Available online at
https://mijuitm.com.my

e-ISSN: 2735-0703

Mathematical Sciences and Informatics Journal 6(2) 2025, 1 – 21.

Mathematical Sciences and Informatics Journal

Hesitant Triangular Fuzzy Generalized Geometric Heronian Mean in MCDM

Wan Syaidatul Izzati binti Wan Abdul Rahman[1], Zahari Md Rodzi[1,2*]

[1,2]*Mathematical Science Studies, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Cawangan Negeri Sembilan, Kampus Seremban, 73000 Negeri Sembilan, Malaysia*
[2]*School of Mathematics Science, Faculty of Science and Technology, University Kebangsaan Malaysia, 43600 UKM, Bangi, Selangor, Malaysia*

ARTICLE INFO

Article history:
Received 2 May 2025
Revised 26 June 2025
Accepted 15 July 2025
Online first
Published 30 July 2025

Keywords:
Hesitant Triangular Fuzzy
MCDM
Heronian Mean
Generalized Geometric Heronian Mean
Hesitant
Triangular Fuzzy

DOI:
10.24191/mij.v6i2.4676

ABSTRACT

This paper introduces a new aggregator, the Hesitant Triangular Fuzzy Generalized Geometric Heronian Mean (HTFGGHM). The hesitant triangular fuzzy set (HTFS) combined with the Generalized Geometric Heronian mean (GGHM), makes the HTFGGHM operator capable of ensuring reasonable aggregation through desirable indexes such as idempotency, monotonicity, and boundedness. As will be shown, it manages to retain the inherent uncertainty and correlation of the criteria while offering clear and coherent rankings. Incorporating the carefulness of hesitant fuzzy sets and the compute-intensive power of GGHM, the HTFGGHM operator improves the decision accuracy and hence serves as a handy tool to tackle vague situations in a multi-attribute decision-making process.

1. INTRODUCTION

Multi-Criteria Decision Making (MCDM), in recent years, has been identified as an important area of research since it can be applied in different fields such as economics, engineering, and management. MCDM problems usually require rating many alternatives based on conflicting criteria, making the decision-making process inherently multidimensional and complex (Liu & Shih, 2024). Those traditional methods work well under some circumscribed circumstances but generally struggle with the uncertainties and imprecision of real-world decision-making (Divsalar et al., 2023).

[1,2*] Corresponding author. *E-mail address*: zahari@uitm.edu.my
https://doi.org/10.24191/mij.v6i2.4676

Fuzzy logic has become an attractive and widely used tool to model and treat uncertainty in decision-making problems (Ying & Xin, 2024). Hesitant fuzzy sets, or fuzzy sets where experts are uncertain about their preferences and thus may have a range of potential values to describe each criterion have already been used in several fuzzy models (Li & Xu, 2024). A very easy but efficient model for this kind of uncertainty is the triangular fuzzy set, which is a special case of hesitant fuzzy sets (Fang, 2023). When paired with more complex aggregation approaches like the Generalized Geometric Heronian mean (GGHM), these models offer a rigorous, analytical framework for solving multi-faceted decision/policy-making problems.

Hesitant fuzzy sets (HFS), introduced by Torra (2010), have become increasingly popular because they effectively capture the uncertainty in decision-making processes, especially when decision-makers are unsure and may offer multiple possible values or judgments for each option. In recent years, researchers have built on Torra's foundational work, applying HFS to a wide range of decision-making scenarios, particularly when preferences are unclear or conflicting. This has led to a deeper understanding of how to handle uncertainty in complex decision-making situations.

Liao and Xu (2014) introduces a method for selecting suppliers in MCDM, where hesitant fuzzy sets are used to represent uncertainty in the decision-makers' preferences. By applying an aggregation operator, the method combines multiple fuzzy values for each supplier, improving the accuracy of the selection process by better capturing the complex relationships between the criteria. Li and Xu (2024) emphasizes on HFS and their development such as normal wiggly based hesitant fuzzy sets (NWHFS) and mixed-normal-based hesitant fuzzy sets MNHFS to accommodate vagueness and include probabilistic data for effective impulse-making in multi-attribute decision-making scenarios. Hasnan et al. (2024) introduces a method that combines triangular fuzzy numbers with the MEREC approach to make decision-making in complex scenarios more accurate and reliable. By addressing uncertainty and ambiguity, the method is demonstrated through halal supplier selection, showcasing its ability to handle real-world challenges and provide clear, informed decisions.

Amman et al. (2024) proposed the concept of Dual-hesitant Fermatean fuzzy set (DHFFS), which applies Hamacher operations to make MCDM more effective through the management of both membership functions as well as non-membership hesitancy levels to aggregate complexity and develop improved solutions for MCDM in complex cases. Wang et al. (2024) fills the gaps of applying HFS in MCDM by proposing NWHFS for modeling to reduce errors of decision and interactions between criteria and alternatives. Xian et al. (2024) introduces a Z hesitant fuzzy linguistic term set (ZHFLTS) accompanied by visualization metric for mapping to T-spherical fuzzy space to MCDM and overcome the problems of ambiguity and randomness, particularly in traditional Chinese medicine (TCM).

In recent years, the use of Hesitant Triangular Fuzzy Numbers (HTFN) in MCDM has gained significant attention. Gholizade et al. (2023) develops the Hesitant Triangular Fuzzy Sorting (HTFFS) method for sorting hesitant fuzzy sets and triangular fuzzy numbers for ranking the alternatives for MCDM with linguistic inputs under uncertainty. Pu et al. (2022) introduces a decision-making method using Hesitant Triangular Fuzzy Power Aggregation (HTFPA) operators to handle uncertainty and attribute correlations, demonstrated through a futures product selection example. Anitha and Vidhya (2023) presents hesitant triangular fuzzy sets (HTFS) and propounds Dombi operation of hesitant triangular fuzzy Dombi weighted averaging and geometric operators: scoring technique to augment multi-attribute decision-making (MADM) process. Fany Helena (2024) proposes two new algorithms for MCDM based on Triangular Hesitant Fuzzy Sets (THFS), introducing mid-value ranking and ambiguity ranking methods to calculate expected values and determine criteria weights. The algorithms effectively handle uncertain and imprecise conditions, demonstrated through a case study on health issues, and highlight the advantages of THFS in addressing vague decision-making scenarios. Sultan et al. (2021) presents the hesitant fuzzy linear regression model (HFLRM) applying the symmetric triangular fuzzy numbers (STFN) to handle hesitant fuzzy data for MCDM and expounding more than the conventional fuzzy linear regression models.

https://doi.org/10.24191/mij.v6i2.4676

Researchers have introduced several aggregation operators specifically designed for HTFN sets to effectively combine diverse opinions from different experts. For instance, Wang et al. (2014) examine the aggregation of hesitant triangular fuzzy data using Bonferroni means, which help capture the interrelationships between the combined values. Rodzi et al. (2021) introduced a hesitant fuzzy MCDM approach that uses a z-score function along with hesitant degrees, deviation values, and weighted algorithms to effectively rank alternatives, as demonstrated in supplier selection scenarios. In addition, Tang et al. (2019) present dual hesitant Pythagorean fuzzy Heronian mean operators, which offer an alternative method for integrating information in hesitant Pythagorean fuzzy settings, providing a unique approach to handling uncertainty in decision-making. Nishad et al. (2023) presents a new aggregation operator for triangular fuzzy numbers in the context of hesitant fuzzy time series forecasting to refine the forecast where environment relations are comprehensive and imprecise. Wei et al. (2018) present the new q-rung orthopair fuzzy generalized Heronian mean (q-ROFGHM) operator is defined based on the generalized geometric Heronian mean (GHM) to verify its flexibility in upgrading MADM. Al-Quran (2021) presents a new MCDM method using T-Spherical Hesitant Fuzzy Sets (T-SHFS) to handle uncertainty more effectively by introducing aggregation operators like T-SHFWA and T-SHFWG, it demonstrates the usefulness in ranking alternatives, illustrated through a mobile phone selection example.

