant limits. Viewed through the lens of APOS theory, this behavior indicates that many students remain restricted to the Action stage, performing mechanical calculations instead of developing a deeper geometric understanding. To address this, instruction must pivot from rote calculation toward pedagogies that emphasize spatial reasoning. Educators are encouraged to integrate dynamic geometry software like GeoGebra and adopt established pedagogical strategies, such as the "floor" and "ceiling" spatial metaphors recommended by Martínez-Planell et al. (2025). These interactive tools and conceptual frameworks allow students to visualize complex surface intersections, helping them build the robust spatial schema necessary to translate geometric realities into precise algebraic boundaries.

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