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Bipolar Pythagorean Neutrosophic Hamacher Heronian Mean Operators: A Framework in MCDM

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ABSTRACT

This paper aims to overcome existing limitations in multi-criteria decision-making (MCDM) framework by integrating Bipolar Pythagorean Neutrosophic Set (BPNS) with advanced aggregation operators specifically the Hamacher and the Heronian mean. These integrations are designed to enhance the effective representation and synthesis of decision criteria. To achieve this, a novel framework, the bipolar pythagorean neutrosophic set-generalized Hamacher Heronian mean (BPNS-GHHM) has been proposed. The proposed method satisfies all properties and demonstrates its effectiveness in solving complex MCDM problems.

1. INTRODUCTION

Decision-making frameworks often influenced by objectives, alternatives, and risks, further challenged by imprecision and uncertainty in human judgment. These challenges can lead to inconsistencies, biases, misinterpretations, and other issues in decision-making processes (Rane et al., 2024). To address these complexities, the Bipolar Pythagorean Neutrosophic Set (BPNS) theory, introduced by Ahmad et al. (2024), integrates bipolarity, neutrosophic logic, and Pythagorean fuzzy sets, offering a robust framework for evaluating both positive and negative membership degrees (Rodzi et al., 2024), as well as indeterminacy and non-membership degrees across multiple criteria. BPNS effectively addresses the shortcomings of

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traditional Fuzzy Sets (FS) and Intuitionistic Fuzzy Sets (IFS) in handling contradictory information (Salamat et al., 2024).

Originating from Neutrosophic Set (NS) theory (Smarandache, 1998), BPNS and its extensions such as Interval Neutrosophic Set (INS) (Wang et al., 2004), Interval-Valued Neutrosophic Set (IVNS) (Wang et al., 2005), Single-Valued Neutrosophic Set (SVNS) (Wang et al., 2010), and more, offer advanced tools for managing ambiguity (Sajid et al., 2024). This framework has been applied in diverse fields, including medical data analysis (Mustapha et al., 2021), big data analysis (Abdulbaqi et al., 2025), finance and management science (Mandala et al., 2025), supply chain (Mohamed et al., 2024), healthcare (Elsayed & Arain, 2024), infrastructure design (Alqahtani et al., 2024), agriculture (Saeed & Shafique, 2024), halal certification (Rodzi et al., 2023) and more, effectively addressing the limitations of traditional set theories.

Aggregation operators (AOs) are essential tools in decision-making, synthesizing multiple criteria or preferences into a unified outcome. A variety of AOs, such as the Maclaurin symmetric mean, Choquet integral, ordered weighted average, and others like, Dombi, Hamacher, Einstein, and Frank operators, offer diverse methods to address the complexities and uncertainties (Iqbal & Kalsoom, 2024) of real-world decision-making. The Hamacher operators introduced by Wolfgang Hamacher in the 1970s, offers a flexible generalized of standard algebraic operators, accommodating complex relationships and uncertainties better than traditional algebraic and Einstein operators (Akram et al., 2023).

The Heronian mean (HM) is notable for capturing the interconnections among input values (Mahmood et al., 2023), making it valuable in decision-making. The Generalized Heronian Mean (GHM) extends HM by offering multiple criteria, adding flexibility to manage complex relationships in decision problems. However, HM and GHM do not account for input weights (Li & Yang, 2021), leading to the development of Generalized Weighted Heronian Mean (GWHM) which incorporates weighting factors to reflect the relative importance of criteria. GWHM enhances the applicability and adaptability of the Heronian mean-based model in MCDM scenarios (Anusha et al., 2023; Ahemad et al., 2023). These foundational aggregation operators have laid the groundwork for novel advancements, particularly in combining them with fuzzy and neutrosophic frameworks to enhance their applicability in MCDM.

Numerous researchers have advanced decision-making methodologies by integrating aggregation operators with fuzzy and neutrosophic set framework to address uncertainty and complexity. Asif et al. (2025) introduces and analyzes Pythagorean fuzzy Hamacher interactive operators for multi-attribute decision-making (MADM), demonstrating the effectiveness, stability, and flexibility in handling ambiguities and enhancing decision-making through detailed sensitivity analysis. Thilagavathy and Mohanaselvi (2024) developed T-spherical fuzzy Hamacher Heronian mean operators, including weighted geometric, ordered geometric, and hybrid geometric variants, for group decision-making. Mei et al. (2023) proposes an improved single-valued neutrosophic Hamacher weighted averaging (ISNHWA) operator integrated with grey relational analysis (GRA) to address limitations in existing aggregation operators, demonstrating its rationality and effectiveness in multi-criteria decision-making (MCGDM).