Matejíčka (2013) discusses the weighted generalized Heronian mean, which extends the Heronian mean, and proves a double inequality containing two positive numbers to obtain optimal bounds for the weighted geometric mean, Seiffert mean, and logarithmic mean. Zhang and Ji (2011) offers the generalized Heronian mean when n-tuple positive real variables are available and establish the Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity of the function in its efforts toward building the basis to study related means in mathematical analysis.

The Generalized Geometric Heronian Mean (GGHM) is a new operator interpolating between the arithmetic and the geometric means, extendable for applications in the MCDM problems since it considers the interactions and weights of the criteria (Chu & Liu, 2015). However, the traditional GGHM, the data assumed to be exact and not affected by either vagueness or hesitancy in evaluation which is not the case in real-world decisions (Li & Li, 2023). To fill this gap, Hesitant Triangular Fuzzy sets (HTFS) have been incorporated into the GGHM model. HTFS enables multiple membership values, which represent the vagueness and uncertainty represented in decision makers' preference systems. This improves the stability of the aggregation process in MCDM and captures more effectively real-world dynamics and vagaries.

Thus, the merger of HTF and GGHM provides a powerful means of solving MCDM problems, especially under conditions of high levels of uncertainty or when the data are conflicting. The HTF-GGHM operator solves the problem of multiple criteria evaluations, and uncertainty in the operator's decision-making process. Incorporating multiple perspectives and uncertainty elements makes this approach increase decision dependability. Integrating HTF can improve the evaluation by better management of different degrees of membership, enhancing the decision models. This research introduces HTF-GGHM as a viable solution in enhancing decision making accuracy under the conditions of uncertainty.

The aim of this study is to propose a new aggregation operator which is called Hesitant Triangular Fuzzy Generalized Geometric Heronian Mean (HTF-GGHM) which combines hesitant triangular fuzzy numbers with the concept of GGHM for MCDM. The HTF-GGHM operator is anticipated to incorporate the inherent uncertainty and hesitancy in the judgments of the decision-makers about the assessments with an effective way of aggregating the fuzzy information. This research will investigate the underlying principles for HTF-GGHM.

This paper is organized into distinct sections to provide a clear and structured presentation. Section 1 introduces the research problem, emphasizing the challenges of uncertainty in Multi-Criteria Decision-Making (MCDM) and the need for advanced aggregation methods. Section 2 outlines the fundamental concepts, including fuzzy sets, hesitant fuzzy sets, and triangular fuzzy numbers, which serve as the theoretical basis for the study. Section 3 proposes the Hesitant Triangular Fuzzy Generalized Geometric

https://doi.org/10.24191/mij.v6i2.4676

Heronian Mean (HTFGGHM) operator, describing its development and properties such as idempotency, monotonicity, and boundedness. Section 4 concludes the paper by summarizing its contributions and suggesting future research directions for expanding its applicability in various decision-making contexts.

2. PRELIMINARIES

Definition 1. Fuzzy set: (Zadeh, 1965): A fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ is defined as

$$A = \{\langle x, \mu_A(x)\rangle x \in X\} \tag{1}$$

where $\mu_A(x): X \to [0,1]$ represents the membership function of (A) where $\mu_A(x)$ specifies the degree of membership of an element$x \in X$ within A. Building on this, the concept of HFS was introduced by Torra (2010), allowing the membership degree of an element to encompass multiple possible values between 0 and 1. HFS effectively capture situations where individuals express varying degrees of uncertainty in their preferences, enhancing decision-making by accommodating hesitancy.

2.1 Hesitant Fuzzy Set (HFS)

Definition 2. Hesitant Fuzzy set (Torra, 2010): Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set. A set E defined in X is represented as

$$E = \{\langle x, h_E(x)\rangle | x \in X\} \tag{2}$$

where $h_E(x)$ is a set of various values within $[0,1]$, representing the potential membership degrees of the element $x \in X$ in the set E. This is known as a hesitant fuzzy set. Additionally, Torra (2010) introduced the concepts of the "empty hesitant fuzzy set" and the "full hesitant fuzzy set" as follows:

$$E^{\circ} = \left\{\langle x, h_{E^{\circ}}(x)\rangle \middle| x \in X\right\} \text{ where } h_{E^{\circ}}(x) = \{0\}\ \forall x \in X,$$
$$E^{*} = \{\langle x, h_{E^{*}}(x)\rangle | x \in X\} \text{ where } h_{E^{*}}(x) = \{1\}\ \forall x \in X$$

2.2 Triangular Fuzzy Numbers

Definition 3. Triangular Fuzzy Numbers (Van Laarhoven & Pedrycz, 1983): A triangular fuzzy number $\tilde{a}$ can be defined by a triplet (a^L, a^M, a^U). The membership function $u_{\tilde{a}}(x)$ is defined as:

$$u_{\tilde{a}}(x) = \begin{cases} 0, & x < a_L \\ \dfrac{x - a_L}{a_M - a_L}, & a_L \le x \le a_M \\ \dfrac{a_U - x}{a_U - a_M}, & a_M \le x \le a_U \\ 0, & x < a_U \end{cases} \tag{3}$$

where $0 < aL \le a_M \le a_U$, a_L and a_U stand for the lower and upper values of the support of $\tilde{a}$, respectively, and a_M is the middle value.

https://doi.org/10.24191/mij.v6i2.4676

Definition 4 (Van Laarhoven & Pedrycz, 1983): Basic operational laws relating to triangular fuzzy numbers:

$$\tilde{a} \oplus \tilde{b} = [a^L, a^M, a^U] \oplus [b^L, b^M, b^U] = [a^L + b^L, a^M + b^M, a^U + b^U] \quad (4)$$
$$\tilde{a} \otimes \tilde{b} = [a^L, a^M, a^U] \otimes [b^L, b^M, b^U] = [a^L b^L, a^M b^M, a^U b^U] \quad (5)$$
$$\lambda \otimes \tilde{a} = \lambda \otimes [a^L, a^M, a^U] = [\lambda a^L, \lambda a^M, \lambda a^U], \lambda > 0 \quad (6)$$

Definition 5 (Xu, 2009): Let $\tilde{b} = [b^L, b^M, b^U]$ and $\tilde{a}= [a^L, a^M, a^U]$ be two triangular fuzzy numbers, then the degree of possibility of a $\geq b$ is defined as

$$p(a \geq b) = \lambda \max\left\{1 - \max\left[\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0\right], 0\right\} + (1 - \lambda)\max\left\{1 - max\left[\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0\right], 0\right\} \quad (7)$$

where the value λ is an index of rating attitude. It reflects the decision maker's risk-bearing attitude. If λ < 0.5, the decision maker is risk averter.

From definition 5, we can easily get the following results:

$$(1)\ 0 \leq p(\tilde{a} \geq \tilde{b}) \leq 1, 0 \leq p(\tilde{b} \geq \tilde{a}) \leq 1; \quad (8)$$
$$(2)\ p(\tilde{a} \geq \tilde{b}) + p(\tilde{b} \geq \tilde{a}) = 1. \quad (9)$$
$$p(\tilde{a} \geq \tilde{a}) = p(\tilde{b} \geq \tilde{b}) = 0.5. \quad (10)$$

2.3 Hesitant Triangular Fuzzy Set (HTFS)

Definition 6 (Zhao et al., 2014): Let X be a fixed set, a hesitant triangular fuzzy set (HTFS) on X is in terms of a function that when applied to each x in X and returns a subset of values in [0,1].

$$E = \{\langle x, h_E(x)\rangle | x \in X\} \quad (11)$$

where $\tilde{h}_{E(x)}$ is a set of some possible triangular fuzzy values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, we call $\tilde{h}_{E(x)} = \tilde{h}_1 = (\gamma L, \gamma M, \gamma R)$ a hesitant triangular fuzzy element (HTFE) and $\tilde{h}$ the set of all HTFEs.

