

# Generalized Explicit Expressions for Second-Order Recurrences

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## ABSTRACT

This article investigates the flexibility of closed-form generalized solutions for complex second-order linear recurrence relations. We derive and examine foundational results, including the generating function and a generalized Binet-type formula, in their most comprehensive forms. These tools are then used to establish new identities. Furthermore, we extend our results to generalize classical identities—such as those of Cassini, Catalan, Vajda, and D’Ocagne—for the first  $n$  terms of such recurrences.

## 1. INTRODUCTION

Originally sequences defined using simple linear recurrence relations—such as the Fibonacci sequence  $F_n = F_{n-1} + F_{n-2}$ ,  $n \geq 2$ , with initial terms  $F_0 = 0, F_1 = 1$  and Lucas sequence  $L_n = L_{n-1} + L_{n-2}$ ,  $n \geq 2, L_0 = 2, L_1 = 1$  have been widely studied and are renowned for their fascinating properties in integer mathematics. Comprehensive treatments of such sequences are available in the books by Vajda (1989) and Koshy (2001), while generalizations of these sequences are presented in Verma (2024, 2025b).

Researchers have explored various generalizations of these sequences by either altering the initial conditions or modifying the recurrence relation ( $X_n = pX_{n-1} + qX_{n-2}$ ,  $n \geq 2$  initial conditions  $X_0 = a, X_1 = b$ ). These modifications have further extended the study of such sequences into the complex domain. Notably, Horadam (1961, 1963), Jordan (1965), and Harman (1981) made pioneering contributions by investigating these sequences over the field of complex numbers.

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The development of Fibonacci sequences and their generalizations in the complex domain reveals how seemingly simple recurrence relations can give rise to a rich tapestry of mathematical exploration. This area of study connects numerous fields and inspires new ideas, with applications in number theory, combinatorics, computer science, and physics. One prominent direction involves the study of complex Fibonacci sequences, where the recurrence relations are extended to the complex plane while preserving their core structure.

Horadam (1961) introduced generalized second-order recurrence sequences with arbitrary initial conditions and complex coefficients, establishing a foundational framework for subsequent algebraic and systematic investigations. Falcon and Plaza (2007) and Edson and Yayenie (2009) further examined  $k$ -Fibonacci sequences, offering new generalizations and extending Binet-type formulas to explore underlying algebraic identities.

Asci and Leel (2017) investigated mathematical properties and identities related to generalized Fibonacci and Lucas sequences, while Benjamin and Quinn (2003) explored these sequences through combinatorial techniques and visual proofs. Berzsenyi (1977) analysed recurrence relations in the context of number theory, and Bilgici (2014) studied applications of Fibonacci numbers within various algebraic structures. Good (1993) provided probabilistic interpretations of Fibonacci numbers in mathematical modelling.

Halici and Oz (2016) developed matrix representations and Binet formulas for  $k$ -Fibonacci sequences. Haji-Esmaili and Ghaderi (2011) examined  $k$ -Fibonacci and  $k$ -Lucas sequences through algebraic identities and generating functions. Harman (1981) offered historical perspectives and insights on the educational use of Fibonacci numbers. Horadam (1963) introduced the now well-known Horadam sequence, a broad generalization with extensive applicability.

Klein (1991) explored computational aspects of recurrence sequences, while Koshy (2001) authored a comprehensive resource on Fibonacci and Lucas numbers. Melham and Shannon (1995) studied properties and identities of higher-order Fibonacci-related sequences. Pethe and Horadam (1986) advanced generalizations and analysed their mathematical behaviour. Sury (2004) examined modular properties and divisibility within Fibonacci and Lucas sequences. Vajda (1989) laid foundational results, offering numerous identities and relations. Verma (2025a, 2025b) contributed to the modern theory of generalized Fibonacci sequences and their combinatorial interpretations.