Nithya Sri et al. (2024) constructed Einstein aggregation operators under the Bipolar Linear Diophantine Fuzzy Hypersoft Set framework for social data. Yaacob et al. (2024) proposed Dombi-based Heronian mean aggregation operators within the bipolar neutrosophic set framework to address the limitations of the existing Dombi operator in the context of bipolar neutrosophic sets. Paul et al. (2023) utilized Pythagorean fuzzy Hamacher aggregation operators for MCDM. Jamil et al. (2019, 2022) proposed Hamacher operations based on bipolar neutrosophic sets, demonstrated their application in modelling complex real-world MCDM problems.

Abdul Halim et al. (2024) introduces the rough neutrosophic Shapley weighted Einstein averaging aggregation operator, combining Shapley fuzzy measures and Einstein operators to handle fuzziness, uncertainty, and interaction in decision-making. These developments demonstrate advanced aggregation operators within fuzzy and neutrosophic frameworks to address uncertainty and ambiguity in MCDM

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problems. Alongside these advanced methodologies, score functions play a pivotal role in enhancing decision-making processes by enabling the ranking of alternatives within fuzzy and neutrosophic frameworks. This study aims to integrate bipolar Pythagorean neutrosophic sets with two aggregation operators: the Hamacher and the Heronian mean operators.

This research study is organized as follows: Section 2 presents the fundamental and essential concepts of BPNS, Hamacher operations, and Heronian mean. Section 3 proposes the BPNS-GHHM aggregation operator. The overall conclusion and future research directions are presented in the last section.

2. PRELIMINARIES

This section provides a comprehensive overview of the fundamental concepts and definitions related to BPNS, Hamacher operators, and Heronian mean operators.

2.1 Bipolar Pythagorean Neutrosophic Set

The following definitions of bipolar neutrosophic set are based on Ahmad et al. (2024).

Definition 1. Let z be a nonvoid set. A BPNS with components D in Z is defined as:

$$D = \{\langle z, \alpha_D^+(z), l_D^+(z), \beta_D^+(z), \alpha_D^-(z), l_D^-(z), \beta_D^-(z) \rangle : z \in Z\} \tag{1}$$

where $\alpha_D^+(z), l_D^+(z), \beta_D^+(z) : Z \to [0,1]$ and $\alpha_D^-(z), l_D^-(z), \beta_D^-(z) : Z \to [-1,0]$. $\alpha_D^+(z), l_D^+(z), \beta_D^+(z)$ represents positive membership degree, while $\alpha_D^-(z), l_D^-(z), \beta_D^-(z)$ represents negative membership degree of an element $z \in Z$. The condition for all $z \in Z$ shown below:

$$0 \le \alpha_D^+(z)^2 + \beta_D^+(z)^2 \le 1$$
$$0 \le \alpha_D^-(z)^2 + \beta_D^-(z)^2 \le 1$$
$$0 \le \alpha_D^+(z)^2 + l_D^+(z)^2 + \beta_D^+(z)^2 \le 2$$
$$0 \le \alpha_D^-(z)^2 + l_D^-(z)^2 + \beta_D^-(z)^2 \le 2$$

2.2 Heronian Mean Operators

The generalized Heronian mean (GHM) aggregation operator presented as follows:

Definition 2. Generalized Heronian mean (GHM) was introduced by Sykora (2009).
Let $\boldsymbol{x_i (i = 1, 2, \dots, n)}$ be a set of non-negative real numbers, and $\boldsymbol{p, q \ge 0}$. Then,

$$GHM^{p,q}(x_1, x_2, \dots, x_n) = \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} {x_i}^p {x_j}^q\right)\right]^{\frac{1}{p+q}} \tag{2}$$

2.3 Hamacher Operations Under BPNSs

Definition 3. Hamacher t-norm and t-conorm (Hamacher, 1978).

Hamacher operators proposed a more generalized t-norm and t-conorm in Archimedean t-norm and t-conorm. The Hamacher product, $\otimes$, is a t-norm and the Hamacher sum, $\oplus$, is a t-conorm, are two real numbers a_i and a_j are defined as follows:

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$$T(a_i, a_j) = a_i \otimes a_j = \frac{a_i a_j}{\varsigma + (1-\varsigma)(a_i + a_j - a_i a_j)}, \varsigma > 0 \tag{3}$$

$$T^*(a_i, a_j) = a_i \oplus a_j = \frac{a_i + a_j - a_i a_j - (1-\varsigma) a_i a_j}{1 - (1-\varsigma) a_i a_j}, \varsigma > 0 \tag{4}$$

Definition 4. A new Hamacher operations of bipolar Pythagorean neutrosophic set.