Given three HTFEs, $\tilde{h} = (\gamma L, \gamma M, \gamma R), \tilde{h}_1 = (\gamma_1^L, \gamma_1^M, \gamma_1^R)$, $\tilde{h}_2 = (\gamma_2^L, \gamma_2^M, \gamma_2^R)$ and $\lambda > 0$, we define their operations as follows:

$$h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}; \quad (12)$$

$$\lambda h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}; \quad (13)$$

$$h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\} \quad (14)$$

$$h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}. \quad (15)$$

https://doi.org/10.24191/mij.v6i2.4676

Definition 7. Score Function (Xia & Xu, 2011): Consider a hesitant fuzzy element h, the score function S of an HFE is defined as:

$$S(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma \tag{16}$$

where $\#h$ represents the total number of elements in h.

Definition 8. Let h_1 and h_2 be two hesitant fuzzy elements with scores $S(h_1)$ and $S(h_2)$, respectively. Then:

(1) h_1 is larger than h_2, denoted by $h_1 > h_2$ if $S(h_1) > S(h_2)$
(2) h_1 is considered equal to h_2 if $S(h_1) = S(h_2)$.

2.4 Heronian Mean

The Heronian Mean (HM) is an aggregation method notable for its capability to capture relationships among the inputs. The HM is defined as follows:

Definition 9. Heronian Mean (Beliakov et al., 2007): For a set of nonnegative real numbers a_i where $i = 1,2,\dots,n$, the Heronian mean is defined as:

$$HM(a_1, a_2, \dots, a_n) = \frac{2}{n(n+1)}\sum_{i,j=1}^{n}\sqrt{a_i a_j}. \tag{17}$$

Definition 10. Geometric Heronian Mean (Yu, 2013): Geometric Heronian Mean (GHM) for a set of nonnegative real numbers a_i, where $i = 1,2,\dots,n$, which is defined by:

$$GHM(a_1, a_2, \dots, a_n) = \prod_{i,j=1}^{n}\left(\frac{a_i + a_j}{2}\right)^{\frac{2}{n(n+1)}}. \tag{18}$$

Definition 11. Generalized Geometric Heronian Mean (Yu, 2013): For nonnegative real numbers a_i where $i = 1,2,\dots,n$ and parameters $p, q \geq 0$ (with p and q not both equal to 0), the Generalized Geometric Heronian Mean is defined as:

$$GGHM^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q}\prod_{i,j=1}^{n}\left(pa_i + qa_j\right)^{\frac{2}{n(n+1)}}. \tag{19}$$

The properties of $GGHM^{p,q}$ are as follows:

(i) $GGHM^{p,q}(0,0,\dots,0) = 0$ and $GGHM^{p,q}(1,1,\dots,1) = 1$;
(ii) $GGHM^{p,q}(a_1, a_2, \dots, a_n) = a$ if all $a_i = a$;
(iii) If $(a_i \leq b_i)\ \forall(i)$, then $GGHM^{p,q}(a_1, a_2, \dots, a_n) \leq GGHM^{p,q}(b_1, b_2, \dots, b_n)$, indicating $GGHM^{p,q}$ is monotonic;
(iv) $\min_i\{a_i\} \leq GGHM^{p,q}(a_1, a_2, \dots, a_n) \leq \max_i\{a_i\}$

In the next section, the GGHM is extended to a hesitant fuzzy context with the following proposed methods:

(i) The hesitant triangular fuzzy generalized geometric Heronian mean (HFGGHM).

https://doi.org/10.24191/mij.v6i2.4676

(ii) The weighted hesitant triangular fuzzy generalized geometric Heronian mean (WHTFGGHM).

3. PROPOSED HESITANT TRIANGULAR FUZZY GENERALIZED GEOMETRIC HERONIAN MEAN (HTFGGHM)

Definition 12. Hesitant Triangular Fuzzy Generalized Geometric Heronian Mean: Let $p, q > 0$ and let $h_i = (l_i, m_i, u_i)$ be a collection of hesitant fuzzy triangular elements. The hesitant triangular fuzzy generalized geometric Heronian mean $HTFGGHM^{p,q}$ is defined as:

$$HTFGGHM^{p,q}(h_1, h_2, \dots, h_n) = \frac{1}{p+q} \bigotimes_{i,j=1;i\leq j}^{n} \left(\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)\right)^{\frac{2}{n(n+1)}}, \tag{20}$$

where $\oplus$ and $\otimes$ denote the operations applied according to the laws of hesitant fuzzy elements.

Theorem 1. Let $p, q > 0$ and let $h_i = (l_i, m_i, u_i)$ be a collection of hesitant triangular fuzzy elements. Then the aggregated value obtained using the $HTFGGHM^{p,q}$ operator is also a hesitant triangular fuzzy element, and

$$HTFGGHM^{p,q}(h_1, h_2, \dots, h_n) = \frac{1}{p+q} \bigotimes_{i,j=1;i\leq j}^{n} \left(\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)\right)^{\frac{2}{n(n+1)}},$$

which can also be expressed as:

$$\begin{aligned}
&= 1 - \left(1 - \prod_{i,j=1;i\leq j}^{n} \eta_{i,j}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \\
&= 1 - \Bigg[1 - \textstyle\prod_{i,j=1;i\leq j}^{n} \Big((1-(1-l_i)^p) + \left(1-(1-l_j)^q\right) - (1-(1-l_i)^p)\cdot\big(1- \\
&\left(1-l_j\right)^q\big)\Big)\cdot\Big(\left(1-\left(1-l_j\right)^p\right) + (1-(1-l_i)^q) - \left(1-\left(1-l_j\right)^p\right)\cdot(1-(1- \\
&l_i)^q)\Big)^{\frac{2}{n(n+1)}}\Bigg]^{\frac{1}{p+q}}, 1 - \Bigg[1 - \textstyle\prod_{i,j=1;i\leq j}^{n} \Big((1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right) - (1- \\
&(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right)\Big)\cdot\Big(\left(1-\left(1-m_j\right)^p\right) + (1-(1-m_i)^q) - \big(1-\big(1- \\
&m_j\big)^p\big)\cdot(1-(1-m_i)^q)\Big)^{\frac{2}{n(n+1)}}\Bigg]^{\frac{1}{p+q}}, 1 - \Bigg[1 - \textstyle\prod_{i,j=1;i\leq j}^{n} \Big((1-(1-u_i)^p) + \big(1-\big(1- \\
&u_j\big)^q\big) - (1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right)\Big)\cdot\Big(\left(1-\left(1-u_j\right)^p\right) + (1-(1-u_i)^q) - \\
&\left(1-\left(1-u_j\right)^p\right)\cdot(1-(1-u_i)^q)\Big)^{\frac{2}{n(n+1)}}\Bigg]^{\frac{1}{p+q}}.
\end{aligned} \tag{21}$$

Here, $\eta_{i,j}$ represents the individual elements within the hesitant fuzzy sets involved in the aggregation.

