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NUMERICAL SOLUTION OF THE HEAT EQUATION USING THE FINITE DIFFERENCE METHOD

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ABSTRACT

This paper focuses on solving the one-dimensional heat equation numerically through the Finite Difference Method (FDM), highlighting the application of the Crank-Nicolson approach. The heat equation, a key partial differential equation in physics and engineering, describes how heat diffuses over time within a material. Although exact analytical solutions exist for simple scenarios, they are often inadequate for more complex problems, making numerical techniques essential. In this study, the continuous domain is discretized in both time and space, converting the heat equation into a set of algebraic expressions. The Crank-Nicolson method, well-regarded for its numerical stability and second-order precision, is applied to examine temperature variations under different types of boundary conditions, such as Dirichlet and Neumann. Implementation is carried out using Wolfram Mathematica, which also enables dynamic visualizations through animated plots and 3D surfaces. The accuracy of the numerical results is checked by comparing them to known exact solutions, using measures like the L_2-Norm and maximum absolute error. The analysis demonstrates that the Crank-Nicolson method is an effective and accurate tool for simulating heat transfer, offering a reliable solution strategy for practical thermal conduction problems.

Keywords: Heat equation, Finite Difference Method, Crank-Nicolson Method, Numerical Solution, Error Analysis

Introduction

The heat equation is a fundamental partial differential equation that models the diffusion of thermal energy through a medium over time. Its applications span across various scientific and engineering domains, including thermal analysis in mechanical systems, environmental simulations, and materials science. Although analytical solutions exist for idealized cases with simple boundary conditions and uniform materials, real-world problems often involve complexities such as irregular geometries, non-homogeneous materials, and mixed or time-dependent boundary conditions that make exact solutions infeasible. In such cases, numerical methods provide an effective alternative. Among these, the Finite Difference Method (FDM) is one of the most widely used techniques for approximating the solution of partial differential equations. By discretizing both space and time, FDM transforms the continuous heat equation into a solvable system of algebraic equations.

This study focuses on the Crank-Nicolson scheme, an implicit finite difference approach known for its second order accuracy and unconditional stability. It is particularly effective in handling transient heat conduction problems and supports a wide range of boundary conditions, including Dirichlet and Neumann types. To investigate the performance of this method, the one-dimensional heat equation was solved using Wolfram Mathematica under different boundary conditions. The numerical results were then validated against analytical solutions using error metrics such as the $L_2 - \text{Norm}$ and maximum absolute error. The findings confirmed the method's high accuracy and robustness, with minimal deviation from the exact solutions. This research demonstrates the Crank-Nicolson method's adaptability and efficiency, offering a reliable computational tool for solving practical heat transfer problems. While the study is limited to one-dimensional cases and fixed boundary conditions, it lays the groundwork for future research in higher dimensions, variable boundaries, and more complex thermal systems

Literature Review

Numerical methods are always employed to resolve the heat equation when analytical solutions are impossible due to complex geometries, time-varying boundary conditions, or inhomogeneous material properties. One of the most well-known and effective is the Finite Difference Method (FDM), which converts the differential equations into algebraic systems by discretizing space and time (Song et al., 2018). For homogeneous equations, FDM yields exact results, especially for steady-state heat conduction problems with uniform materials like copper and aluminium (Loskor & Sarkar, 2022). For non-homogeneous equations with internal heat sources or non-uniform thermal properties, more advanced extensions of FDM such as the Crank-Nicolson method demonstrate improved accuracy and stability (Safari, 2024).