Recent works have significantly contributed to this domain by introducing new formulations, exploring cross-relations, and extending classical identities. Kaya and Özimamoğlu (2022) introduced the generalized Gauss  $k$ -Pell numbers, along with their polynomial extensions, and derived corresponding Binet formulas and recurrence relations. Rihane (2024) investigated the  $k$ -Fibonacci and  $k$ -Lucas sequences, identifying conditions under which they can be expressed as a product of two Pell numbers, enriching the interplay between different families of linear recurrence sequences. Verma (2025b) presented a thorough treatment of generalized Fibonacci sequences, deriving closed-form Binet-type expressions and proving novel summation identities in matrix and polynomial form. Daşdemir and Varol (2025) investigated conditions under which Lucas numbers can be expressed as products of two Pell numbers.

Collectively, these contributions have facilitated the derivation of closed-form expressions, complex Binet-type formulas, generating functions, and innovative identities that extend classical results—such as those of Cassini, Catalan, Vajda, and D’Ocagne—into the complex domain. These advancements provide a robust theoretical and computational framework for further investigations into the rich and captivating world of generalized complex sequences derived from second-order recurrences and their elegant simplifications.

## 2. GENERALIZED SECOND-ORDER RECURRENCE RELATIONS

**Definition 1.** Generalized second-order recurrence relation is given by:

$$C_n(p, q, C_1, C_2) = pC_{n-1} + iqC_{n-2}, \quad n \geq 3, \quad i = \sqrt{-1}, \quad (1)$$

where  $C_1 = a_1 + ia_2$ ,  $C_2 = b_1 + ib_2$ ,  $p = p_1 + ip_2$ ,  $q = q_1 + iq_2 (\neq 0)$ ,  $a_j, b_j, p_j$ , and  $q_j$  ( $j=1,2$ ) are arbitrary real numbers.

### 2.1 Generalized Complex Sequence

First few terms of the generalized complex sequence  $C_n(p, q, C_1, C_2)$  are tabulated below:

$$C_n(n, p, q, C_1, C_2) = \left\{ \begin{array}{l} a_1 + ia_2, b_1 + ib_2, pb_1 - qa_2 + i(pb_2 + qa_1), \\ p^2b_1 - pqa_2 - qb_2 + i(p^2b_2 + pqa_1 + qb_1), \\ p^3b_1 - p^2qa_2 - 2pqb_2 - q^2a_1 + i(p^3b_2 + p^2qa_1 + 2pqb_1 - q^2a_2), \\ p^4b_1 - p^3qa_2 - 3p^2qb_2 - 2pq^2a_1 - q^2b_1 \\ + i(p^4b_2 + p^3qa_1 + 3p^2qb_1 - 2pq^2a_2 - q^2b_2), \\ \dots \end{array} \right\}. \quad (2)$$

For brevity, at many places, in this article, notation  $C_n(p, q, C_1, C_2)$  will refer to as  $C_n$ .

### 2.2 Special Case

If we take,  $C_1 = 1+i, C_2 = 1-i, p = 2+3i, q = -i$  then complex sequence obtained is  $V_n(n, p, q, V_1, V_2) = \left\{ 1+i, -i, -1+2i, -2+2i, -4+i, -6-i, -7-5i, \dots \right\}$ .

If we take,  $C_1 = 0+0i, C_2 = 1+0i, p = 1+0i, q = \frac{1+0i}{i}$  then generalized complex sequence reduce to  $C_n(n, p, q, V_1, V_2) = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$  which is the classical Fibonacci sequence.

If we take,  $C_1 = 2+0i, C_2 = 1+0i, p = 1+0i, q = \frac{1+0i}{i}$  then generalized complex sequence reduce to  $C_n(n, p, q, C_1, C_2) = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \dots\}$  which is the Lucas sequence.

