MATHEMATICS AND STATISTICS

UNDERGRADUATE RESEARCH PROCEEDINGS 2025

UiTM CAWANGAN NEGERI SEMBILAN

APPROXIMATE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING TRANSFORMATION AND PERTURBATION METHOD

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Abstract

Fractional calculus extends basic calculus, specifically differentiation and integration to non-integer orders which include real and complex numbers, enabling more accurate modelling of actual systems. The fractional differential equation (FDEs) is one kind of fractional calculus that has been widely implemented in a variety of fields, including biology, engineering, and physics. FDE extends ordinary and partial differential equations of non-integer order. This makes them more difficult to solve, since there are very few well-developed analytical methods available to solve FDE that have a complicated solution. This study explores the numerical methods, the Homotopy Perturbation Transform Method (HPTM) in solving FDE, specifically time-fractional diffusion equations, using the Riemann-Liouville definition of fractional derivatives. HPTM combines the Laplace transform as well as Homotopy Perturbation Method (HPM) that simplifies the equation and approaches the solution efficiently. Thus, the findings indicate that the suggested method is highly efficient and is able to simplify the solution of FDE. The solutions were also analyzed using Maple software and have been plotted for various fractional derivative orders, α and several numerical illustrations are provided.

Keywords: fractional derivative equation, fractional diffusion equation, homotopy perturbation method, laplace transform

1. Introduction

Fractional calculus is a generalization of basic calculus, specifically in differentiation and integration (Fernandez and Fahad, 2022). The theory covers differentiation and integration functions to non-integer orders, which include real and complex numbers (Ma et al., 2018). This is different from classical calculus which is only limited to integer order of derivatives and integrals. The theory of fractional calculus has existed since 1695. Its origins can be traced back to the interaction between L'Hôpital and Leibniz over the non-integer order of derivatives (Munoz-Pacheco et al., 2023). Fractional calculus can be applied in various areas (Khan et al., 2012). This makes it a valuable tool for researchers exploring systems with non-integer characteristics.

Fractional differential equations (FDE) belong to fractional calculus, which extends ordinary and partial differential equations to non-integer orders. This kind of equation is needed in various complex systems in real life. It is important in modelling nonlinear mechanisms and structures in various fields (Khan et al., 2012). For example, in medicine, biology, chemistry, mathematics, physics, engineering, and economics (including control theory and the oil industry) (Hailat et al., 2019). FDE is also a useful tool for characterizing long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviour, power laws, and even allometric scaling laws (Zheng et al., 2019).

When referring to the fractional derivatives, several mathematicians approach this problem from different perspectives and provide various definitions. It includes fractional derivatives of Riemann-Liouville, Caputo, Hadamard, Weyl, and Riesz (Oliveira and Machado, 2014). This study will focus on fractional derivative in the Riemann-Liouville approach.

It is important to highlight that the FDE structure is more challenging to solve than the classical differential equation and is also very difficult to solve analytically. This because FDE solutions, which have non-integer order, may not have simple closed-form expressions. Therefore, numerical solutions are usually used to approximate the solution of this equation. There are many numerical methods mentioned in past research, among them are homotopy perturbation method (HPM), the variation iteration method

(VIM), the Adomian decomposition method (ADM), and the Bessel collocation method (BCM) (Hailat et al., 2019).

This study will focus on the homotopy perturbation transform method (HPTM) which refers to the combination of the Laplace transform and the homotopy perturbation method (HPM). HPTM will be used to obtain an approximate solution for FDE. Additionally, time-fractional diffusion equation is a form of FDE that describes the free movement of particles modelled by the classical diffusion equation (Kumar et al., 2012). Particularly, in this study, HPTM will be used to solve the time-fractional diffusion equations in the Riemann-Liouville sense. The general form of the time-fractional diffusion equation is

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \psi \frac{\partial u}{\partial x}, \tag{1}$$

where ψ is the diffusion coefficient. In fractional diffusion, the order of the derivative, α is a non-integer, allowing for more flexibility in modeling anomalous diffusion phenomena.

The structure of this article is as follows. In the following sections, we will briefly introduce a few preliminary concepts, such as the homotopy perturbation method (HPM), the Laplace transform and the Riemann-Liouville derivative. The details of the proposed method are presented in a subsequent section. Following that, some examples are provided to demonstrate the application of the suggested method in solving fractional-order diffusion equations using the Riemann-Liouville derivative. We also presented the numerical experiments that show the validity of the proposed method. The final section emphasises the conclusion and we made some recommendations.