The Crank-Nicolson scheme, which was presented by Crank and Nicolson in 1947, is an implicit technique that provides a trade-off between second order accuracy in time and space and numerical stability (Liu & Hao, 2022). Besides that, the Crank-Nicolson scheme is particularly suitable for transient problems and, due to its trapezoidal time-stepping, is ideal for simulations with fine spatial meshes or long times (Mohebbi & Dehghan, 2010). The boundary conditions are also crucial for effective heat transfer modelling. Studies have shown its effectiveness in both homogeneous and non-homogeneous heat equations, as well as its adaptability to boundary conditions such as Dirichlet and Neumann types (Mojumder et al., 2023). Dirichlet conditions yield prescribed temperatures, and Neumann conditions yield a constant heat flux. Both conditions have been utilized effectively in FDM simulations (Hajrulla et al., 2024). The Crank-Nicolson scheme has also been noted to be extremely stable when used under mixed boundary conditions or on non-regular domains (Chai et al., 2020). Besides, optimization techniques such as adaptive mesh refinement, hybrid schemes, and iterative solvers such as the Gauss-Seidel or Conjugate Gradient methods have been employed to ensure computational efficiency (Tafrikan & Ghani, 2022). Computational software such as Wolfram

Mathematica also enables the use of FDM via symbolic computation and advanced visualization (Narahari et al., 2013). The literature as a whole show that FDM, especially the Crank-Nicolson variation, is a strong, accurate, and flexible way to represent one-dimensional heat transfer problems in many different situations.

Methodology

The process of solving the one-dimensional heat equation in this study begins with the selection of the equation, expressed as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where T = temperature, t = time, x = spatial coordinates, and α = thermal diffusivity constant.
The equation is then solved using the Finite Difference Method (FDM), a numerical approach that discretizes the continuous problem. To enhance accuracy and ensure stability, the Crank-Nicolson method which is a semi-implicit scheme, is used for time discretization. After that, the heat equation is used on a one-dimensional model, which is solved using the discretized scheme. Numerical results are then generated, and visual representations such as surface and line plots are produced using Wolfram Mathematica to illustrate the temperature distribution over time. Finally, the accuracy of the numerical solution is evaluated through error analysis, including the $L_2 -$Norm and maximum absolute error, and comparing the results with the exact analytical solution to validate the effectiveness of the Crank-Nicolson method.

Result and Discussion

This model problem examines the temperature distribution along a one-dimensional rod subjected to mixed boundary conditions, where one end is Dirichlet and the other is Neumann. Using Wolfram Mathematica and the Crank-Nicolson finite difference scheme, the simulation visualizes heat transfer through surface plots and animations. The domain considered is $0 \le x \le 10,\ t > 0$ and the numerical solution is compared with the analytical solution. The analysis evaluates the effect of spatial and temporal step sizes on accuracy and stability, using error metrics such as absolute error, $L_2 -$Norm, and maximum norm. The findings show that even with mixed boundary conditions, the Crank-Nicolson approach yields reliable and accurate approximations.

The heat equation, along with its initial and boundary conditions for model problem is formulated as:

$$u_t = \alpha^2 u_{xx} \text{ for } 0 \le x \le 10, t > 0 \tag{2}$$

Initial condition

$$u(x,0) = 3\sin\left(\frac{5x}{2}\right) \quad (3)$$

Boundary condition

$$\text{Dirichlet: } u(0,t) = 0 \quad (4)$$

$$\text{Neumann: } u_x(10,t) = 0 \quad (5)$$

The simulation was implemented in Wolfram Mathematica over a domain of length $L = 10$ and run until $T_{max} = 1.0$. The spatial and temporal grids were discretized into 200 steps each, resulting in $dx = 0.05$ and $dt = 0.005$. A key dimensionless parameter $r = \alpha \frac{\partial t}{\partial x^2}$ was used to form the tridiagonal matrix in the Crank-Nicolson scheme. These discretization choices provided a fine resolution for capturing heat transfer dynamics efficiently and accurately, particularly near the boundaries.