## 3. GENERATING FUNCTION

**Theorem 1.** For  $C_n(p, q, C_1, C_2) = pC_{n-1} + iqC_{n-2}, n \geq 3, i = \sqrt{-1}$  where  $C_1 = a_1 + ia_2, C_2 = b_1 + ib_2, p = p_1 + ip_2$  and  $q = q_1 + iq_2, i = \sqrt{-1}, a_j, b_j, p_j$ , and  $q_j \neq 0$  ( $j=1,2$ ) are arbitrary real numbers. Then generating function is

$$C(z) = \frac{(a_1 + ia_2)z + (b_1 + ib_2 - pz(a_1 + ia_2))z^2}{[1 - (p_1 + ip_2)z - i(q_1 + iq_2)z^2]}. \quad (3)$$

**Proof:** For any  $p, q, C_1$  and  $C_2$ , let the generating function be:

$$C(z) = \sum_{n=1}^{\infty} C_n(p, q, V_1, V_2) z^n, \quad (4)$$

$$C(z) = C_1 z + C_2 z^2 + \sum_{n=3}^{\infty} C_n(p, q, V_1, V_2) z^n. \quad (5)$$

Now,

$$\sum_{n=3}^{\infty} C_n(p, q, V_1, V_2) z^n = \sum_{n=3}^{\infty} p C_{n-1}(p, q, V_1, V_2) z^n + \sum_{n=3}^{\infty} i q C_{n-2}(p, q, V_1, V_2) z^n \quad (6)$$

$$= p \sum_{n=3}^{\infty} C_{n-1}(p, q, V_1, V_2) z^n + i q \sum_{n=3}^{\infty} C_{n-2}(p, q, V_1, V_2) z^n,$$

$$\sum_{n=3}^{\infty} C_{n-1}(p, q, V_1, V_2) z^n = z \sum_{m=2}^{\infty} C_m z^m = z(C(z) - z C_1), \quad (7)$$

$$\sum_{n=3}^{\infty} C_{n-2}(p, q, V_1, V_2) z^n = z^2 \sum_{k=1}^{\infty} C_k z^k = z^2 C(z). \quad (8)$$

On combining equations (5)-(8), we obtain

$$\begin{aligned} \sum_{n=3}^{\infty} C_n(p, q, V_1, V_2) z^n &= p z (C(z) - z C_1) + i q z^2 C(z) \\ C(z) &= C_1 z + C_2 z^2 + p z (C(z) - z C_1) + i q z^2 C(z) \\ C(z) [1 - p z - i q z^2] &= C_1 z + C_2 z^2 - p z^2 C_1 \\ C(z) &= \frac{C_1 z + (C_2 - p z C_1) z^2}{[1 - p z - i q z^2]}. \end{aligned} \quad (9)$$

On considering the  $p, q, C_1$  and  $C_2$  from definition (1), we have

$$C(z) = \frac{(a_1 + i a_2) z + [b_1 + i b_2 - p z (a_1 + i a_2)] z^2}{[1 - (p_1 + i p_2) z - i (q_1 + i q_2) z^2]}.$$

Therefore, the stated theorem is confirmed.

**Theorem 2. Explicit sum formula complex generalized recurrence relation:**

The  $n^{\text{th}}$  term  $C_n(p, q, C_1, C_2)$  of the sequence defined in expression (1) is

$$C_n(p, q, C_1, C_2) = (C_2 - p C_1) \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \binom{n-k-2}{k} p^{n-2k-2} (i q)^k + C_1 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k} p^{n-2k} (i q)^k, \quad n \geq 2. \quad (10)$$

**Proof:** Since the generating function (9) derived in the above theorem, we have

$$\begin{aligned} C(z) &= \frac{C_1 z + (C_2 - p C_1) z^2}{1 - p z - i q z^2} \\ &= [C_1 z + (C_2 - p C_1) z^2] (1 - p z - i q z^2)^{-1}, \\ &= [C_1 z + (C_2 - p C_1) z^2] [1 - (p z + i q z^2)]^{-1}, \\ &= [C_1 z + (C_2 - p C_1) z^2] \sum_{n=0}^{\infty} (p + i q z)^n z^n \\ &= [C_1 z + (C_2 - p C_1) z^2] \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} p^{n-k} (i q)^k z^{n+k}, \end{aligned}$$

$$\begin{aligned}
&= [C_1 z + (C_2 - pC_1)z^2] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+k}{k} p^n (iq)^k z^{n+2k}, \quad (n \rightarrow n+k) \\
&= [C_1 z + (C_2 - pC_1)z^2] \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k} (iq)^k z^n, \quad (n \rightarrow n-2k) \\
&= (C_2 - pC_1) \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k} (iq)^k z^{n+2} + C_1 \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k} (iq)^k z^{n+1}.
\end{aligned}$$

