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SOLVING HEAT EQUATION USING FINITE DIFFERENCE METHOD

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Abstract

This paper presents a numerical solution of the one-dimensional heat equation using the Crank-Nicolson finite difference method. The heat equation is a second-order partial differential equation that models the distribution of heat over time in a given medium. Due to the limitations of analytical methods in solving problems with complex boundary conditions, numerical approaches are essential. In this study, we focus on a test problem involving Dirichlet boundary conditions at both ends of the domain. The continuous heat equation is discretized in both space and time to form a system of linear algebraic equations. The Crank-Nicolson scheme is selected for its stability and accuracy. Numerical implementation is carried out using Wolfram Mathematica, which also enables the generation of 3D surface plots for better visualization. Error analysis is performed by comparing the numerical solution to the exact analytical solution using the L_2 norm and maximum absolute error. Results indicate that the method yields accurate and stable approximations for the heat distribution over time.

Keywords: One-dimensional, heat equation, Crank-Nicolson scheme, Dirichlet boundary condition, numerical method

1. Introduction

The heat equation is one of the most widely studied partial differential equations in applied mathematics and physics due to its role in modeling the diffusion of heat over time. It has many real-world applications such as in thermal conduction, fluid dynamics, and material sciences (Mebrate, 2015). The analytical solutions for the heat equation are often limited to simple geometries and boundary conditions, which may not be suitable for more complex or practical situations (Nolasco et al., 2019).

To address these limitations, numerical methods such as the Finite Difference Method (FDM) are commonly used. Among various FDM techniques, the Crank-Nicolson method is known for being unconditionally stable and second-order accurate in both time and space (Mohebbi & Dehghan, 2010). This makes it particularly effective for solving parabolic equations like the heat equation. In this study, the Crank-Nicolson method is used to simulate a one-dimensional heat conduction problem with Dirichlet boundary conditions. The implementation and visualization are carried out using Wolfram Mathematica. The numerical results are then validated through error analysis by comparing them with the known analytical solution.

In this study, the Crank-Nicolson scheme is applied to solve a one-dimensional heat conduction problem with Dirichlet boundary conditions imposed at both ends of the domain. The numerical experiment is carried out using Wolfram Mathematica, which provides a flexible environment for both computation and visualization. The heat distribution is examined through surface plots and the accuracy of the method is evaluated using error metrics such as the L_2 norm and $L\infty$ norm. The goal is to demonstrate the effectiveness of the Crank-Nicolson approach in producing accurate numerical results for heat conduction problems.

2. Literature review

The heat equation is a fundamental partial differential equation (PDE) that models the distribution of temperature over time. Due to the complexity of real-world problems, such as irregular geometries and time-varying boundary conditions, analytical solutions are often impractical, making numerical methods essential (Mebrate, 2015). One of the most widely used techniques is the Finite Difference Method (FDM), which transforms the continuous heat equation into a system of algebraic equations through spatial and temporal discretization (Brezzi et al., 2005; Adamiec-Wojcik et al., 2013). Among numerical schemes, the Crank-Nicolson method, stands out for its second-order accuracy and unconditional stability in both space and time (Liu & Hao, 2022; Mohebbi & Dehghan, 2010). It offers a balanced approach between the explicit and implicit methods, making it especially effective for transient heat conduction. Studies have shown its advantages over Forward Time Centered Space (FTCS) schemes, which suffer from stability limitations (Mojumder et al., 2023; Loskor & Sarkar, 2022). Although Crank-Nicolson requires solving a system of equations at each time step, its accuracy and robustness justify the computational effort.

Boundary conditions also significantly impact simulation accuracy. Dirichlet conditions, which fix temperature at boundaries and Neumann conditions, which fix heat flux are both commonly applied in FDM based heat simulations (Hajrulla et al., 2023; Nolasco et al., 2019). Chai et al. (2020) successfully used finite difference techniques with level set and ghost fluid methos to handle complex domains with mixed boundary conditions, proving the adaptability of FDM in irregular geometries. Furthermore, the distinction between homogeneous and non-homogeneous heat equations is vital. Homogeneous equations, with uniform properties and no internal heat source, are easier to solve and often used in steady-state simulations (Ahmad & Gunel, 2023). In contrast, non-homogeneous equations involve variable material properties or internal heat sources and require more advanced discretization methods. Safari (2024) demonstrated the effectiveness of combining Crank-Nicolson with meshless techniques to handle irregular material distributions and boundary complexities.

In summary, the FDM particularly the Crank-Nicolson scheme remains one of the most reliable, accurate and flexible methods for solving both homogeneous and non-homogeneous heat equations under a variety of boundary conditions.

3. Methodology

The process of solving that one-dimensional heat equation in this study begins with the formulation of the standard heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $T =$ temperature, $t =$ time, $x =$ spatial coordinates, and $\alpha =$ thermal diffusivity are constant.

In real-world applications, exact analytical solutions are often only available for very simple initial and boundary conditions. When dealing with more complex or practical cases, such as irregular domains or mixed boundary types, numerical methods like the Finite Difference Method (FDM) become essential. These methods allow the approximation of solutions by discretizing the continuous domain, enabling simulation and analysis that would otherwise be impossible.

To solve this, the Finite Difference Method (FDM) is used to discretize the spatial and temporal domains. The Crank-Nicolson method is applied as the chosen scheme for time discretization. It is known for its second-order accuracy in both time and space as well as its numerical stability. The equation is thus transformed into a system of algebraic equations that can be solved computationally.

