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A COMPARATIVE STUDY OF FIRST ORDER DIFFERENTIAL EQUATION BY USING BUTCHER'S FIFTH ORDER METHOD AND DORMAND PRINCE METHOD

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Abstract

Nonlinear differential equations present considerable challenges when tackled with fundamental approaches due to their inherent complexity. Numerical methods like Euler, Taylor, and Runge-Kutta provide effective alternatives for solving these equations. The Butcher's fifth-order method and Dormand Prince method are part of the Runge-Kutta family that have been used to address higher-order differential equations and have been applied in science and mathematics fields. Both methods are known for their high accuracy in solving such problems, as they can produce better approximation to the exact solutions. However, there are limited studies on solving the first-order Bernoulli and Riccati differential equations using these methods. Therefore, this study attempts to solve both differential equations using the Butcher's fifth-order and Dormand Prince methods with a fixed step size of h = 0.1. Both methods were analysed by comparing their absolute errors against the exact solution. The results showed that both Butcher's and Dormand Prince methods are capable of solving first-order Bernoulli and Riccati differential equations. However, in terms of accuracy, the findings highlight that the Butcher's fifth-order method is better at solving problems 1 and 3, while the Dormand Prince method is suitable for solving problems 2 and 4, demonstrating more accurate approximations due to smaller errors. It proved that both methods are accurate and suitable for solving these problems. However, the accuracy of both methods depends on the specific problems. Future research could investigate their application to higher-order nonlinear differential equations with different step sizes. The accuracy and efficiency of both methods could be further evaluated by comparing relative errors, computational work, or execution time.

Keywords*:* Differential equation, Bernoulli and Riccati problems, Error analysis, Butcher's fifth order method, Dormand Prince method

Introduction

Many real-world situations in science and engineering can be modelled using differential equations. Differential equations are particularly crucial to scientists and engineers, as they serve as essential tools for the mathematical modelling of any circumstances requiring a rate of change [1]. A differential equation is known as an equation that involves an independent variable, a dependent variable, and the derivative of the dependent variable with respect to the independent variable. These differential equations can be classified into two categories: ordinary differential equations (ODEs) and partial differential equations (PDEs) [2]. An ODE is a mathematical equation that expresses one or more functions of an independent variable along with its derivatives. ODEs can be categorised into linear and nonlinear equations. There are various types of nonlinear differential equations such as second-order Abel's differential equation, Chrystal's equation, and Lagrange's equation. However, this study will focus on first-order nonlinear differential equations, specifically Bernoulli and Riccati differential equations.

Bernoulli's differential equation, introduced by Jacob Bernoulli, can be transformed from a nonlinear equation into a linear equation by substitution, where $z = y^{1-n}$, where $n \neq 0$ and $n \neq 1$ [3]. This equation is still used in real-life applications which are covered in mathematics, engineering and physics [4]. Applications include the analysis of free vibration of Euler-Bernoulli beams modelled by spatial-fractional differential equations [5], identifying the properties of q-differential equations of higher order using q-Bernoulli polynomials [6], and identifying daily traffic flow patterns based on the nonlinear grey Bernoulli model [7]. Meanwhile, Riccati differential equation is named after Jacopo Francesco Riccati. This equation has also been applied to various fields including mathematics, physics and engineering, such as optimal control, robust stability, and finding numerical solutions of Riccati differential equations using Euler and Runge-Kutta methods [8]. This equation can be solved analytically by various numerical methods such as Adomian decomposition method [9], He's variational iteration method (VIM), the homotopy perturbation method (HPM) and the iterated He's HPM [10], and a piecewise VIM [11].

In this study, Butcher's fifth order and Dormand Prince methods are used to solve the first order Bernoulli and Riccati differential equations and the results obtained will be compared to the exact solutions by calculating their errors based on absolute error.

Methodology

Two methods Butcher's fifth order and Dormand Prince methods will be represented in this section for solving first order Bernoulli and Riccati differential equations.