Equating the coefficient of  $z^n$  on both sides, we obtain

$$C_n(p, q, C_1, C_2) = (C_2 - pC_1) \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-k-2}{k} p^{n-2k-2} (iq)^k + C_1 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} p^{n-2k-1} (iq)^k.$$

This substantiates the stated theorem.

#### 4. BINET TYPE FORMULA FOR COMPLEX GENERALIZED RECURRENCE RELATION

**Theorem 3.** The Binet Type formula for complex generalized recurrence relation  $C_n(p, q, C_1, C_2) = pC_{n-1} + iqC_{n-2}$  is

$$C_n(p, q, C_1, C_2) = \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) m_1 m_2 C_1. \quad (11)$$

**Proof:** Let  $C_n = m^n$ , then  $C_n(p, q, C_1, C_2) = pC_{n-1} + iqC_{n-2}$  is  $m^n = pm^{n-1} + iqm^{n-2}$ .

Divide  $m^{n-2}$  ( $m \neq 0$ ), we obtain characteristic equation of the generalized complex recurrence relation (1) as:

$$m^2 - pm - iq = 0. \quad (12)$$

On solving characteristic equation, we have

$$m_1, m_2 = \frac{p \pm \sqrt{p^2 + 4iq}}{2}. \quad (13)$$

Let the general solution of the recurrence is

$$C_n = Pm_1^n + Qm_2^n, \quad (14)$$

where  $P$  and  $Q$  are obtained from the initial conditions:

$$C_1 = Pm_1 + Qm_2, \quad (15)$$

$$C_2 = Pm_1^2 + Qm_2^2. \quad (16)$$

On solving equations (15) and (16), we obtain

$$P = \frac{C_2 - C_1 m_2}{m_1(m_1 - m_2)}, Q = \frac{C_1 m_1 - C_2}{m_2(m_1 - m_2)}. \quad (17)$$

On combining equations (14) and (17), we obtain formula for the  $n^{\text{th}}$  term complex generalized recurrence

$$C_n(p, q, V_1, V_2) = \frac{C_2 - C_1 m_2}{m_1(m_1 - m_2)} m_1^n + \frac{C_1 m_1 - C_2}{m_2(m_1 - m_2)} m_2^n,$$

which can be written as:

$$C_n(p, q, C_1, C_2) = \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) m_1 m_2 C_1.$$

This substantiates the stated theorem.

## 5. SUMMATION FORMULA FOR FIRST THE $n$ TERMS

**Theorem 4.** For complex generalized recurrence relation  $C_n(p, q, C_1, C_2) = pC_{n-1} + iqC_{n-2}$ , the  $n^{\text{th}}$  term is

$$C_n(p, q, C_1, C_2) = \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) m_1 m_2 C_1.$$

Then, sum of first  $n$  terms:

$$\begin{aligned} S(n, p, q, C_1, C_2) &= \sum_{k=1}^n C_k(p, q, C_1, C_2) \\ &= \frac{1}{1-p-iq} \left( -C_{n+1}(p, q, C_1, C_2) - iqC_n(p, q, C_1, C_2) \right). \end{aligned} \quad (18)$$