2. Preliminaries

We will concentrate on the following time-fractional order diffusion equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(F(x)u), \tag{2}$$

where the fractional derivative term is in the Riemann-Liouville derivative sense. The parameter α refers to order of the time-fractional derivative term with $0 < \alpha < 1$.

2.1. *Riemann-Liouville derivative*

In this section, we will briefly present some basic definitions related to the Riemann-Liouville fractional derivative.

Definition 2.1. The Riemann-Liouville fractional derivative is defines as

$$^{RL}D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt \tag{3}$$

Equation (3) is the most widely known definition of the fractional derivative; it is usually called the Riemann–Liouville definition.

Definition 2.2. The Laplace transform of the Riemann-Liouville fractional derivative is defines as

$$\mathscr{L}\{D_x^\alpha f(x)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k D_x^{\alpha-k-1} f(0) \tag{4}$$

where $n - 1 < \alpha \leq n$.

2.2. *Laplace transform*

Laplace transform will be applied to simplify the equation in the first phase. The following steps will describe how to apply Laplace transform to simplify the FDE, and most of these steps are taken from Kumar et al. (2012).

For an overview of the HPTM procedure, the following is a nonlinear fractional differential equation:

$$D_t^\alpha u(x,t) + R(x)u(x,t) + N(x)u(x,t) = q(x,t), \;\; t > 0, \tag{5}$$

where N is the nonlinear operator, R is the linear operator and $q(x,t)$ is continuous function.

Laplace transform will be applied to both sides of (5):

$$\mathscr{L}\{D_t^\alpha u(x,t)\} + \mathscr{L}\{R(x)u(x,t) + N(x)u(x,t)\} = \mathscr{L}\{q(x,t)\}. \tag{6}$$

The derivative property of the Laplace transform (4) is applied in (6)

$$\mathscr{L}\{u(x,t)\} = s^{-\alpha} D_t^{\alpha-1}u(x,0) + s^{-\alpha}\mathscr{L}\{q(x,t)\} - s^{-\alpha}\mathscr{L}\{R(x)u(x,t) + N(x)u(x,t)\}. \tag{7}$$

The inverse Laplace transform is then applied to both sides to get the solution

$$u(x,t) = G(x,t) - \mathscr{L}^{-1}\big\{s^{-\alpha}\mathscr{L}\{R(x)u(x,t) + N(x)u(x,t)\}\big\}, \tag{8}$$

where $G(x,t)$ defined the term arising form the source term and the prescribed initial conditions.

2.3. *Homotopy perturbation method*

Homotopy perturbation method (HPM) is one of the analytical approaches to estimate the solution of FDE (Khan et al., 2012). HPM is needed to obtain efficient analytical and numerical methods since it is known for its efficiency and simplicity (Javeed et al., 2019).

In the second phase, HPM will be implemented to approximate the solution. The solution can be expressed as a power series in p:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{9}$$

where the homotopy parameter, p is considered as a small parameter.

The nonlinear term can be expressed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{10}$$

where H_n is the He's polynomial

$$H_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right], \; n = 0, 1, 2, 3, \ldots. \tag{11}$$

Substituting (9), (10) and (11) into (8) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\mathscr{L}^{-1}\big\{s^{-\alpha}\mathscr{L}\{R\sum_{n=0}^{\infty} p^n u_n(x,t) + N\sum_{n=0}^{\infty} p^n H_n(u)\}\big\} \tag{12}$$

Equating terms to the same power of p on both sides will get

$$p^0 : \; u_0(x,t) = G(x,t) \tag{13}$$

$$p^n : \; u_n(x,t) = \mathscr{L}^{-1}\big\{s^{-\alpha}\mathscr{L}\{Ru_{n-1}(x,t) + H_{n-1}(u)\}\big\}, \; n > 0. \tag{14}$$

3. Homotopy Perturbation Transform Method

This Homotopy perturbation transformation method (HPTM) is a method which combines two specific methods, the Laplace transform and the homotopy perturbation method (HPM). Laplace transform is applied to simplify the equation. Then, we implement the homotopy perturbation method to approximate the solution of fractional differential equations (FDE).

There are three phases in solving fractional differential equations (FDE). The first phase is to apply the Laplace transform to simplify the equation. The second phase is to apply homotopy perturbation method (HPM) to approximate the solution. The next section explains how HPTM is used to solve the equations.