Let $A_1 = \langle t_1^+, i_1^+, f_1^+, t_1^-, i_1^-, f_1^- \rangle$ and $A_2 = \langle t_2^+, i_2^+, f_2^+, t_2^-, i_2^-, f_2^- \rangle$, be any two BPNSs and $\delta > 0$ be any real number, then we define basic Hamacher operators with $\tau > 0$.

$$A_1 \oplus_H A_2 = \left\langle \begin{array}{c} \sqrt{\frac{(t_1^+)^2 + (t_2^+)^2 - (t_1^+)^2 (t_2^+)^2 - (1-\tau)(t_1^+)^2 (t_2^+)^2}{1-(1-\tau)(t_1^+)^2 (t_2^+)^2}}, \frac{i_1^+ i_2^+}{\tau + (1-\tau)(i_1^+ + i_2^+ - i_1^+ i_2^+)}, \frac{f_1^+ f_2^+}{\tau + (1-\tau)(f_1^+ + f_2^+ - f_1^+ f_2^+)}, \\ \frac{-t_1^- t_2^-}{\tau + (1-\tau)(t_1^- + t_2^- - t_1^- t_2^-)}, \frac{-(-i_1^- - i_2^- - i_1^- i_2^- - (1-\tau) i_1^- i_2^-)}{1-(1-\tau) i_1^- i_2^-}, -\sqrt{\frac{(f_1^-)^2 + (f_2^-)^2 - (f_1^-)^2 (f_2^-)^2 - (1-\tau)(f_1^-)^2 (f_2^-)^2}{1-(1-\tau)(f_1^-)^2 (f_2^-)^2}} \end{array} \right\rangle \tag{5}$$

$$A_1 \otimes_H A_2 = \left\langle \begin{array}{c} \frac{t_1^+ t_2^+}{\tau + (1-\tau)(t_1^+ + t_2^+ - t_1^+ t_2^+)}, \frac{i_1^+ + i_2^+ - i_1^+ i_2^+ - (1-\tau) i_1^+ i_2^+}{1-(1-\tau) i_1^+ i_2^+}, \sqrt{\frac{(f_1^+)^2 + (f_2^+)^2 - (f_1^+)^2 (f_2^+)^2 - (1-\tau)(f_1^+)^2 (f_2^+)^2}{1-(1-\tau)(f_1^+)^2 (f_2^+)^2}}, \\ -\sqrt{\frac{(t_1^-)^2 + (t_2^-)^2 - (t_1^-)^2 (t_2^-)^2 - (1-\tau)(t_1^-)^2 (t_2^-)^2}{1-(1-\tau)(t_1^-)^2 (t_2^-)^2}}, \frac{i_1^- i_2^-}{\tau + (1-\tau)(i_1^- + i_2^- - i_1^- i_2^-)}, \frac{-(f_1^- f_2^-)}{\tau + (1-\tau)(f_1^- + f_2^- - f_1^- f_2^-)} \end{array} \right\rangle \tag{6}$$

$$\delta \cdot A_1 = \left\langle \begin{array}{c} \sqrt{\frac{(1+(\tau-1)(t_1^+)^2)^\delta - (1-(t_1^+)^2)^\delta}{(1+(\tau-1)(t_1^+)^2)^\delta + (\tau-1)(1-(t_1^+)^2)^\delta}}, \frac{\tau (i_1^+)^\delta}{(1+(\tau-1)(1-i_1^+))^\delta + (\tau-1)(i_1^+)^\delta}, \frac{\tau (f_1^+)^\delta}{(1+(\tau-1)(1-f_1^+))^\delta + (\tau-1)(f_1^+)^\delta}, \\ \frac{-\tau |t_1^-|^\delta}{(1+(\tau-1)(1+t_1^-))^\delta + (\tau-1)|t_1^-|^\delta}, \frac{-\tau |i_1^-|^\delta}{(1+(\tau-1)(1+i_1^-))^\delta + (\tau-1)|i_1^-|^\delta}, -\sqrt{\frac{(1+(\tau-1)(f_1^-)^2)^\delta - (1+(f_1^-)^2)^\delta}{(1+(\tau-1)(f_1^-)^2)^\delta - (\tau-1)(1+(f_1^-)^2)^\delta}} \end{array} \right\rangle \tag{7}$$