**Proof:** Since

$$\sum_{k=1}^n C_k(p, q, C_1, C_2) = \sum_{k=1}^n \left[ \left( \frac{m_1^{k-1} - m_2^{k-1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{k-2} - m_2^{k-2}}{m_1 - m_2} \right) m_1 m_2 C_1 \right], \quad (19)$$

$$\sum_{k=1}^n \left( \frac{m_1^{k-1} - m_2^{k-1}}{m_1 - m_2} \right) C_2 = \frac{-1}{(1-(m_1+m_2)+m_1 m_2)} \left( \left( \frac{m_1^n - m_2^n}{m_1 - m_2} \right) - \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) m_1 m_2 + 1 \right) C_2 \quad (20)$$

$$\begin{aligned} &\sum_{k=1}^n \left( \frac{m_1^{k-2} - m_2^{k-2}}{m_1 - m_2} \right) m_1 m_2 C_1, \\ &= \frac{1}{(1-(m_1+m_2)+m_1 m_2)} \left( \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) m_1^2 m_2^2 - \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) m_1 m_2 + (m_1 + m_2 - 1) \right) C_1. \end{aligned} \quad (21)$$

On combining equations (19), (20) and (21), we obtain

$$\begin{aligned} \sum_{k=1}^n C_k(p, q, C_1, C_2) &= \sum_{k=1}^n \left[ \left( \frac{m_1^{k-1} - m_2^{k-1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{k-2} - m_2^{k-2}}{m_1 - m_2} \right) m_1 m_2 C_1 \right], \\ &= \frac{-1}{(1-(m_1+m_2)+m_1 m_2)} \left( \left( \frac{m_1^n - m_2^n}{m_1 - m_2} \right) - \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) m_1 m_2 + 1 \right) C_2 \\ &\quad - \frac{1}{(1-(m_1+m_2)+m_1 m_2)} \left( \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) m_1^2 m_2^2 - \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) m_1 m_2 + (m_1 + m_2 - 1) \right) C_1. \end{aligned}$$

On further simplification, we obtain

$$= \frac{1}{(1 - (m_1 + m_2) + m_1 m_2)} \left[ \begin{aligned} & - \left( \frac{m_1^n - m_2^n}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) m_1 m_2 C_1 \\ & + \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) m_1 m_2 C_1 \\ & + C_2 + (m_1 + m_2 - 1) C_1 \end{aligned} \right],$$

$$= \frac{1}{(1 - p - iq)} \left[ \begin{aligned} & - \left( \frac{m_1^n - m_2^n}{m_1 - m_2} \right) C_2 + iq \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) C_1 \\ & + \left( \frac{m_1^{n-1} - m_2^{n-1}}{m_1 - m_2} \right) C_2 + iq \left( \frac{m_1^{n-2} - m_2^{n-2}}{m_1 - m_2} \right) C_1 \\ & + C_2 + (p - 1) C_1 \end{aligned} \right].$$

$$S(n, p, q, C_1, C_2) = \sum_{k=1}^n C_k(p, q, C_1, C_2)$$

$$= \frac{1}{1 - p - iq} \left( -C_{n+1}(p, q, C_1, C_2) - iq C_n(p, q, C_1, C_2) \right),$$

where  $m_1, m_2 = \frac{p \pm \sqrt{p^2 + 4iq}}{2}$ . This substantiates the stated sum formula.

## 6. GENERALIZATION OF CASSINI'S, CATALAN'S VAJDA'S AND D'OCAGNE'S IDENTITIES FORMULA

This section extends several known identities of second-order recurrence relations into the complex domain and establishes new identities corresponding to the general recurrence relation (1). These identities can be derived either from the general definition (1) or from the Binet-type formula stated in Theorem 4.1. In the results that follow, we primarily employ the general Binet-type formula given in equation (11).

In the following identities,  $m_1, m_2 = \frac{p \pm \sqrt{p^2 + 4iq}}{2}$ , and therefore  $m_1 - m_2 = \sqrt{p^2 + 4iq}$ ,  $m_1 + m_2 = p$  and  $m_1 m_2 = -iq$ .