Butcher's fifth order method

Butcher's fifth order method is an extension of the Runge-Kutta fourth order (RK4) method, which consists of six function evaluations of intermediate slopes in obtaining the average weighted function values at different points. The article by Hossain et al. (2017) was referred to obtain the solution of the differential equations. The mathematical formula of Butcher's method is demonstrated as follows:

$$y_{n+1} = y_n + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right) \text{ for } n = 0,1,2,\ldots \tag{1}$$

where:

$$\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1 h) \\
k_3 &= f(x_n + \frac{1}{4}h, y_n + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h) \\
k_4 &= f(x_n + \frac{1}{2}h, y_n - \frac{1}{2}k_2 h + k_3 h) \\
k_5 &= f(x_n + \frac{3}{4}h, y_n + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h) \\
k_6 &= f(x_n + h, y_n - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h)
\end{aligned} \tag{2}$$

Dormand Prince method

The Dormand Prince method is one of the Runge-Kutta (RK) methods that has been widely applied in real-life applications, especially in the fields of science and chemistry, due to its high accuracy. This method involves a seven-stage function evaluations of intermediate slopes at different points to obtain the estimated solutions. However, this method has its own property known as First Same As Last (FSAL) property, only six function evaluations need to be calculated [13]. The mathematical formula of Dormand Prince method was referred from Keong and Hamzah (2018) and is presented below:

$$y_{n+1} = y_n + \frac{35}{384}k_1 + \frac{500}{1113}k_3 + \frac{125}{192}k_4 - \frac{2187}{6784}k_5 + \frac{11}{86}k_6 \text{ for } n = 0,1,2,\ldots \tag{3}$$

where:

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \tfrac{1}{5}h, y_n + \tfrac{1}{5}k_1) \\
k_3 &= hf(x_n + \tfrac{3}{10}h, y_n + \tfrac{3}{40}k_1 + \tfrac{9}{40}k_2) \\
k_4 &= hf(x_n + \tfrac{4}{5}h, y_n + \tfrac{44}{45}k_1 - \tfrac{56}{15}k_2 + \tfrac{32}{9}k_3) \\
k_5 &= hf(x_n + \tfrac{8}{9}h, y_n + \tfrac{19372}{6561}k_1 - \tfrac{25360}{2187}k_2 + \tfrac{64448}{6561}k_3 - \tfrac{212}{729}k_4) \\
k_6 &= hf(x_n + h, y_n + \tfrac{9017}{3168}k_1 - \tfrac{355}{33}k_2 + \tfrac{46732}{5247}k_3 + \tfrac{49}{176}k_4 - \tfrac{5103}{18656}k_5) \\
k_7 &= hf(x_n + h, y_n + \tfrac{35}{384}k_1 + \tfrac{500}{1113}k_3 + \tfrac{125}{192}k_4 - \tfrac{2187}{6784}k_5 + \tfrac{11}{84}k_6)
\end{aligned} \tag{4}$$

Error Analysis

The results obtained from the proposed methods will be compared with the exact solutions based on absolute error. The purpose of this comparison is to determine which method is more accurate and suitable for solving the first order Bernoulli and Riccati differential equations. The accuracy of the results can be assessed by observing the smaller errors produced by the proposed methods. The formula for absolute error is presented below:

$$E = \left| y - \bar{y} \right| \tag{5}$$

where:

E = Absolute error
y = Actual value
$\bar{y}$ = Predicted value

Illustrative Problem

The effectiveness of the proposed methods for solving first order Bernoulli and Riccati equations will be presented in this section. All the computations are performed using Maple 18 software, with the step size fixed at $h = 0.1$.

Problem 1

Consider the Bernoulli differential equation taken from [14].

$$y' = x^3y^3 - xy \tag{6}$$

with the initial conditions $y(0)=1$, and the exact solution is given by $y(x) = \dfrac{1}{1+x^2}$.