$$A_1{}^\delta = \left\langle \begin{array}{c} \frac{(t_1^+)^\delta}{(1+(\tau-1)(1-t_1^+))^\delta + (\tau-1)(t_1^+)^\delta}, \frac{(1+(\tau-1) i_1^+)^\delta - (1-i_1^+)^\delta}{(1+(\tau-1) i_1^+)^\delta + (\tau-1)(1-i_1^+)^\delta}, \sqrt{\frac{(1+(\tau-1)(-f_1^+)^2)^\delta - (1-(-f_1^+)^2)^\delta}{(1+(\tau-1)(-f_1^+)^2)^\delta + (\tau-1)(1-(-f_1^+)^2)^\delta}}, \\ -\sqrt{\frac{(1+(\tau-1)(t_1^-)^2)^\delta - (1-(t_1^-)^2)^\delta}{(1+(\tau-1)(t_1^-)^2)^\delta + (\tau-1)(1-(t_1^-)^2)^\delta}}, \frac{-\tau |i_1^-|^\delta}{(1+(\tau-1)(1+i_1^-))^\delta + (\tau-1)|i_1^-|^\delta}, \frac{-\tau |f_1^-|^\delta}{(1+(\tau-1)(1+f_1^-))^\delta + (\tau-1)|f_1^-|^\delta} \end{array} \right\rangle \tag{8}$$

3. PROPOSED GENERALIZED HAMACHER HERONIAN MEAN OPERATOR FOR BPNS

This section outlines the proposed bipolar Pythagorean neutrosophic set-generalized Hamacher Heronian mean (BPNS-GHHM) operator. Additionally, the associated properties and proofs for the aggregation operator are presented.

3.1 BPNS-GHHM Operator

Definition 5. Let $\boldsymbol{p, q \geq 0}$ and $\boldsymbol{A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1, 2, \ldots, n)}$ be a collection of BPNSs.

The BPNS-GHHM operator defined as follows:

$$BPNS - GHHM^{p,q}(A_1, A_2, \ldots, A_n) = \left[\frac{2}{n(n+1)} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (A_i)^p \otimes (A_j)^q \right) \right]^{\frac{1}{p+q}} \tag{9}$$

Theorem 1. Let $p, q \geq 0$ and $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2, \ldots, n)$ be a collection of BPNSs. Based on the Hamacher operations in Definition 4, the aggregated value of the BPNS-GHHM operator under BPNSs information, as defined in Definition 5, is given as follows:

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$$BPNS-GHHM^{p,q}(A_1,A_2,\dots,A_n)=\left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i)^p\otimes(A_j)^q\right)\right]^{\frac{1}{p+q}}=$$

$$\left(\begin{array}{c}\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i^+)^p(t_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-t_i^+))^p(1+(\tau-1)(1-t_j^+))^q\right)^{\frac{2(p+q)}{n(n+1)}}-(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i^+)^p(t_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}},\\ \frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)i_i^+)^p(1+(\tau-1)i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-i_i^+)^p(1-i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)i_i^+)^p(1+(\tau-1)i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-i_i^+)^p(1-i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}},\\ \sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(-f_i^+)^2)^p(1+(\tau-1)(-f_j^+)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(-f_i^+)^2)^p(1-(-f_j^+)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(-f_i^+)^2)^p(1+(\tau-1)(-f_j^+)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(-f_i^+)^2)^p(1-(-f_j^+)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}},\\ -\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^-)^2)^p(1+(\tau-1)(t_j^-)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^-)^2)^p(1-(t_j^-)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^-)^2)^p(1+(\tau-1)(t_j^-)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^-)^2)^p(1-(t_j^-)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}},\\ \frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^p|i_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^-))^p(1+(\tau-1)(1+i_j^-))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^p|i_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}},\\ \frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^-|^p|f_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+f_i^-))^p(1+(\tau-1)(1+f_j^-))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^-|^p|f_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}}\end{array}\right) \quad (10)$$