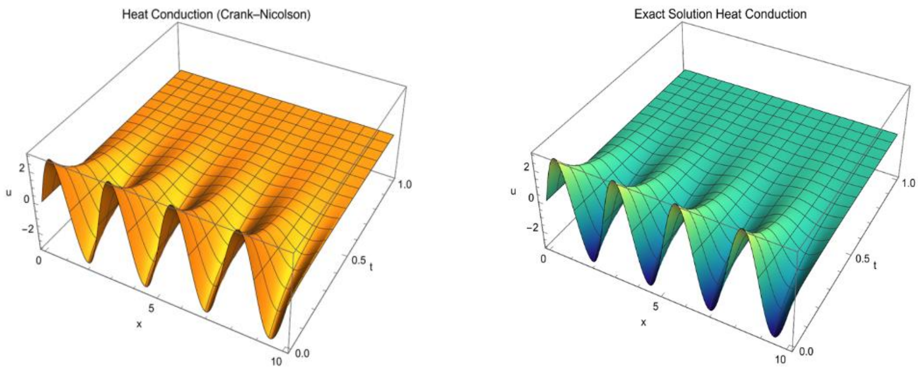

Figure 1: 3D-plot of Crank-Nicolson Solution and the Exact Solution

The temperature changes over time and along the rod are clearly illustrated through surface and line plots. Figure 1 above show that the Crank-Nicolson method produces results that closely match the exact solution, highlighting its accuracy. The 3D plots reveal smooth and consistent heat flow, and the visible symmetry and wave patterns further support the method's reliability in solving heat transfer problems effectively.

Table 1: Comparison of Exact Solution and Crank-Nicolson Solution with Error

x	Exact Solution	FDM (Ucn)	Absolute Error
0	0.0000000	0	0.0000000
1	0.0034660	0.0034925	0.0000265
2	-0.0055535	-0.0055960	0.0000425
3	0.0054323	0.0054741	0.0000418
4	-0.0031506	-0.0031691	0.0000185
5	-0.0003841	-0.0002969	0.0000872
6	0.0037661	0.0046497	0.0008836
7	-0.0037661	-0.0010337	0.0046165

8	0.0052872	0.0190347	0.0137475
9	-0.0028214	0.0152562	0.0180776
10	-0.0007665	0.0005732	0.0013397
$L_2 - \text{Norm}$	-	-	0.1042487
$L_\infty - \text{Norm}$	-	-	0.0186432

Table 1 shows a comparison between the exact analytical solution and the numerical results from the Crank-Nicolson method for the one-dimensional heat equation with mixed boundary conditions. The table shows the temperature at different points along the rod at a certain period. Overall, the numerical solution closely matches the exact values, particularly near the center of the domain. Small differences begin to appear near the edges, peaking at $x = 9$ where the maximum deviation is recorded. The method was verified to be accurate with the $L_2 - \text{Norm}$ and the maximum absolute error, which were 0.1042 and 0.0186 respectively. These relatively low error values indicate that the numerical scheme is both accurate and stable throughout the simulation. The smooth progression of temperature over time also reflects the Crank-Nicolson method's strong stability and second-order accuracy. In summary, the method effectively replicates heat transfer under the boundary conditions imposed, with minimal errors near the boundaries due to boundary behavior. The use of small spatial grid size and an appropriate time step assisted in enhancing the precision of the results.

Conclusion

This study reports that the Crank-Nicolson finite difference scheme we found to be a stable and accurate tool for the solution of the one-dimensional heat equation under mixed Dirichlet and Neumann boundary conditions. Low $L_2 - \text{Norm}$ and maximum absolute error values verified that the simulation findings, which were implemented and visualized using Wolfram Mathematica, closely matched analytical answers. Although there were slight variations in the area of the Neumann boundary, the overall accuracy was within acceptable limits. These results confirm that the Crank-Nicolson method's reliability as a useful instrument for real-world thermal analysis, especially when boundary conditions are irregular. Kubacka and Ostrowski (2021) demonstrated that the Crank-Nicolson method is stable and accurate even when they are applied in Robin-type boundary condition. The method may be extended to two- or three-dimensional systems in future research, or adaptive time-stepping may be investigated.

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