$ph_i \oplus qh_j =$

https://doi.org/10.24191/mij.v6i2.4676

By referring to equation (13), we'll get

$$ph_i = \{1-(1-l_i)^p, 1-(1-m_i)^p, 1-(1-u_i)^p\}$$
$$qh_j = \left\{1-\left(1-l_j\right)^q, 1-\left(1-m_j\right)^q, 1-\left(1-u_j\right)^q\right\}$$

$$ph_j = \left\{1-\left(1-l_j\right)^p, 1-\left(1-m_j\right)^p, 1-\left(1-u_j\right)^p\right\}$$
$$qh_i = \{1-(1-l_i)^q, 1-(1-m_i)^q, 1-(1-u_i)^q\}$$

Proof: Using equation (14), we get

$\boldsymbol{Addition}(\oplus)$: $ph_i \oplus qh_j$

$$= \begin{pmatrix} (1-(1-l_i)^p) + \left(1-\left(1-l_j\right)^q\right) - (1-(1-l_i)^p)\cdot\left(1-\left(1-l_j\right)^q\right), \\ (1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right) - (1-(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right), \\ (1-(1-u_i)^p) + \left(1-\left(1-u_j\right)^q\right) - (1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right) \end{pmatrix}$$

$\boldsymbol{Addition}(\oplus)$: $ph_j \oplus qh_i$

$$= \begin{pmatrix} (1-\left(1-l_j\right)^p) + (1-(1-l_i)^q) - (1-\left(1-l_j\right)^p)\cdot(1-(1-l_i)^q), \\ (1-\left(1-m_j\right)^p) + (1-(1-m_i)^q) - (1-\left(1-m_j\right)^p)\cdot(1-(1-m_i)^q), \\ (1-\left(1-u_j\right)^p) + (1-(1-u_i)^q) - (1-\left(1-u_j\right)^p)\cdot(1-(1-u_i)^q) \end{pmatrix}$$

By equation (15), we get

$$\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right) =$$

$$\begin{aligned}\Big[\Big((1-(1-l_i)^p) &+ \left(1-\left(1-l_j\right)^q\right) - (1-(1-l_i)^p)\cdot\left(1-\left(1-l_j\right)^q\right)\Big) \\ &\cdot\left(\left(1-\left(1-l_j\right)^p\right) + (1-(1-l_i)^q) - \left(1-\left(1-l_j\right)^p\right)\right. \\ &\left.\cdot(1-(1-l_i)^q)\right), \Big((1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right) \\ &- (1-(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right)\Big) \\ &\cdot\left(\left(1-\left(1-m_j\right)^p\right) + (1-(1-m_i)^q) - \left(1-\left(1-m_j\right)^p\right)\right. \\ &\left.\cdot(1-(1-m_i)^q)\right), \Big((1-(1-u_i)^p) + \left(1-\left(1-u_j\right)^q\right) \\ &- (1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right)\Big) \\ &\cdot\left((1-\left(1-u_j\right)^p) + (1-(1-u_i)^q) - (1-\left(1-u_j\right)^p)\cdot(1\right. \\ &\left.-(1-u_i)^q)\right)\Big]\end{aligned}$$

https://doi.org/10.24191/mij.v6i2.4676

From equation (12),

$$\left(\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)\right)^{\frac{2}{n(n+1)}} =$$

$$\Big[\Big((1-(1-l_i)^p) + \left(1-\left(1-l_j\right)^q\right) - (1-(1-l_i)^p)\cdot\left(1-\left(1-l_j\right)^q\right)\Big)$$
$$\cdot\Big(\left(1-\left(1-l_j\right)^p\right) + (1-(1-l_i)^q) - \left(1-\left(1-l_j\right)^p\right)$$
$$\cdot(1-(1-l_i)^q)\Big)^{\frac{2}{n(n+1)}}, \Big((1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right)$$
$$-(1-(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right)\Big)$$
$$\cdot\Big(\left(1-\left(1-m_j\right)^p\right) + (1-(1-m_i)^q) - \left(1-\left(1-m_j\right)^p\right)$$
$$\cdot(1-(1-m_i)^q)\Big)^{\frac{2}{n(n+1)}}, \Big((1-(1-u_i)^p) + \left(1-\left(1-u_j\right)^q\right)$$
$$-(1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right)\Big)$$
$$\cdot\Big((1-\left(1-u_j\right)^p) + (1-(1-u_i)^q) - (1-\left(1-u_j\right)^p)\cdot(1$$
$$-(1-u_i)^q)\Big)^{\frac{2}{n(n+1)}}\Big]$$

By referring to equation (15),

$$\prod_{i,j=1;i\le j}^{n}\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)^{\frac{2}{n(n+1)}} =$$

$$\prod_{i,j=1;i\le j}^{n}\Big((1-(1-l_i)^p) + \left(1-\left(1-l_j\right)^q\right) - (1-(1-l_i)^p)\cdot\left(1-\left(1-l_j\right)^q\right)\Big)\cdot \Big(\left(1-\left(1-l_j\right)^p\right) + (1-(1-l_i)^q) - \left(1-\left(1-l_j\right)^p\right)\cdot(1-(1-l_i)^q)\Big)^{\frac{2}{n(n+1)}}, \prod_{i,j=1;i\le j}^{n}\Big((1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right) - (1-(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right)\Big)\cdot\Big(\left(1-\left(1-m_j\right)^p\right) + (1-(1-m_i)^q) - \left(1-\left(1-m_j\right)^p\right)\cdot(1-(1-m_i)^q)\Big)^{\frac{2}{n(n+1)}}, \prod_{i,j=1;i\le j}^{n}\Big((1-(1-u_i)^p) + \left(1-\left(1-u_j\right)^q\right) - (1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right)\Big)\cdot\Big((1-\left(1-u_j\right)^p) + (1-(1-u_i)^q) - (1-\left(1-u_j\right)^p)\cdot(1-(1-u_i)^q)\Big)^{\frac{2}{n(n+1)}}$$

By referring equation (13), we'll get

$$\frac{1}{p+q}\bigotimes_{i,j=1;i\le j}^{n}\left(\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)\right)^{\frac{2}{n(n+1)}}$$

$$= 1 - \left(1 - \prod_{i,j=1;i\le j}^{n} \eta_{i,j}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$$

https://doi.org/10.24191/mij.v6i2.4676

$= 1 - \left[1 - \prod_{i,j=1;i\le j}^{n}\left((1-(1-l_i)^p) + \left(1-\left(1-l_j\right)^q\right) - (1-(1-l_i)^p)\cdot\left(1-\left(1-l_j\right)^q\right)\right)\cdot \left(\left(1-\left(1-l_j\right)^p\right) + (1-(1-l_i)^q) - \left(1-\left(1-l_j\right)^p\right)\cdot(1-(1-l_i)^q)\right)^{\frac{2}{n(n+1)}}\right]^{\frac{1}{p+q}}, 1 - \left[1 - \prod_{i,j=1;i\le j}^{n}\left((1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right) - (1-(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right)\right)\cdot\left(\left(1-\left(1-m_j\right)^p\right) + (1-(1-m_i)^q) - \left(1-\left(1-m_j\right)^p\right)\cdot(1-(1-m_i)^q)\right)^{\frac{2}{n(n+1)}}\right]^{\frac{1}{p+q}}, 1 - \left[1 - \prod_{i,j=1;i\le j}^{n}\left((1-(1-u_i)^p) + \left(1-\left(1-u_j\right)^q\right) - (1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right)\right)\cdot\left((1-\left(1-u_j\right)^p) + (1-(1-u_i)^q) - \left(1-\left(1-u_j\right)^p\right)\cdot(1-(1-u_i)^q)\right)^{\frac{2}{n(n+1)}}\right]^{\frac{1}{p+q}}$

Equation (21) has been proven. □

Proposition 2. Let h_{α_i} and h_{β_i} be two collections of HTEs, $\eta_{i,j,i<j} = (ph_{\alpha_i} \oplus qh_{\alpha_j}) \otimes (ph_{\alpha_j} \oplus qh_{\alpha_i})$ and $\eta_{i,j,i<j} = (ph_{\beta_i} \oplus qh_{\beta_j}) \otimes (ph_{\beta_j} \oplus qh_{\beta_i})$. If for any $\gamma_{\alpha_i} \in h_{\alpha_i}$ and $\gamma_{\beta_i} \in h_{\beta_i}$ $(i,j = 1,2,\ldots,n; i \ne j)$, we have $\gamma_{\alpha_i} \le \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \le \gamma_{\beta_j}$, then $\eta_{\alpha_{i,j,i<j}} \le \eta_{\beta_{i,j,i<j}}$.