**Theorem 5. Cassini's identity (sometimes called Simson's identity):** For  $n \geq 1$ ,

$$C_{n+1} C_{n-1} - C_n^2 = -(m_1 m_2)^{n-2} (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (22)$$

**Theorem 6. Catalan's identity (Catalan's identity is the generalization of Cassini Identity):**

For  $n \geq r$ ,

$$C_{n+r} C_{n-r} - C_n^2 = -(m_1 m_2)^{n-r-1} \left( \frac{m_1^r - m_2^r}{m_1 - m_2} \right)^2 (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (23)$$

**Theorem 7. Vajda's identity (Vajda's identity is the generalization of Catalan's identity):**

For integers  $n, k$  and  $l$

$$C_{n+k}C_{n+l} - C_nC_{n+k+l} = (m_1m_2)^{n-1} \left( \frac{m_1^k - m_2^k}{m_1 - m_2} \right) \left( \frac{m_1^l - m_2^l}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2). \quad (24)$$

**Theorem 8. D'Ocagne's identity:** For integers,  $n, k, k-n \geq 0$

$$C_{n+1}C_k - C_{k+1}C_n = (m_1m_2)^{n-1} \left( \frac{m_1^{k-n} - m_2^{k-n}}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2). \quad (25)$$

The above identities can also be obtained from the following identity by choosing  $n, r$  and  $k$  appropriately.

**Theorem 9.** For choosing  $n, r$  and  $k, n+r \geq k$ ,

$$\begin{aligned} C_kC_n - C_{k-r}C_{n+r} \\ = (m_1m_2)^{m-r-1} \left( \frac{m_1^{n+r-k} - m_2^{n+r-k}}{m_1 - m_2} \right) \left( \frac{m_1^r - m_2^r}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2). \end{aligned} \quad (26)$$

**Theorem 10.** For negative subscripts  $n$ , we obtain the identity

$$C_{-n} = -(m_1m_2)^{-(n-1)} \left[ \left( \frac{m_1^{n+1} - m_2^{n+1}}{m_1 - m_2} \right) C_2 - \left( \frac{m_1^{n+2} - m_2^{n+2}}{m_1 - m_2} \right) C_1 \right]. \quad (27)$$

**Theorem 11.** For  $n, k \geq 1$ ,

$$C_kC_{-n+1} - C_{k+1}C_{-n} = (m_1m_2)^{-(n+1)} \left( \frac{m_1^{k+n} - m_2^{k+n}}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2). \quad (28)$$

**Theorem 12.** For  $n \geq 1$ ,

$$C_{n+3}^2 - C_nC_{n+6} = (m_1m_2)^{n-1} \left( \frac{m_1^3 - m_2^3}{m_1 - m_2} \right)^2 (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2). \quad (29)$$

**Theorem 13.** For  $n \geq 1$ ,

$$\begin{aligned} C_{n+1}C_{n+2}C_{n+6} - C_{n+3}^3 \\ = \frac{(m_1m_2)^{n-1}}{(m_1 - m_2)^2} (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2) \left( (m_1 - m_2)^2 (m_1 + m_2)C_{n+6} - (m_1^3 - m_2^3)^2 C_{n+3} \right). \end{aligned} \quad (30)$$

**Theorem 14.** For integers  $n, k$  and  $k-n \geq 0$ ,

$$C_nC_k - C_{n-1}C_{k+1} = (m_1m_2)^{n-2} \left( \frac{m_1^{k-n+1} - m_2^{k-n+1}}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2).$$

On replacing  $n$  by  $n+1$  the above identity can also be written as:

$$C_{n+1}C_k - C_nC_{k+1} = (m_1m_2)^{n-1} \left( \frac{m_1^{k-n} - m_2^{k-n}}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2)C_1C_2 + m_1m_2C_1^2). \quad (31)$$

**Theorem 15.** For  $n \geq 1$ ,

$$C_n C_{n-1} - C_{n-2} C_{n+1} = (m_1 m_2)^{n-3} (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (32)$$

**Theorem 16.** For integers  $n, k$ ,

$$C_{n+k} C_{n+1} - C_n C_{n+k+1} = (m_1 m_2)^{n-1} \left( \frac{m_1^k - m_2^k}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (33)$$

Now taking  $n+k=r$  in the identity (33), we obtain the following identity.