The implementation is carried out in a one-dimensional spatial domain with uniform grid spacing. Wolfram Mathematica is utilized to perform numerical computations and to generate visual outputs, including line plots and surface plots to illustrate the evolution of temperature over time. The accuracy of the results is evaluated through error analysis by comparing the numerical solution with the exact analytical solution using both the L_2 norm and the maximum absolute error, L∞ norm.

To evaluate the accuracy of the numerical solution, two error metrics were used:

- The L2 norm, which measures the root mean square error

$$L_2\ norm = \sqrt{\sum_{i=1}^{N_X} (T_{numerical}(i,t) - T_{analytical}(i,t))^2} \tag{2}$$

- The maximum absolute error (L∞ norm) which identifies the largest deviation between the numerical and analytical solutions

$$L_\infty\ norm = max_i \left| T_{numerical}(i,t) - T_{analytical}(i,t)) \right| \tag{3}$$

4. Results and Discussion

This study examines the temperature distribution along one-dimensional rod governed by the heat equation:

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{dx^2} \qquad \text{for } 0 \le x \le L,\ t \ge 0 \tag{4}$$

Initial condition $\qquad \phi(x,0) = f(x)$ (5)

Boundary condition $\qquad \phi(0,t) = \phi_0$ (6)

$$\phi(L,t) = \phi_L \tag{7}$$

The numerical simulation was carried out using Wolfram Mathematica. The spatial domain is divided uniformly from $x = 0$ to $x = 10$, and the Crank-Nicolson scheme is applied at each time step to compute the temperature distribution. Both ends of the rod are held at fixed temperatures (Dirichlet boundary conditions). Figure 1 displays a combined 3D surface plot of the temperature distribution obtained using the Crank-Nicolson method and the exact analytical solution. The visual comparison indicates that both surfaces are nearly identical, highlighting the accuracy of the numerical approach.

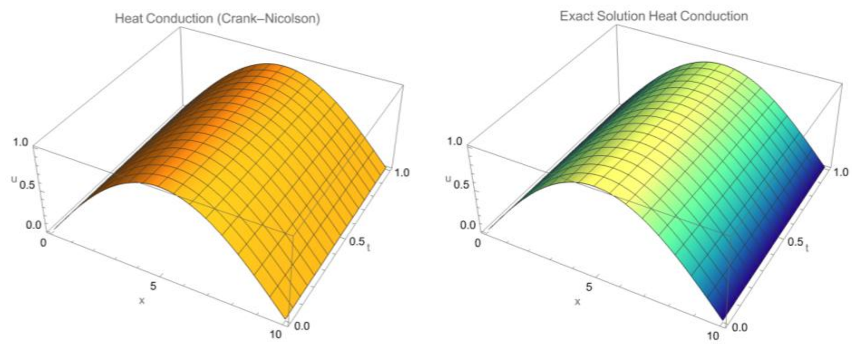

Figure 1: 3D-plot of Crank-Nicolson Solution and the Exact Solution

To evaluate the accuracy of the numerical method, results at selected spatial points and a fixed time level were compared to the exact values. Table 1 presents the exact solution, numerical results and the corresponding absolute errors.

Table 1: Comparison of Crank-Nicolson and Exact Solutions with Absolute Error

x	**Exact Solution**	**FDM (Ucn)**	**Absolute Error**
0	0.0000000	0	0.0000000
1	0.2799750	0.2799755	5.6760409×10^{-7}
2	0.5325441	0.5325451	1.0796471×10^{-6}
3	0.7329840	0.7329855	1.4860068×10^{-6}
4	0.8616744	0.8616761	1.7469058×10^{-6}
5	0.9060181	0.9060199	1.8368054×10^{-6}
6	0.8616744	0.8616761	1.7469058×10^{-6}
7	0.7329840	0.7329855	1.4860068×10^{-6}
8	0.5325441	0.5325451	1.0796471×10^{-6}
9	0.2799750	0.2799755	5.6760409×10^{-7}
10	0.0000000	0	0.0000000
$L_2 - norm$	−	−	0.0000184
$L\infty - norm$	−	−	1.8368054×10^{-6}

The computed error metrics are:
- L_2 norm: 0.0000184
- L∞ norm: 1.8368054×10^{-6}

These small errors further confirm that the numerical solution achieved in this project is both accurate and stable. One reason for the small error in Problem 2 could be due to its Dirichlet boundary conditions, which generally provide better numerical stability by controlling the boundary temperature effectively. Further, discretization parameters including spatial step also contributed significantly to the minimization of numerical errors.

To summarize, the outcome Problem 2 using the graphical comparison, data table analysis, and error norm calculation indicate that the Crank-Nicolson finite difference finds a suitable and reasonable numerical scheme to solve one dimensional heat conduction problem with the Dirichlet boundary conditions.

5. Conclusion

This study demonstrated how the Finite Difference Method (FDM), specifically the Crank-Nicolson scheme, is highly accurate and stable approach for solving the one-dimensional heat equation under Dirichlet boundary conditions. Using Wolfram Mathematica for implementation and visualization, the numerical solution was compared against the exact analytical solution. The results showed strong agreement, supported by low L_2 norm (0.0000184) and L_∞ error norm (1.8368054×10^{-6}) highlighting the method's numerical stability and second order accuracy in both time and space. Overall, the Crank-Nicolson method proves to be a robust and reliable approach for modeling heat conduction problems with fixed boundary temperatures. For future work, this approach can be extended to more complex scenarios such as time-varying or Robin boundary conditions, non-uniform grids and higher dimensional problems.

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