Proposition 3. Let $h_i (i = 1,2,\ldots,n)$ be a collection of HTFEs, $\eta_{ij,i<j} = \left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)$ and $h_i^- = \cup_{\gamma_i \in h_i}(min\{\gamma_i^L\}, min\{\gamma_i^M\}, min\{\gamma_i^U\})$, $h_i^+ = \cup_{\gamma_i \in h_i}(max\{\gamma_i^L\}, max\{\gamma_i^M\}, max\{\gamma_i^U\})$, $i,j \in \{1,2,\ldots,n\}$; then

$$\bigcup_{\gamma^- \in h_i^-}((1-(1-\gamma^{-L})^{p+q})^{\mathbf{2}}, (1-(1-\gamma^{-M})^{p+q})^{\mathbf{2}}, (1-(1-\gamma^{-U})^{p+q})^{\mathbf{2}}) \le \bigcup_{\gamma^+ \in h_i^+}((1-(1-\gamma^{+L})^{p+q})^{\mathbf{2}}, (1-(1-\gamma^{+M})^{p+q})^{\mathbf{2}}, (1-(1-\gamma^{+U})^{p+q})^{\mathbf{2}}) \tag{22}$$

Theorem 4. Idempotency property: $A = (l_i, m_{i,} u_i)(i = 1,2,\ldots,n)$ represent a set of HTFEs. If $A_i = (l_i, m_{i,} u_i)$ for all i, then $HTF - GGHM^{p,q}\left(A_{1,} A_2, \ldots, A_n\right) = a$.

Proven since $A_i = (l_i, m_{i,} u_i)(i = 1,2,\ldots,n)$, then,

$$GGHM^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q}\prod_{i,j=1}^{n}\left(pa_i + qa_j\right)^{\frac{2}{n(n+1)}} \tag{23}$$

$$= \frac{1}{p+q}\left[\prod_{i,j=1}^{n}(pa+qa)\right]^{\frac{2}{n(n+1)}}$$

$$= \frac{1}{p+q}\left[\prod_{i,j=1}^{n}(a(pa+qa))^{\frac{n(n+1)}{2}}\right]^{\frac{2}{n(n+1)}}$$

https://doi.org/10.24191/mij.v6i2.4676

$$= \frac{1}{p+q}\left[\left(a(p+q)\right)^{\frac{n(n+1)}{2}}\right]^{\frac{2}{n(n+1)}}$$
$$= a$$

Theorem 5. Monotonicity property: Let h_{α_i} and h_{β_i} $(i = 1,2, \dots, n)$ be two collections of HTFEs; if , for any $\gamma_{\alpha_i} \le h_{\alpha_i}$ and $\gamma_{\beta_j} \le h_{\beta_j} (i, j = 1,2, \dots, n; i \ne j)$, one has $\gamma_{\alpha_i} \le \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \le \gamma_{\beta_j}$. Then,

$$HTF - GGHM^{p,q}(h_{\alpha_1}, h_{\alpha_2}, \dots, h_{\alpha_n}) \le HTF - GGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \dots, h_{\beta_n})$$

Proof: By proposition 2, we get $\eta_{\alpha_{ij}} \le \eta_{\beta_{ij}}, i, j \in \{1,2, \dots, n\}, i \ne j$
Then

$$1 - \left(1 - \prod_{i,j=1; i\le j}^{n} \eta_{\alpha_{i,j}}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \le 1 - \left(1 - \prod_{i,j=1; i\le j}^{n} \eta_{\beta_{i,j}}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \tag{24}$$

By definition 7, we acquire:

$$\begin{aligned}
&HTF - GGHM^{p,q}\left(h_{\alpha_1}, h_{\alpha_2}, \dots, h_{\alpha_n}\right) \\
&\quad = \frac{1}{p+q} \bigotimes_{i,j=1; i\le j}^{n} \left(\eta_{\alpha_{ij}}\right)^{\frac{2}{n(n+1)}} \\
&\quad = \bigcup_{\eta_{i,j}\in\sigma_i, i\le j} \left\{ 1 - \left(1 - \prod_{i,j=1; i\le j}^{n} \eta_{\alpha_{i,j}}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \right\} \\
&\quad \le \bigcup_{\eta_{i,j}\in\sigma_i, i\le j} \left\{ 1 - \left(1 - \prod_{i,j=1; i\le j}^{n} \eta_{\beta_{i,j}}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \right\} \\
&\quad = \frac{1}{p+q} \bigotimes_{i,j=1; i\le j}^{n} \left(\eta_{\beta_{ij}}\right)^{\frac{2}{n(n+1)}} \\
&\quad = HTF - GGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \dots, h_{\beta_n})
\end{aligned} \tag{25}$$

Theorem 6. Boundedness: Let $h_i (i = 1,2, \dots, n\ )$ be a collection of HTEs, $h_i^- = \cup_{\gamma_i \in h_i} (min\{\gamma_i^L\}, min\{\gamma_i^M\}, min\{\gamma_i^U\}), h_i^+ = \cup_{\gamma_i\in h_i}(max\{\gamma_i^L\}, max\{\gamma_i^M\}, max\{\gamma_i^U\}), i, j \in \{1,2, \dots, n\}$; then

https://doi.org/10.24191/mij.v6i2.4676

$$\bigcup_{\gamma^- \in h_i^-}\left(((1-(1-\gamma^{-L})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{-M})^{p+q})^2)^{\frac{1}{p+q}}, ((1 - (1-\gamma^{-U})^{p+q})^2)^{\frac{1}{p+q}}\right) \le HTFGGHM^{p,q}(h_1, h_2, \dots, h_n) \le \bigcup_{\gamma^+ \in h_i^+}\left(((1-(1-\gamma^{+L})^{p+q})^2)^{\frac{1}{p+q}}, ((1 - (1-\gamma^{+M})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{+U})^{p+q})^2)^{\frac{1}{p+q}}\right) \tag{26}$$

Proof: By Proposition 3, we have

$$\bigcup_{\gamma^- \in h_i^-}((1-(1-\gamma^{-L})^{p+q})^2, (1-(1-\gamma^{-M})^{p+q})^2, (1-(1-\gamma^{-U})^{p+q})^2) \le \eta_{ij,i<j} \le \bigcup_{\gamma^+ \in h_i^+}((1-(1-\gamma^{+L})^{p+q})^2, (1-(1-\gamma^{+M})^{p+q})^2, (1 - (1-\gamma^{+U})^{p+q})^2)$$

So

$$\bigcup_{\gamma^- \in h_i^-}\left(((1-(1-\gamma^{-L})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{-M})^{p+q})^2)^{\frac{1}{p+q}}, ((1 - (1-\gamma^{-U})^{p+q})^2)^{\frac{1}{p+q}}\right) \le 1 - \left(1 - \prod_{i,j=1; i\le j}^{n} \eta_{i,j}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \le \bigcup_{\gamma^+ \in h_i^+}\left(((1-(1-\gamma^{+L})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{+M})^{p+q})^2)^{\frac{1}{p+q}}, ((1 - (1-\gamma^{+U})^{p+q})^2)^{\frac{1}{p+q}}\right) \tag{27}$$

By definition 7, we complete the proof.