4. Numerical Examples

The purpose of using Homotopy perturbation method (HPTM) to solve FDE is to prove its simplicity and efficiency. In order to verify the efficiency of the proposed method, a comparison between the approximate solution obtained and the exact solution is carried out. Approximate solutions will be evaluated at different values of α. Moreover, the results of this study will also be analysed using Maple software. Therefore, in this article, apart from manual calculations, Maple software will also be used to test the efficiency of HPTM by comparing the approximate solutions at different α values with the exact solutions.

There are two examples to demonstrate how HPTM was being applied to solve time-fractional diffusion equations.

Example 4.1. Consider the following time-fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial}{\partial x} x u(x,t) \tag{15}$$

with the initial condition $u(x,0) = 1$.

Applying the steps as explained in Section 2 and using $\alpha = 0.5$, we get

$$u(x,t) = \frac{2}{\pi} + \mathscr{L}^{-1}\{s^{-1/2}\mathscr{L}(D_x^2 u(x,t) + D_x x u(x,t))\}. \tag{16}$$

Substituting (12) into (16) and equating terms to the same order of p we get

$$\begin{aligned}
p^0 &: \ u_0(x,t) = \frac{2}{\pi} \\
p^1 &: \ u_1(x,t) = \frac{4\sqrt{t}}{\pi^{3/2}} \\
p^2 &: \ u_2(x,t) = \frac{2t}{\pi} \\
p^3 &: \ u_3(x,t) = \frac{t^2}{\pi} \\
&\vdots
\end{aligned}$$

As shown above, the sample solution for Example 4.1 with $\alpha = 0.5$ by using HPTM has been achieved. We also show our numerical result graphically in Figure 1 for the comparison of approximation between different values of fractional order.

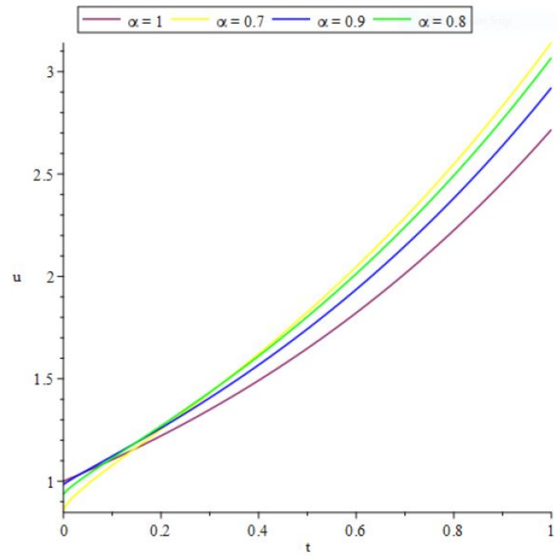

Figure 1: Comparison of solution $u(x,t)$ at different values of α in Example 4.1

From Figure 4.1, it is observed that as α decreases, the solution $u(x,t)$ increases for the same values of x and t. This indicated that when the value of $\alpha < 1$, the equation incorporates fractional effects, which increase $u(x,t)$.

Example 4.2. Consider the following time-fractional diffusion equation

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \tag{17}$$

with the initial condition $u(x,y,0) = \sin x \sin y$.

Applying the steps as explained in Section 2 and using $\alpha = 0.5$, we get

$$u(x,y,t) = \frac{2\sin x \sin y}{\pi} + \mathscr{L}^{-1}\{s^{-1/2}\mathscr{L}(D_x^2 u(x,t) + D_y^2 u(x,t))\}\}. \tag{18}$$

Substituting (12) into (18) and equating terms to the same order of p we get

$$\begin{aligned}
p^0 &: \ u_0(x,t) = \frac{2\sin x \sin y}{\pi} \\
p^1 &: \ u_1(x,t) = \frac{-8\sin x \sin y\sqrt{t}}{\pi^{3/2}} \\
p^2 &: \ u_2(x,t) = \frac{8t\sin x \sin y}{\pi} \\
p^3 &: \ u_3(x,t) = \frac{-64t^2\sin x \sin y}{3\pi^{3/2}} \\
&\vdots
\end{aligned}$$

The calculation above is the sample solution for Example 4.2 when $\alpha = 0.5$ by using HPTM. The comparison of approximation between different values of fractional order, α is presented graphically in Figure 2 below.