**Theorem 17.** Let  $n+k=r$ , and  $r-n \geq 0$ , we obtain

$$C_r C_{n+1} - C_n C_{r+1} = (m_1 m_2)^{n-1} \left( \frac{m_1^{r-n} - m_2^{r-n}}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (34)$$

**Theorem 18.** Vajda's type identity. For integers  $n, k$  and  $r$ ,

$$C_n C_{n+k-r} - C_{n-r} C_{n+k} = (m_1 m_2)^{n-r-1} \left( \frac{m_1^k - m_2^k}{m_1 - m_2} \right) \left( \frac{m_1^r - m_2^r}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (35)$$

**Theorem 19.** For integers  $n, k$  and  $r$ , taking  $k = m - n + r$ ,

$$C_n C_m - C_{n-r} C_{m+r} = (m_1 m_2)^{n-r-1} \left( \frac{m_1^{m-n+r} - m_2^{m-n+r}}{m_1 - m_2} \right) \left( \frac{m_1^r - m_2^r}{m_1 - m_2} \right) (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2). \quad (36)$$

**Theorem 20.** For  $n \geq 1$ ,

$$\begin{aligned} & C_{n-2} C_{n-1} C_{n+1} C_{n+2} - C_n^4 \\ &= (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2)^2 (m_1 + m_2)^2 (m_1 m_2)^{2n-5} \\ &+ (m_1 m_2)^{n-3} (C_2^2 - (m_1 + m_2) C_1 C_2 + m_1 m_2 C_1^2) (m_1^2 + 3m_1 m_2 + m_2^2) C_n^2. \end{aligned} \quad (37)$$

## 7. CONCLUSION

In conclusion, this study demonstrates the effectiveness and flexibility of closed-form generalized solutions in addressing complex second-order linear recurrence relations. Through the derivation of comprehensive generating functions and a generalized Binet-type formula, we have developed powerful analytical tools for examining such recurrences. The successful formulation of new identities, along with the generalization of classical results—including those of Cassini, Catalan, Vajda, and D'Ocagne—as well as a sum formula for the first  $n$  terms of the sequences, highlights the strength and broad applicability of our approach. Collectively, these findings offer a unified framework for understanding and extending the properties of a wide class of second-order linear recurrence relations.

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## 9. CONFLICT OF INTEREST STATEMENT

The authors declared that there is no conflict of interest in this paper.

## 10. AUTHORS' CONTRIBUTIONS

Author contributed work and manuscript.