Theorem 7. Suppose $p, q > 0$ and let $h_i = (l_i, m_i, u_i)$ represent a set of hesitant triangular fuzzy elements, accompanied by a weight vector $w = (w_1, w_2, \dots, w_n)^T$ where $w_i \ge 0, i = 1,2, \dots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregated value derived using the $\boldsymbol{WHTFGGHM^{p,q}}$ operator is also a hesitant triangular fuzzy element, and

$$\boldsymbol{WHTFGGHM^{p,q}}(h_1, h_2, \dots, h_n)s = \frac{1}{p+q} \bigotimes_{i,j=1; i\le j}^{n} \left((ph_i \oplus qh_j) \otimes (ph_j \oplus qh_i)\right)^{\frac{2w_i w_j}{(1+w_i)}},$$

https://doi.org/10.24191/mij.v6i2.4676

which can also be expressed as:

$$= \bigcup_{\eta_{i,j}\in\sigma i,j;i\le j} \left\{ 1-\left(1-\prod_{i,j=1;i\le j}^{n} \eta_{i,j}^{\frac{2w_iw_j}{(1+w_i)}}\right)^{\frac{1}{p+q}} \right\}$$

$$= 1-\left[1-\prod_{i,j=1;i\le j}^{n}\Big((1-(1-l_i)^p)+\big(1-(1-l_j)^q\big)-(1-(1-l_i)^p)\cdot\big(1-(1-l_j)^q\big)\Big)\cdot\Big(\big(1-(1-l_j)^p\big)+(1-(1-l_i)^q)-\big(1-(1-l_j)^p\big)\cdot(1-(1-l_i)^q)\Big)^{\frac{2w_iw_j}{(1+w_i)}}\right]^{\frac{1}{p+q}}, 1-\left[1-\prod_{i,j=1;i\le j}^{n}\Big((1-(1-m_i)^p)+\big(1-(1-m_j)^q\big)-(1-(1-m_i)^p)\cdot\big(1-(1-m_j)^q\big)\Big)\cdot\Big(\big(1-(1-m_j)^p\big)+(1-(1-m_i)^q)-\big(1-(1-m_j)^p\big)\cdot(1-(1-m_i)^q)\Big)^{\frac{2w_iw_j}{(1+w_i)}}\right]^{\frac{1}{p+q}}, 1-\left[1-\prod_{i,j=1;i\le j}^{n}\Big((1-(1-u_i)^p)+\big(1-(1-u_j)^q\big)-(1-(1-u_i)^p)\cdot\big(1-(1-u_j)^q\big)\Big)\cdot\Big(\big(1-(1-u_j)^p\big)+(1-(1-u_i)^q)-\big(1-(1-u_j)^p\big)\cdot(1-(1-u_i)^q)\Big)^{\frac{2w_iw_j}{(1+w_i)}}\right]^{\frac{1}{p+q}} \quad (28)$$

Here, $\eta_{i,j}$ represents the individual elements within the hesitant fuzzy sets involved in the aggregation.

$ph_i \oplus qh_j$=

By referring to equation (13), we get

$$\begin{aligned} ph_i &= \{1-(1-l_i)^p, 1-(1-m_i)^p, 1-(1-u_i)^p\} \\ qh_j &= \{1-(1-l_j)^q, 1-(1-m_j)^q, 1-(1-u_j)^q\} \\ ph_j &= \{1-(1-l_j)^p, 1-(1-m_j)^p, 1-(1-u_j)^p\} \\ qh_i &= \{1-(1-l_i)^q, 1-(1-m_i)^q, 1-(1-u_i)^q\} \end{aligned}$$

Proof: Using equation (14), we get

Addition$(\oplus)$: $ph_i \oplus qh_j$

$$= \begin{pmatrix} (1-(1-l_i)^p)+\big(1-(1-l_j)^q\big)-(1-(1-l_i)^p)\cdot\big(1-(1-l_j)^q\big), \\ (1-(1-m_i)^p)+\big(1-(1-m_j)^q\big)-(1-(1-m_i)^p)\cdot\big(1-(1-m_j)^q\big), \\ (1-(1-u_i)^p)+\big(1-(1-u_j)^q\big)-(1-(1-u_i)^p)\cdot\big(1-(1-u_j)^q\big) \end{pmatrix}$$

https://doi.org/10.24191/mij.v6i2.4676

$\boldsymbol{Addition}(\oplus)$: $ph_j \oplus qh_i$

$$= \begin{pmatrix} (1-(1-l_j)^p) + (1-(1-l_i)^q) - (1-(1-l_j)^p)\cdot(1-(1-l_i)^q), \\ (1-(1-m_j)^p) + (1-(1-m_i)^q) - (1-(1-m_j)^p)\cdot(1-(1-m_i)^q), \\ (1-(1-u_j)^p) + (1-(1-u_i)^q) - (1-(1-u_j)^p)\cdot(1-(1-u_i)^q) \end{pmatrix}$$

By equation (15), we'll get

$$\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right) =$$

$$\begin{aligned} &\Big[\Big((1-(1-l_i)^p) + \left(1-(1-l_j)^q\right) - (1-(1-l_i)^p)\cdot\left(1-(1-l_j)^q\right)\Big) \\ &\cdot\Big(\left(1-(1-l_j)^p\right) + (1-(1-l_i)^q) - \left(1-(1-l_j)^p\right) \\ &\cdot(1-(1-l_i)^q)\Big), \Big((1-(1-m_i)^p) + \left(1-(1-m_j)^q\right) \\ &- (1-(1-m_i)^p)\cdot\left(1-(1-m_j)^q\right)\Big) \\ &\cdot\Big(\left(1-(1-m_j)^p\right) + (1-(1-m_i)^q) - \left(1-(1-m_j)^p\right) \\ &\cdot(1-(1-m_i)^q)\Big), \Big((1-(1-u_i)^p) + \left(1-(1-u_j)^q\right) \\ &- (1-(1-u_i)^p)\cdot\left(1-(1-u_j)^q\right)\Big) \\ &\cdot\Big((1-(1-u_j)^p) + (1-(1-u_i)^q) - (1-(1-u_j)^p)\cdot(1 \\ &- (1-u_i)^q)\Big)\Big] \end{aligned}$$

From equation (12),

$$\left(\left(ph_i \oplus qh_j\right) \otimes \left(ph_j \oplus qh_i\right)\right)^{\frac{2w_iw_j}{(1+w_i)}} =$$

$$\Bigg[\Big((1-(1-l_i)^p)+\big(1-(1-l_j)^q\big)-(1-(1-l_i)^p)\cdot\big(1-(1-l_j)^q\big)\Big)$$
$$\cdot\Big(\big(1-(1-l_j)^p\big)+(1-(1-l_i)^q)-\big(1-(1-l_j)^p\big)$$
$$\cdot(1-(1-l_i)^q)\Big)^{\frac{2w_iw_j}{(1+w_i)}},\Big((1-(1-m_i)^p)+\big(1-(1-m_j)^q\big)$$
$$-(1-(1-m_i)^p)\cdot\big(1-(1-m_j)^q\big)\Big)$$
$$\cdot\Big(\big(1-(1-m_j)^p\big)+(1-(1-m_i)^q)-\big(1-(1-m_j)^p\big)$$
$$\cdot(1-(1-m_i)^q)\Big)^{\frac{2w_iw_j}{(1+w_i)}},\Big((1-(1-u_i)^p)+\big(1-(1-u_j)^q\big)$$
$$-(1-(1-u_i)^p)\cdot\big(1-(1-u_j)^q\big)\Big)$$
$$\cdot\big((1-(1-u_j)^p\big)+(1-(1-u_i)^q)-(1-\big(1-u_j)^p\big)\cdot(1$$
$$-(1-u_i)^q)\big)^{\frac{2w_iw_j}{(1+w_i)}}\Bigg]$$

By referring to equation (15),

$$\prod_{i,j=1;i\le j}^{n}\left(ph_i\oplus qh_j\right)\otimes\left(ph_j\oplus qh_i\right)^{\frac{2w_iw_j}{(1+w_i)}}=$$

https://doi.org/10.24191/mij.v6i2.4676

$$\prod_{i,j=1;i\leq j}^{n}\left((1-(1-l_i)^p)+\left(1-\left(1-l_j\right)^q\right)-(1-(1-l_i)^p)\cdot\left(1-\left(1-l_j\right)^q\right)\right)$$
$$\cdot\left(\left(1-\left(1-l_j\right)^p\right)+(1-(1-l_i)^q)-\left(1-\left(1-l_j\right)^p\right)\right.$$
$$\left.\cdot(1-(1-l_i)^q)\right)^{\frac{2w_iw_j}{(1+w_i)}},\prod_{i,j=1;i\leq j}^{n}\left((1-(1-m_i)^p)+\left(1-\left(1-m_j\right)^q\right)\right.$$
$$\left.-(1-(1-m_i)^p)\cdot\left(1-\left(1-m_j\right)^q\right)\right)$$
$$\cdot\left(\left(1-\left(1-m_j\right)^p\right)+(1-(1-m_i)^q)-\left(1-\left(1-m_j\right)^p\right)\right.$$
$$\left.\cdot(1-(1-m_i)^q)\right)^{\frac{2w_iw_j}{(1+w_i)}},\prod_{i,j=1;i\leq j}^{n}\left((1-(1-u_i)^p)+\left(1-\left(1-u_j\right)^q\right)\right.$$
$$\left.-(1-(1-u_i)^p)\cdot\left(1-\left(1-u_j\right)^q\right)\right)$$
$$\cdot\left((1-\left(1-u_j\right)^p)+(1-(1-u_i)^q)-(1-\left(1-u_j\right)^p)\cdot(1\right.$$
$$\left.-(1-u_i)^q)\right)^{\frac{2w_iw_j}{(1+w_i)}}$$