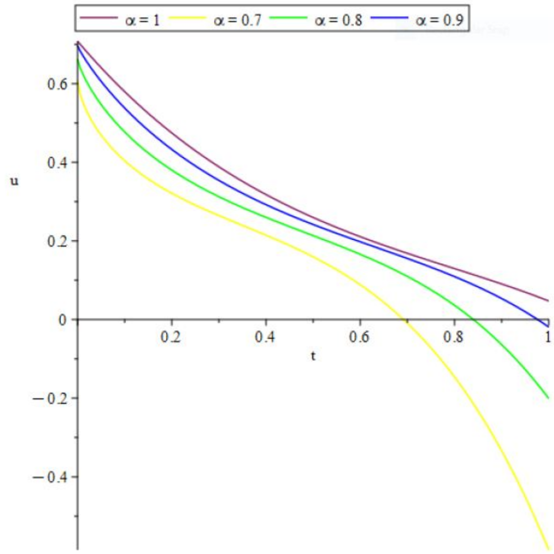

Figure 2: Comparison of solution $u(x, t)$ at different values of α in Example 4.2

From Figure 2, the solution obtained for Example 4.2 with the HPTM method appears to decrease very quickly with the increase in α at the value of $x = y = 1$. We can see a similar behaviour in the approximate solution, $u(x, y, t)$ for this example, same as in Example 4.1 when α has evolved.

Figure 3 illustrates the comparative solution between the exact solution and the approximate solution using HPTM for $\alpha = 1$.

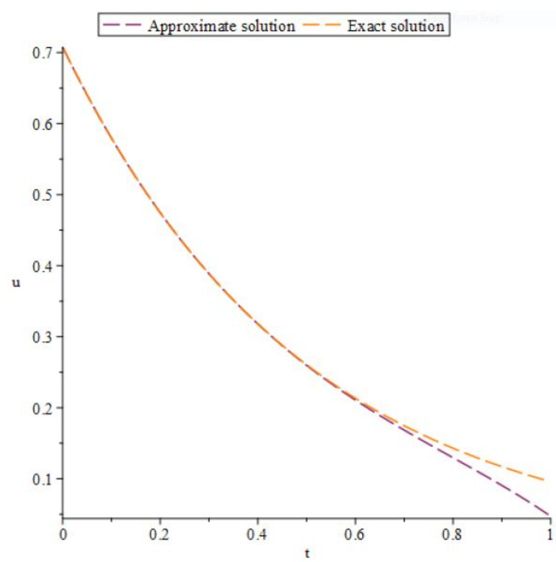

Figure 3: Comparison of exact solution and approximate solution for $\alpha = 1$

Here, we compare our solution, (18) with the exact solution, i.e. $u(x, t) = e^{-2t} \sin x \sin y$ having $\alpha = 1$ (Kumar et al., 2012). From the numerical result, we obtained a good result in comparison to the exact solution.

5. Conclusion

This study's main contribution is implementing a method to solve FDE in the Riemann-Liouville sense. In this section, the convergence of the FDE solution can be shown by using HPTM, and it proves that this method is easy to use than other methods. The efficiency of the FDE solution obtained by using HPTM with different values of α. The graph shown in Figure 1 and Figure 2 is a comparison graph of the approximate solution for different values of $\alpha = 0.7, 0.8, 0.9, 1$. When $\alpha = 1$, the equation corresponds to a standard differential equation of integer order. As $\alpha < 1$, the equation incorporates fractional derivative effects, which increase or decrease $u(x,t)$. This behaviour in fractional derivative equations shows that the order of derivative, α governs the influence of past states or memory effects on the current states of the equation.

The main goal of this study was to apply the Homotopy Perturbation Transformation Method (HPTM) to find approximate solutions for fractional differential equations (FDE). The achievement of this objective was eased by the application of the method that based on the Riemann-Liouville derivative on a specific class of FDE.

The selection of HPTM resulted from its unique feature: it balances analytical and numerical efficiency. HPTM is considered the convenient method for dealing with FDE problems. This method offers the advantage of a simpler calculation steps when compared to several other numerical and analytical techniques. The Variational Iteration Method (VIM) provides a particularly useful example for comparison. Selecting an appropriate multiplier for the mathematical problem is important in the VIM. This selection process can necessitate meaningful manual adjustment of the initial conditions. This study reveals the important potential of HPTM as an effective alternative for solving FDE, thus contributing to at least some progress in several future applications within the field of FDE.

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