Table 3.1 Solutions of problem 1 with a step size of h = 0.1 and $x \in [0,1]$

x	Exact solution	Butcher's Fifth Order Method	Dormand Prince Method	Errors of Butcher's Method	Errors of Dormand Prince Method
0.0	1.0000000000	1.0000000000	1.0000000000	0	0
0.1	0.9900990099	0.9950371888	0.9950371905	4.9382×10^{-3}	4.9382×10^{-3}
0.2	0.9615384615	0.9805806735	0.9805806762	1.9042×10^{-2}	1.9042×10^{-2}
0.3	0.9174311927	0.9578262830	0.9578262858	4.0395×10^{-2}	4.0395×10^{-2}
0.4	0.8620689655	0.9284766890	0.9284766914	6.6408×10^{-2}	6.6408×10^{-2}
0.5	0.8000000000	0.8944271893	0.8944271914	9.4427×10^{-2}	9.4427×10^{-2}
0.6	0.7352941176	0.8574929238	0.8574929260	1.2220×10^{-1}	1.2220×10^{-1}
0.7	0.6711409396	0.8192319180	0.8192319206	1.4809×10^{-1}	1.4809×10^{-1}
0.8	0.6097560976	0.7808688061	0.7808688094	1.7111×10^{-1}	1.7111×10^{-1}
0.9	0.5524861878	0.7432941421	0.7432941461	1.9081×10^{-1}	1.9081×10^{-1}
1.0	0.5000000000	0.7071067763	0.7071067810	2.0711×10^{-1}	2.0711×10^{-1}

Problem 2

Consider the Bernoulli differential equation taken from [15].

$$y' = 2xy + 2x^3 y^2 \tag{7}$$

with the initial condition $y(0)$=1, and the exact solution is given by $y(x) = \dfrac{1}{1-x^2}$.

Table 3.2 Solutions of problem 2 with a step size of h = 0.1 and $x \in [0, 0.9]$

x	Exact solution	Butcher's Fifth Order Method	Dormand Prince Method	Errors of Butcher's Method	Errors of Dormand Prince Method
0.0	1.0000000000	1.0000000000	1.0000000000	0	0
0.1	1.0101010100	1.0101010190	1.0101010100	9.0000×10^{-9}	0
0.2	1.0416666670	1.0416666890	1.0416666650	2.2000×10^{-8}	2.0000×10^{-9}
0.3	1.0989010990	1.0989011470	1.0989010940	4.8000×10^{-8}	5.0000×10^{-9}
0.4	1.1904761900	1.1904763010	1.1904761790	1.1100×10^{-7}	1.1000×10^{-8}
0.5	1.3333333330	1.3333336150	1.3333332950	2.8200×10^{-7}	3.8000×10^{-8}
0.6	1.5625000000	1.5625008090	1.5624998480	8.0900×10^{-7}	1.5200×10^{-7}
0.7	1.9607843140	1.9607867560	1.9607834970	2.4420×10^{-6}	8.1700×10^{-7}
0.8	2.7777777780	2.7777757860	2.7777733360	1.9920×10^{-6}	4.4420×10^{-6}
0.9	5.2631578950	5.2616611750	5.2635061440	1.4967×10^{-3}	3.4825×10^{-4}

Problem 3

Consider the Riccati differential equation taken from [16].

$$y' = -\sin x + \left(\cos^2 x\right)y - \left(\cos x\right)y^2 \tag{8}$$

with the initial condition $y(0)$=1, and the exact solution is given by $y(x) = \cos x$.