By referring to equation (13), we'll get

$$\frac{1}{p+q}\bigotimes_{i,j=1;i\leq j}^{n}\left(\left(ph_i\oplus qh_j\right)\otimes\left(ph_j\oplus qh_i\right)\right)^{\frac{2w_iw_j}{(1+w_i)}}$$

$$=1-\left(1-\prod\nolimits_{i,j=1;i\leq j}^{n}\eta_{i,j}^{\frac{2w_iw_j}{(1+w_i)}}\right)^{\frac{1}{p+q}}$$

https://doi.org/10.24191/mij.v6i2.4676

$$= 1 - \left[1 - \prod_{i,j=1; i \le j}^{n} \left((1-(1-l_i)^p) + \left(1-\left(1-l_j\right)^q\right) - (1-(1-l_i)^p) \cdot \left(1-\left(1-l_j\right)^q\right)\right) \cdot \left(\left(1-\left(1-l_j\right)^p\right) + (1-(1-l_i)^q) - \left(1-\left(1-l_j\right)^p\right) \cdot (1-(1-l_i)^q)\right)^{\frac{2w_iw_j}{(1+w_i)}}\right]^{\frac{1}{p+q}}, 1$$

$$- \left[1 - \prod_{i,j=1; i \le j}^{n} \left((1-(1-m_i)^p) + \left(1-\left(1-m_j\right)^q\right) - (1-(1-m_i)^p) \cdot \left(1-\left(1-m_j\right)^q\right)\right) \cdot \left(\left(1-\left(1-m_j\right)^p\right) + (1-(1-m_i)^q) - \left(1-\left(1-m_j\right)^p\right) \cdot (1-(1-m_i)^q)\right)^{\frac{2w_iw_j}{(1+w_i)}}\right]^{\frac{1}{p+q}}, 1$$

$$- \left[1 - \prod_{i,j=1; i \le j}^{n} \left((1-(1-u_i)^p) + \left(1-\left(1-u_j\right)^q\right) - (1-(1-u_i)^p) \cdot \left(1-\left(1-u_j\right)^q\right)\right) \cdot \left((1-\left(1-u_j\right)^p) + (1-(1-u_i)^q) - (1-\left(1-u_j\right)^p) \cdot (1 - (1-u_i)^q)\right)^{\frac{2w_iw_j}{(1+w_i)}}\right]^{\frac{1}{p+q}}$$

Equation (28) has been proven.

Theorem 8. Idempotency property: $A = \left(l_i, m_{i,} u_i\right)(i = 1,2, \dots, n)$ represent a set of HTFEs. If $A_i = \left(l_i, m_{i,} u_i\right)$ for all i, then $HTF - GGHM^{p,q}\left(A_{1,} A_2, \dots, A_n\right) = a$.

This theorem is proved in the same way as theorem 4.

https://doi.org/10.24191/mij.v6i2.4676

Theorem 9. Monotonicity property: Let h_{α_i} and h_{β_i} $(i = 1,2, \dots, n)$ be two collections of HTFEs; if , for any $\gamma_{\alpha_i} \leq h_{\alpha_i}$ and $\gamma_{\beta_j} \leq h_{\beta_j}(i, j = 1,2, \dots, n; i \neq j)$, one has $\gamma_{\alpha_i} \leq \gamma_{\beta_i}$ and $\gamma_{\alpha_j} \leq \gamma_{\beta_j}$, then $HTF - GGHM^{p,q}(h_{\alpha_1}, h_{\alpha_2}, \dots, h_{\alpha_n}) \leq HTF - GGHM^{p,q}(h_{\beta_1}, h_{\beta_2}, \dots, h_{\beta_n})$.

This theorem is proved in the same way as theorem 5.

Theorem 10. Boundedness: Let $h_i(i = 1,2, \dots, n\ )$ be a collection of HTEs, $h_i^- = \cup_{\gamma_i \in h_i}(min\{\gamma_i^L\}, min\{\gamma_i^M\}, min\{\gamma_i^U\}), h_i^+ = \cup_{\gamma_i \in h_i}(max\{\gamma_i^L\}, max\{\gamma_i^M\}, max\{\gamma_i^U\}), i, j \in \{1,2, \dots, n\}$, then

$$\cup_{\gamma^- \in h_i^-}\left(((1-(1-\gamma^{-L})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{-M})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{-U})^{p+q})^2)^{\frac{1}{p+q}}\right) \leq 1-\left(1-\prod_{i,j=1; i\leq j}^{n} \eta_{i,j}^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \leq \cup_{\gamma^+ \in h_i^+}\left(((1-(1-\gamma^{+L})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{+M})^{p+q})^2)^{\frac{1}{p+q}}, ((1-(1-\gamma^{+U})^{p+q})^2)^{\frac{1}{p+q}}\right) \tag{29}$$

This theorem is proved in the same way as theorem 6.

Finally, the WHTFGGHM operator's properties are well-established. It is similar to the HTFGGHM operator's reasoning. We have therefore, decided to omit the comprehensive proof.

4. CONCLUSIONS

The purpose of this work is to create a sophisticated and effective HTFGGHM operator to be used in MCDM contexts with vague and fuzzy data. The operator proposed here, as a combination of HTFS and GGHM, retains hesitancy and the correlation among criteria while fulfilling other important properties such as idempotence, monotonicity, and boundedness. The HTFGGHM operator enhances real-world decision-making benefits and performs well where ambiguity and complexity exist, and other mathematical methods fail to work. Due to its ability to compile and sort alternatives, this tool is particularly useful for decision-makers. Future studies can compare it with other fuzzy extensions, see how it performs when applied in dynamic decision-making problems, include other fuzzy extensions and run it on larger datasets. Comparisons with other advanced operators and applications in healthcare, supply chain, and environmentally friendly issues can place additional emphasis on scalability as well as applicability.

5. ACKNOWLEDGEMENTS/FUNDING

The authors would like to acknowledge the generous support provided by the UiTM Endowment Scholarship, Universiti Teknologi MARA (UiTM) and Universiti Teknologi MARA (UiTM), Cawangan Negeri Sembilan, Kampus Seremban, for providing the facilities that contributed to the successful completion of this research.

6. CONFLICT OF INTEREST STATEMENT

The authors agree that this research was conducted in the absence of any self-benefit, commercial or financial conflicts, and declare the absence of conflicting interests with the funders.

https://doi.org/10.24191/mij.v6i2.4676

7. AUTHORS' CONTRIBUTIONS

The authors jointly developed the concept and structure of this study. The formulation of the proposed HTFGGHM operator, along with its theoretical analysis and properties, was collaboratively carried out. Both authors contributed to the writing, refinement, and critical review of the manuscript to ensure clarity, coherence, and academic quality. The final version of the paper reflects the shared effort and mutual agreement of both authors.

