UNIVERSITI TEKNOLOGI MARA

MATHEMATICS AND STATISTICS

UNDERGRADUATE RESEARCH PROCEEDINGS 2025

UiTM CAWANGAN NEGERI SEMBILAN

COEFFICIENT BOUND OF A GENERALIZED CLASS OF CLOSE-TO-CONVEX FUNCTIONS USING HERGLOTZ REPRESENTATION THEOREM

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Abstract

In this article, we introduce a generalized class of close-to-convex functions, $G_K(\alpha,\delta,t)$ *which denoted as* $\operatorname{Re}\left(e^{i\alpha}\dfrac{f'(z)}{g'(z)}\right)>\delta,(z\in E)$ *where* $|\alpha|\le\pi,\ \cos\alpha>\delta,\ 0\le\delta\le1,\ -1<t\le1$ *and* $g'(z)=\dfrac{1+tz}{1-z}$. *Furthermore, we find the representation theorem and coefficient bound for* $G_K(\alpha,\delta,t)$ *using Herglotz Representation Theorem.*

Keywords*:* Coefficient bound, Close-to-convex, Herglotz representation theorem

Introduction

Complex analysis is the poetry of mathematics which studies a subfield of mathematics that investigates complex numbers, including their functions, derivatives, limits, manipulation, and other mathematical properties. Geometric Function Theory (GFT) is a branch of complex analysis that studies the geometric properties of one-to-one analytic functions. It was started by the German mathematician Ludwig Bieberbach in 1916. According to Duren [1], an analytic function $f(z)$ maps one region to another in the complex plane one-to-one.

Let $\mathbb{C}$ represent complex numbers, and let $f(z)$ be a complex-valued function of the complex variable z. A function f is called univalent in a domain $D\in\mathbb{C}$ if it is one-to-one, meaning that if $z_1,z_2\in D$, $f(z_1)=f(z_2)$, then $z_1=z_2$. (Goodman, [2]). Additionally, a function is called analytic at a point $m\in D$, if it can be differentiated at every point in a neighbourhood around z_0. Let A denoted the class of function of the form

$$f(z)=z+a_2z^n+a_3z^n+\ldots+a_nz^n=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the unit disc, $E=\{z:|z|<1\}$. The function $f(z)$ is also known as normalized univalent function if it satisfies conditions of $f(z)=f'(0)-1=0$ are fixed is denoted by S.

Based on Kaharudin [3], if $f \in S$ is given by (1.1) and $f \in G_K(\alpha,\delta)$, then

$$|a_n| = \frac{2}{n}\left(\frac{1}{2} + A_{\alpha\delta}(n-1)\right), \quad n = 2,3,4...$$

where the functions in this class satisfy the condition

$$\operatorname{Re}\left\{e^{i\alpha}\frac{f'(z)}{g'(z)}\right\} > \delta, \ (z \in E)$$

with $|\alpha| < \pi, \ \cos\alpha > \delta, \quad g'(z) = \frac{1}{1-z}$.

In addition, Yahya, Soh, and Mohamad [4] stated that if $f \in S$ and $f \in G(\alpha,\delta)$ then

$$|a_n| = \frac{2}{n}\left(\frac{1}{2} + \frac{A_{\alpha\delta}(n-1)}{2}\right), \ n = 2,3,4,...$$

where the functions in this class satisfy the condition

$$\operatorname{Re}\left\{e^{i\alpha}\frac{f'(z)}{g'(z)}\right\} > \delta, \ (z \in E)$$

with $|\alpha| < \pi, \ \cos\alpha > \delta, \quad g'(m) = \frac{z}{1-z^2}$.

In the present paper, we focused on the generalized class of close-to-convex functions, $G_K(\alpha,\delta,t)$ which is denoted as S and satisfies the condition

$$\operatorname{Re}\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > \delta, (z \in E) \tag{1.2}$$

where $|\alpha| \le \pi, \ \cos\alpha > \delta, \ 0 \le \delta \le 1, \ -1 < t \le 1$ and $g'(z) = \frac{1+tz}{1-z}$. The main objectives in this study are to find representation theorem and to determine the coefficient bound for the $G_K(\alpha,\delta,t)$ by using Herglotz Representation Theorem.

Preliminaries

To obtain the main result, we applied the Herglotz Representation Theorem to derive the Representation Theorem for $G_K(\alpha,\delta,t)$.