Table 3.3 Solutions of problem 3 with a step size of h = 0.1 and $x \in [0,1]$

x	Exact solution	Butcher's Fifth Order Method	Dormand Prince Method	Errors of Butcher's Method	Errors of Dormand Prince Method
0.0	1.0000000000	1.0000000000	1.0000000000	0	0
0.1	0.9950041653	0.9950041644	0.9950041660	9.0000×10^{-10}	7.0000×10^{-10}
0.2	0.9800665778	0.9800665764	0.9800665792	1.4000×10^{-9}	1.4000×10^{-9}
0.3	0.9553364891	0.9553364873	0.9553364910	1.8000×10^{-9}	1.9000×10^{-9}
0.4	0.9210609940	0.9210609920	0.9210609963	2.0000×10^{-9}	2.3000×10^{-9}
0.5	0.8775825619	0.8775825600	0.8775825646	1.9000×10^{-9}	2.7000×10^{-9}
0.6	0.8253356149	0.8253356132	0.8253356178	1.7000×10^{-9}	2.9000×10^{-9}
0.7	0.7648421873	0.7648421858	0.7648421902	1.5000×10^{-9}	2.9000×10^{-9}
0.8	0.6967067093	0.6967067081	0.6967067122	1.2000×10^{-9}	2.9000×10^{-9}
0.9	0.6216099683	0.6216099672	0.6216099710	1.1000×10^{-9}	2.7000×10^{-9}
1.0	0.5403023059	0.5403023049	0.5403023084	1.0000×10^{-9}	2.5000×10^{-9}

Problem 4

Consider the Riccati differential equation taken from [17].

$$y' = 1 + y^2 \tag{9}$$

with the initial condition $y(0)=0$, and the exact solution is given by $y(x) = \tan x$.

Table 3.4 Solutions of problem 4 with a step size of h = 0.1 and $x \in [0,1]$

x	Exact solution	Butcher's Fifth Order Method	Dormand Prince Method	Errors of Butcher's Method	Errors of Dormand Prince Method
0.0	0.0000000000	0.0000000000	0.0000000000	0	0
0.1	0.1003346721	0.1003346723	0.1003346720	2.0000×10^{-10}	1.0000×10^{-10}
0.2	0.2027100355	0.2027100366	0.2027100354	1.1000×10^{-9}	1.0000×10^{-10}
0.3	0.3093362496	0.3093362523	0.3093362494	2.7000×10^{-9}	2.0000×10^{-10}
0.4	0.4227932187	0.4227932240	0.4227932183	5.3000×10^{-9}	4.0000×10^{-10}
0.5	0.5463024898	0.5463024987	0.5463024887	8.9000×10^{-9}	1.1000×10^{-9}
0.6	0.6841368083	0.6841368221	0.6841368053	1.3800×10^{-8}	3.0000×10^{-9}
0.7	0.8422883805	0.8422884002	0.8422883722	1.9700×10^{-8}	8.3000×10^{-9}
0.8	1.0296385570	1.0296385820	1.0296385330	2.5000×10^{-8}	2.4000×10^{-8}
0.9	1.2601582180	1.2601582360	1.2601581510	1.8000×10^{-8}	6.7000×10^{-8}
1.0	1.5574077250	1.5574076800	1.5574075250	4.5000×10^{-8}	2.0000×10^{-7}

The results from the proposed methods are displayed in Tables 3.1 – 3.4. The numerical solutions and errors are calculated with a fixed step size of 0.1 by using Maple 18 software. The results are compared to the exact solutions. Butcher's fifth order method gives a better approximation to the exact solution for problems 1 and 3, while Dormand Prince method demonstrates a close approximation to the exact solution for problems 2 and 4. In terms of accuracy, Butcher's fifth order method is more accurate and suitable for solving problems 1 and 3. For problems 2 and 4, Dormand Prince method is more accurate and suitable, presenting a better approximation with smaller errors produced.

Conclusion

In this paper, Butcher's fifth order and Dormand Prince methods have been applied to solve first order Bernoulli and Riccati differential equations. It can be concluded that both methods are capable of solving all the problems. However, in terms of accuracy, the accuracy of both methods varies depending on the specific problem.

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