## REFERENCES

- Asci, M., & Lee, G. Y. (2017). Generalized Gaussian Fibonacci numbers and sums by matrix methods. *Utilitas Mathematica*, 102, 349–357.
- Benjamin, A. T., & Quinn, J. J. (2003). The Fibonacci numbers exposed more discretely. *Mathematics Magazine*, 76(3), 182–192. <https://doi.org/10.2307/3219319>
- Berzsenyi, G. (1977). Gaussian Fibonacci numbers. *Fibonacci Quarterly*, 15(3), 233–236. <https://doi.org/10.1080/00150517.1977.12430445>
- Bilgici, G. (2014). New generalization of Fibonacci and Lucas sequences. *Applied Mathematical Sciences*, 8(29), 1429–1437. <https://doi.org/10.12988/ams.2014.4162>
- Daşdemir, A., & Varol, M. (2025). Lucas numbers which are products of two Pell numbers. *Fibonacci Quarterly*, 63(1), 78–83. <https://doi.org/10.1080/00150517.2024.2412958>
- Edson, M., & Yayenie, O. (2009). A new generalization of Fibonacci sequences and extended Binet formula. *Integers*, 9, 639–654. <https://doi.org/10.1515/integ.2009.051>
- Falcon, S., & Plaza, A. (2007). The k-Fibonacci sequences and their relationship with the Pascal triangle. *Chaos, Solitons & Fractals*, 32(5), 1615–1625. <https://doi.org/10.1016/j.chaos.2006.09.022>
- Good, I. J. (1993). Complex Fibonacci and Lucas numbers, continued fractions, and the square root of the golden ratio. *Fibonacci Quarterly*, 31(1), 7–20. <https://doi.org/10.1080/00150517.1993.12429315>
- Halici, S., & Oz, S. (2016). On some Gaussian Pell and Pell-Lucas numbers. *Ordu University Science and Technology Journal*, 6(1), 8–18.
- Haji-Esmaili, E., & Ghaderi, S. H. (2011). A closed-form expression for second-order recurrences. *Acta Mathematica Universitatis Comenianae*, 80(2), 251–253.
- Harman, C. J. (1981). Complex Fibonacci numbers. *Fibonacci Quarterly*, 9(1), 82–86. <https://doi.org/10.1080/00150517.1981.12430133>
- Horadam, A. F. (1961). A generalized Fibonacci sequence. *American Mathematical Monthly*, 68(5), 455–459. <https://doi.org/10.1080/00029890.1961.11989696>
- Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *American Mathematical Monthly*, 70(3), 289–291. <https://doi.org/10.2307/2313129>

- Jordan, J. H. (1965). Gaussian Fibonacci and Lucas numbers. *Fibonacci Quarterly*, 3(4), 315–318. <https://doi.org/10.1080/00150517.1965.12431411>
- Kaya, A., & Özimamoğlu, H. (2022). On a new class of the generalized Gauss k-Pell numbers and their polynomials. *Notes on Number Theory and Discrete Mathematics*, 28(4), 593–602. <https://doi.org/10.7546/nntdm.2022.28.4.593-602>
- Klein, S. T. (1991). Efficient generation of Fibonacci words. *Theoretical Computer Science*, 87(2), 281–291. [https://doi.org/10.1016/0304-3975\(91\)90045-A](https://doi.org/10.1016/0304-3975(91)90045-A)
- Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*. Wiley. <https://doi.org/10.1002/9781118033067>
- Melham, R. S., & Shannon, A. G. (1995). A generalization of the Catalan identity and some consequences. *Fibonacci Quarterly*, 33(1), 82–84. <https://doi.org/10.1080/00150517.1995.12429178>
- Pethe, S., & Horadam, A. F. (1986). Generalized Gaussian Fibonacci numbers. *Bulletin of the Australian Mathematical Society*, 33(1), 37–48. <https://doi.org/10.1017/s0004972700002847>
- Rihane, S. E. (2024). On k-Fibonacci and k-Lucas numbers written as a product of two Pell numbers. *Boletín de la Sociedad Matemática Mexicana*, 30(1), 20. <https://doi.org/10.1007/s40590-024-00593-9>
- Sury, B. (2004). A variant of Binet's formula. *Mathematics Magazine*, 77(4), 308–310. <https://doi.org/10.1080/0025570X.2004.11953271>
- Vajda, S. (1989). *Fibonacci and Lucas numbers, and the golden section*. Ellis Horwood. ISBN 978-0745807157
- Verma, K. L. (2024). A comprehensive generalization of classical Fibonacci sequences, Binet formula and identities. *Journal of Applied and Pure Mathematics*, 6(5–6), 283–299. <https://doi.org/10.23091/JAPM.2024.283>
- Verma, K. L. (2025a). On the matrix representation of generalized Fibonacci sequences and applications. *Journal of Mathematical Sciences and Computational Mathematics*, 6(1), 134–144. <https://doi.org/10.15864/jmscm.6109>
- Verma, K. L. (2025b). The mathematics of generalized Fibonacci sequences: Binet's formula and identities. *Mathematica Moravica*, 29(1), 113–124. <https://doi.org/10.5937/MatMor2501113V>



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