REFERENCES

Al-Quran, A. (2021). A new multi-attribute decision-making method based on the T-spherical hesitant fuzzy sets. *IEEE Access, 9*, 156200–156210. https://doi.org/10.1109/ACCESS.2021.3128953

Amman, M., Rashid, T., Ali, A., Albalawi, O., & Alharthi, A. M. (2024). Dual-hesitant fermatean fuzzy Hamacher aggregation operators and TOPSIS with their application to multi-criteria decision-making. *PLOS ONE, 19*(10), e0311580. https://doi.org/10.1371/journal.pone.0311580

Anitha, B., & Vidhya, Muneeswaren (2023). Hesitant triangular fuzzy Dombi operators and its applications. In *Proceedings of the 2023 Fifth International Conference on Electrical, Computer and Communication Technologies (ICECCT)* (pp. 1–8). IEEE. https://doi.org/10.1109/ICECCT56650.2023.10179707

Beliakov, G., Pradera, A., & Calvo, T. (2007). *Aggregation functions: A guide for practitioners*. Springer. https://doi.org/10.1007/978-3-540-73721-6

Chu, Y., & Liu, P. (2015). Some two-dimensional uncertain linguistic Heronian mean operators and their application in multiple-attribute decision making. *Neural Computing and Applications, 26*(6), 1461–1480. https://doi.org/10.1007/s00521-014-1813-8

Divsalar, M., Ahmadi, M., Ghaedi, M., & Ishizaka, A. (2023). An extended TODIM method for hyperbolic fuzzy environments. *Computers & Industrial Engineering, 185*, 109655. https://doi.org/10.1016/j.cie.2023.109655

Fang, B. (2023). Some uncertainty measures for probabilistic hesitant fuzzy information. *Information Sciences, 625*, 255–276. https://doi.org/10.1016/j.ins.2022.12.101

Fany Helena, E. (2024). Decision-making problem for triangular hesitant fuzzy set. *Communications on Applied Nonlinear Analysis, 31*(5s). https://doi.org/10.52783/cana.v31.1036

Gholizade, M., Rahmanimanesh, M., Soltanizadeh, H., & Sana, S. S. (2023). Hesitant triangular fuzzy FlowSort method: The multi-criteria decision-making problems. *International Journal of Systems Science: Operations & Logistics, 10*(1). https://doi.org/10.1080/23302674.2023.2259293

Hasnan, Q. H., Rodzi, Z. M., Kamis, N. H., Al-Sharqi, F., Al-Quran, A., & Romdhini, M. U. (2024). Triangular fuzzy MEREC (TFMEREC) and its applications in multi-criteria decision making. *Journal of Fuzzy Extension and Applications, 5*(4), 505–532. https://doi.org/10.22105/jfea.2024.446557.1399

https://doi.org/10.24191/mij.v6i2.4676

Li, L., & Xu, Y. (2024). An extended hesitant fuzzy set for modeling multi-source uncertainty and its applications in multiple-attribute decision-making. *Expert Systems with Applications, 238*, Article 121834. https://doi.org/10.1016/j.eswa.2023.121834

Li, Y., & Li, J. (2023). Processing of real-world data in traditional Chinese medicine. *MEDS Public Health and Preventive Medicine, 3*(2), 1–6. https://doi.org/10.23977/PHPM.2023.030201

Liao, H., & Xu, Z. (2014). Some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision-making environment. *Journal of Intelligent & Fuzzy Systems*, *26*(4), 1601–1617. https://doi.org/10.3233/IFS-130841

Liu, F.-H. F., & Shih, S.-C. (2024). *Algorithms for multi-criteria decision-making and efficiency analysis problems* [*Preprint*]. arXiv. https://arxiv.org/abs/2406.06090

Matejíčka, L. (2013). Sharp bounds for the weighted geometric mean of the first Seiffert and logarithmic means in terms of weighted generalized Heronian mean. *Abstract and Applied Analysis, 2013*, Article 721539. https://doi.org/10.1155/2013/721539

Nishad, A. K., Aggarwal, G., & Abhishekh. (2023). Hesitant fuzzy time series forecasting model of higher order based on one and two-factor aggregate logical relationship. *Engineering Applications of Artificial Intelligence, 126*, Article 106897. https://doi.org/10.1016/j.engappai.2023.106897

Pu, D., Yu, M., & Yuan, G. (2022). Multi-attribute decision-making method based on hesitant triangular fuzzy power average operator. *Advances in Fuzzy Systems, 2022*, Article 4467548. https://doi.org/10.1155/2022/4467548

Rodzi, Z. M., Ahmad, A. G., Ismail, N. F., & Abdullah, N. L. (2021). Z-score functions of hesitant fuzzy sets. *Mathematics and Statistics*, *9*(4), 445–455. https://doi.org/10.13189/ms.2021.090405

Sultan, A., Sałabun, W., Faizi, S., & Ismail, M. (2021). Hesitant fuzzy linear regression model for decision making. *Symmetry, 13*(10), Article 1846. https://doi.org/10.3390/sym13101846

Tang, M., Wang, J., Lu, J., Wei, G., Wei, C., & Wei, Y. (2019). Dual hesitant Pythagorean fuzzy Heronian mean operators in multiple attribute decision making. *Mathematics, 7*(4), Article 344. https://doi.org/10.3390/math7040344

Torra, V. (2010). Hesitant fuzzy sets. *International Journal of Intelligent Systems*, *25*(6), 529–539. https://doi.org/10.1002/int.20418

Van Laarhoven, P. J. M., & Pedrycz, W. (1983). A fuzzy extension of Saaty's priority theory. *Fuzzy Sets and Systems*, *11*(1–3), 229–241. https://doi.org/10.1016/S0165-0114(83)80082-7

Wang, C., Li, Q., Zhou, X., & Yang, T. (2014). Hesitant triangular fuzzy information aggregation operators based on Bonferroni means and their application to multiple attribute decision making. *The Scientific World Journal, 2014*, Article 648516. https://doi.org/10.1155/2014/648516

Wang, Q., Wu, M., Zhang, D., & Wang, P. (2024). Correlation coefficients between normal wiggly hesitant fuzzy sets and their applications. *Scientific Reports, 14*(1), 1–18. https://doi.org/10.1038/s41598-024-67961-3

https://doi.org/10.24191/mij.v6i2.4676

Wei, G., Gao, H., & Wei, Y. (2018). Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(7), 1426–1458. https://doi.org/10.1002/int.21985

Xia, M., & Xu, Z. (2011). Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning*, *52*(3), 395–407. https://doi.org/10.1016/j.ijar.2010.09.002

Xian, S., Ma, D., & Feng, X. (2024). Z hesitant fuzzy linguistic term set and their applications to multi-criteria decision making problems. *Expert Systems with Applications, 238*, Article 121786. https://doi.org/10.1016/j.eswa.2023.121786

Xu, Z. (2009). Fuzzy harmonic mean operators. *International Journal of Intelligent Systems*, *24*(2), 152–172. https://doi.org/10.1002/int.20330

Ying, L., & Xin, G. (2024). Mixed correlation coefficient between probability hesitation fuzzy sets and applications. *International Journal of Fuzzy Systems, 26*(1), 154–167. https://doi.org/10.1007/s40815-023-01581-3

Yu, D. (2013). Intuitionistic fuzzy geometric Heronian mean aggregation operators. *Applied Soft Computing, 13*(2), 1235–1246. https://doi.org/10.1016/j.asoc.2012.09.021

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

Zhang, T. Y., & Ji, A. P. (2011). Schur-convexity of generalized Heronian mean. In Z. Liu, X. Fei, & X. Wang (Eds.), *Communications in computer and information science: Vol. 244. Information computing and applications (Part II, pp. 25–33)*. Springer. https://doi.org/10.1007/978-3-642-27452-7_4

Zhao, X., Lin, R., & Wei, G. (2014). Hesitant triangular fuzzy information aggregation based on Einstein operations and their applications to multiple attribute decision making. *Expert Systems with Applications, 41*(4), 1086–1094. https://doi.org/10.1016/j.eswa.2013.07.104