Theorem 1

Let $f \in S$ and $f \in G_K(\alpha,\delta,t)$, then

$$\frac{e^{i\alpha}\dfrac{f'(z)(1-z)}{1+tz} - \delta - i\sin\alpha}{\cos\alpha - \delta} = p(z).$$

Proof.

Based on (1.2), let $\frac{f'(z)(1-z)}{1+tz}$ be written as

$$\frac{f'(z)(1-z)}{1+tz}=1+\sum_{n=1}^{\infty}p_n z^n . \tag{2.1}$$

Then, (2.1) can be written as

$$e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta=1+\sum_{n=1}^{\infty}p_n z^n .$$

By applying into the relation *P*, we have

$$e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta=e^{i\alpha}\left(1+\sum_{n=1}^{\infty}p_n z^n\right)-\delta . \tag{2.2}$$

By expanding the equation (2.2)

$$e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta=e^{i\alpha}+\sum_{n=1}^{\infty}e^{i\alpha}p_n z^n-\delta .$$

We can apply the definition of $e^{i\alpha}=\cos\alpha+i\sin\alpha$ into the equation,

$$e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta=\cos\alpha+i\sin\alpha+\sum_{n=1}^{\infty}e^{i\alpha}p_n z^n-\delta .$$

We divide the equation with $\cos\alpha-\delta$ to obtain P,

$$\frac{e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta-i\sin\alpha}{\cos\alpha-\delta}=\frac{\cos\alpha-\delta+\sum_{n=1}^{\infty}e^{i\alpha}p_n z^n}{\cos\alpha-\delta} .$$

Therefore,

$$\frac{e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta-i\sin\alpha}{\cos\alpha-\delta}=1+\frac{\sum_{n=1}^{\infty}\left(e^{i\alpha}p_n\right)z^n}{\cos\alpha-\delta},$$

by applying $b_n=\frac{e^{i\alpha}p_n}{\cos\alpha-\delta}$, we have

$$\frac{e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta-i\sin\alpha}{\cos\alpha-\delta}=1+\sum_{n=1}^{\infty}b_n z^n .$$

We related the function in *P* with

$$\frac{e^{i\alpha}\frac{f'(z)(1-z)}{1+tz}-\delta-i\sin\alpha}{\cos\alpha-\delta}=p(z). \tag{2.3}$$

So that, $f\in G_K(\alpha,\delta,t)$ if and only if $p(z)\in P$. Based on (2.3), it is noted that $\cos\alpha-\delta$ should always be positive which brings about the condition $\cos\alpha>\delta$ in the definition of the class $G_K(\alpha,\delta,t)$. In addition, by using an approach of Herglotz representation theorem for function in P give a representation function for $G_K(\alpha,\delta,t)$.

Now, we shall prove our main result.

Main Results

In this section, we shall focus on the coefficient bound of $G_K(\alpha,\delta,t)$.

Theorem 2

If $f \in G_K(\alpha,\delta,t)$, *then*

$$|a_n| \le \frac{1}{n}\big(t(1-2A_{\alpha\delta}n)+1+2A_{\alpha\delta}(n-1)\big), \qquad n=2,3,4,...$$

and equality is attained for each n when f is an extreme point of $G_K(\alpha,\delta,t)$.

Proof.

Given

$$p \in P \Leftrightarrow p(z)=\int_x \frac{1+xz}{1-xz}d\mu, \qquad |x|=1,$$

for some probability measure μ on the unit *X*. Rearranging Theorem 1 to make as the subject,

$$e^{i\alpha}\frac{\frac{f'(z)}{g'(z)}-\delta-i\sin\alpha}{\cos\alpha-\delta}=p(z).$$

Then,

$$e^{i\alpha}\frac{f'(z)}{g'(z)}=p(z)(\cos\alpha-\delta)+\delta+i\sin\alpha,$$

by replacing $\cos\alpha-\delta=A_{\alpha\delta}$, we have

$$e^{i\alpha}f'(z)=g'(z)\left[A_{\alpha\delta}p(z)+\delta+i\sin\alpha\right].$$

Therefore,

$$f'(z)=e^{-i\alpha}\left[g'(z)\left[A_{\alpha\delta}p(z)+\delta+i\sin\alpha\right]\right] \tag{3.1}$$

which implies $A_{\alpha\delta}>0$. From (3.1), we have,

$$f'(z)=e^{-i\alpha}\left[g'(z)\right]\left[(\cos\alpha-\delta)\int_x\frac{1+xz}{1-xz}d\mu(x)+\delta+i\sin\alpha\right].$$

Then, follows that

$$f(z)=e^{-i\alpha}\int_0^z\left[g'(\phi)\right]\left[(\cos\alpha-\delta)\int_x\frac{1+x\phi}{1-x\phi}d\mu(x)+\delta+i\sin\alpha\right]d\phi.$$

Thus,

$$f(z)=\int_0^z\left[\int_x\left(\frac{1+xt\phi}{1-x\phi}\right)\left(\frac{1+x\phi\left[e^{-i\alpha}(\cos\alpha-\delta)-e^{-i\alpha}(\delta+i\sin\alpha)\right]}{1-x\phi}\right)d\mu(x)\right]d\phi,$$

and

$$f(z)=\int_0^z\left[\int_x\left(\frac{1+xt\phi}{1-x\phi}\right)\left(\frac{1+x\phi\left[e^{-i2\alpha}-2\delta e^{-i\alpha}\right]}{1-x\phi}\right)d\mu(x)\right]d\phi.$$

Then,

$$f(z)=\int_0^z\left[\int_x\left(\frac{(1+xt\phi)\left[1+x\phi\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\right]}{(1-x\phi)^2}\right)d\mu(x)\right]d\phi.$$

Now, let $j=e^{-i2\alpha}-2\delta e^{-i\alpha}$,

$$f(z)=\int_0^z\left[\int_x\left(\frac{(1+xt\phi)(1+x\phi j)}{(1-x\phi)^2}\right)d\mu(x)\right]d\phi. \tag{3.2}$$

Based on (3.2), we apply partial fraction to simplify this equation. Therefore

$$f(z)=\int_0^z\left[\int_x\left(jt-\frac{t(2j+1)+j}{1-k}+\frac{t(1+j)+j+1}{(1-k)^2}\right)d\mu(x)\right]d\phi.$$

Therefore, substitute $k=x\phi$ and $j=e^{-i2\alpha}-2\delta e^{-i\alpha}$,

$$\begin{aligned}f(z)=\int_0^z\Bigg[\int_x t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)&\\ +\frac{1}{1-x\phi}\Big(-t\left(2\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)+1\right)-\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)&\\ +\frac{t\left(1+e^{-i2\alpha}-2\delta e^{-i\alpha}\right)+1+e^{-i2\alpha}-2\delta e^{-i\alpha}}{1-x\phi}\Big)d\mu(x)\Bigg]d\phi.&\end{aligned}$$

Since,

$$1+e^{-i2\alpha}-2\delta e^{-i2\alpha}=2A_{\alpha\delta}e^{-i\alpha},$$

therefore, we have

$$f(z)=\int_0^z\left[\int_x t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\right.$$
$$+\frac{1}{1-x\phi}\left(-2t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)-t-\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\right. \tag{3.3}$$
$$\left.\left.+\frac{2tA_{\alpha\delta}e^{-i\alpha}+2A_{\alpha\delta}e^{-i\alpha}}{1-x\phi}\right)d\mu(x)\right]d\phi$$

and

$$f(z)=\int_0^z\left[t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\int_x d\mu(x)\right.$$
$$+\left(\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)(-2t-1)-t\right)\int_x\sum_{n=0}^{\infty}(x\phi)^n d\mu(x) \tag{3.4}$$
$$\left.+\left(2A_{\alpha\delta}e^{-i\alpha}(t+1)\right)\int_x\sum_{n=0}^{\infty}(n+1)(x\phi)^n d\mu(x)\right]d\phi.$$

From (3.4), substitute $n=0$, then

$$\begin{aligned} f(z)&=t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)+\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)(-2t-1)-t+2A_{\alpha\delta}e^{-i\alpha}(t+1)\\ &=-t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)-\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)-t+2A_{\alpha\delta}e^{-i\alpha}(t+1)\\ &=2A_{\alpha\delta}e^{-i\alpha}-\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\end{aligned}$$

Substitute $A_{\alpha\delta}=\cos\alpha-\delta$ and $\cos\alpha=\dfrac{e^{i\alpha}+e^{-i\alpha}}{2}$, thus $f(z)=1$.

Then, rewrite equation (3.4) will yield

$$f(z)=\int_0^z\left[1+t\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\int_x d\mu(x)\right.$$
$$+\left(\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)(-2t-1)-t\right)\int_x\sum_{n=1}^{\infty}(x\phi)^n d\mu(x)$$
$$\left.+\left(2A_{\alpha\delta}e^{-i\alpha}(t+1)\right)\int_x\sum_{n=1}^{\infty}(n+1)(x\phi)^n d\mu(x)\right]d\phi.$$

Integrating with respect to ϕ gives us,

$$f(z) = z\left(1 + t\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)\right) + \left(\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)(-2t-1) - t\right)\int_x \sum_{n=1}^{\infty} (x)^n d\mu(x)\left(\frac{z^{n+1}}{n+1}\right) + \left(2A_{\alpha\delta}e^{-i\alpha}(t+1)\right)\int_x \sum_{n=1}^{\infty}(n+1)(x)^n d\mu(x)\left(\frac{z^{n+1}}{n+1}\right)$$

and

$$f(z) = z\left(1 + t\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)\right) + \left(\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)(-2t-1) - t\right)\int_x \sum_{n=2}^{\infty} (x)^{n-1} d\mu(x)\left(\frac{z^{n}}{n}\right) + \left(2A_{\alpha\delta}e^{-i\alpha}(t+1)\right)\int_x \sum_{n=2}^{\infty}(n)(x)^{n-1} d\mu(x)\left(\frac{z^{n}}{n}\right).$$

Rearranging the equation, we have

$$f(z) = z\left(1 + t\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)\right) + \sum_{n=2}^{\infty}\left[\left(\frac{\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)(-2t-1) - t}{n}\right)\int_x (x)^{n-1} d\mu(x) + \left(\frac{2A_{\alpha\delta}e^{-i\alpha}(t+1)}{n}\right)\int_x (n)(x)^{n-1} d\mu(x)\right]z^n \tag{3.5}$$

and from (1.1), we have that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. By comparing (3.5) with (1.1), we have found that

$$a_n = \left[\left(\frac{\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)(-2t-1) - t}{n}\right)\int_x (x)^{n-1} d\mu(x) + \left(\frac{2A_{\alpha\delta}e^{-i\alpha}(t+1)}{n}\right)\int_x (n)(x)^{n-1} d\mu(x)\right]z^n.$$

Rearranging the equation,

$$|a_n| = \frac{\left|-2t\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right) - e^{-i2\alpha} + 2\delta e^{-i\alpha} - t + 2tA_{\alpha\delta}e^{-i\alpha}n + 2A_{\alpha\delta}e^{-i\alpha}n\right|}{n}\int_x \left|(x)^{n-1}\right| d\mu(x).$$

Again since $-e^{-i2\alpha}+2\delta e^{-i\alpha}=1-2A_{\alpha\delta}e^{-i\alpha}$ and $\left|e^{-i\alpha}\right|=1$,

$$\left|a_n\right|=\frac{\left|2t\left(1-2A_{\alpha\delta}e^{-i\alpha}\right)+1-2A_{\alpha\delta}e^{-i\alpha}-t+2tA_{\alpha\delta}e^{-i\alpha}n+2A_{\alpha\delta}e^{-i\alpha}n\right|}{n}\int_x\left|(x)^{n-1}\right|d\mu(x)$$

$$=\frac{\left|t+1+2A_{\alpha\delta}e^{-i\alpha}(n-1)-2tA_{\alpha\delta}e^{-i\alpha}n\right|}{n}\int_x\left|(x)^{n-1}\right|d\mu(x).$$

Thus,

$$\left|a_n\right|=\frac{\left|t\left(1-2A_{\alpha\delta}e^{-i\alpha}n\right)+1+2A_{\alpha\delta}e^{-i\alpha}(n-1)\right|}{n}\int_x\left|(x)^{n-1}\right|d\mu(x)$$

$$\leq\frac{t\left(1-2A_{\alpha\delta}n\right)+1+2A_{\alpha\delta}(n-1)}{n}\int_x\left|(x)^{n-1}\right|d\mu(x)$$

$$=\frac{1}{n}\left(t\left(1-2A_{\alpha\delta}n\right)+1+2A_{\alpha\delta}(n-1)\right)$$

as required. Equality holds when f is an extreme function of $G_K(\alpha,\delta,t)$.

Based on the findings in Theorem 1 and Theorem 2, the representation theorem and coefficient bounds have been established for the class of functions $G_K(\alpha,\delta,t)$.

Conclusion

In conclusion, there are two purposes of this paper, which are to find representation theorem and to determine the coefficient bound for the $G_K(\alpha,\delta,t)$. We believe that we have achieved all the objectives that we have highlighted.

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