Proof: Using Eq. (8), we get

$$(A_i)^p=\left(\begin{array}{c}\frac{(t_i^+)^p}{(1+(\tau-1)(1-t_i^+))^p-(\tau-1)(t_i^+)^p},\frac{(1+(\tau-1)i_i^+)^p-(1-i_i^+)^p}{(1+(\tau-1)i_i^+)^p+(\tau-1)(1-i_i^+)^p},\sqrt{\frac{(1+(\tau-1)(-f_i^+)^2)^p-(1-(-f_i^+)^2)^p}{(1+(\tau-1)(-f_i^+)^2)^p+(\tau-1)(1-(-f_i^+)^2)^p}},\\ -\sqrt{\frac{(1+(\tau-1)(t_i^-)^2)^p-(1-(t_i^-)^2)^p}{(1+(\tau-1)(t_i^-)^2)^p+(\tau-1)(1-(t_i^-)^2)^p}},\frac{-\tau|i_i^-|^p}{(1+(\tau-1)(1+i_i^-))^p+(\tau-1)|i_i^-|^p},\frac{-\tau|f_i^-|^p}{(1+(\tau-1)(1+f_i^-))^p+(\tau-1)|f_i^-|^p}\end{array}\right)$$

$$(A_j)^q=\left(\begin{array}{c}\frac{(t_j^+)^q}{(1+(\tau-1)(1-t_j^+))^q-(\tau-1)(t_j^+)^q},\frac{(1+(\tau-1)i_j^+)^q-(1-i_j^+)^q}{(1+(\tau-1)i_j^+)^q+(\tau-1)(1-i_j^+)^q},\sqrt{\frac{(1+(\tau-1)(-f_j^+)^2)^q-(1-(-f_j^+)^2)^q}{(1+(\tau-1)(-f_j^+)^2)^q+(\tau-1)(1-(-f_j^+)^2)^q}},\\ -\sqrt{\frac{(1+(\tau-1)(t_j^-)^2)^q-(1-(t_j^-)^2)^q}{(1+(\tau-1)(t_j^-)^2)^q+(\tau-1)(1-(t_j^-)^2)^q}},\frac{-\tau|i_j^-|^q}{(1+(\tau-1)(1+i_j^-))^q+(\tau-1)|i_j^-|^q},\frac{-\tau|f_j^-|^q}{(1+(\tau-1)(1+f_j^-))^q+(\tau-1)|f_j^-|^q}\end{array}\right)$$

By referring to Eq. (6), then

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$$(A_i)^p \otimes (A_j)^q = \begin{pmatrix} \frac{(t_i^+)^p (t_j^+)^q}{(1+(\tau-1)(1-t_i^+))^p (1+(\tau-1)\left(1-t_j^+\right))^q - (\tau-1)(t_i^+)^p (t_j^+)^q}, \\ \frac{(1+(\tau-1)i_i^+)^p (1+(\tau-1)i_j^+)^q - (1-i_i^+)^p (1-i_j^+)^q}{(1+(\tau-1)i_i^+)^p (1+(\tau-1)i_j^+)^q + (\tau-1)(1-i_i^+)^p (1-i_j^+)^q}, \\ \sqrt{\frac{(1+(\tau-1)\left(-f_i^+\right)^2)^p (1+(\tau-1)\left(-f_j^+\right)^2)^q - (1-\left(-f_i^+\right)^2)^p (1-\left(-f_j^+\right)^2)^q}{(1+(\tau-1)\left(-f_i^+\right)^2)^p (1+(\tau-1)\left(-f_j^+\right)^2)^q + (\tau-1)(1-\left(-f_i^+\right)^2)^p (1-\left(-f_j^+\right)^2)^q}}, \\ -\sqrt{\frac{(1+(\tau-1)(t_i^-)^2)^p (1+(\tau-1)(t_j^-)^2)^q - (1-\left(t_i^-\right)^2)^p (1-\left(t_j^-\right)^2)^q}{(1+(\tau-1)(t_i^-)^2)^p (1+(\tau-1)(t_j^-)^2)^q + (\tau-1)(1-\left(t_i^-\right)^2)^p (1-\left(t_j^-\right)^2)^q}}, \\ \frac{-\tau |i_i^-|^p |i_j^-|^q}{(1+(\tau-1)(1+i_i^-))^p (1+(\tau-1)\left(1+i_j^-\right))^q + (\tau-1)|i_i^-|^p |i_j^-|^q}, \\ \frac{-\tau |f_i^-|^p |f_j^-|^q}{(1+(\tau-1)(1+f_i^-))^p (1+(\tau-1)\left(1+f_j^-\right))^q + (\tau-1)\left|f_i^-\right|^p \left|f_j^-\right|^q} \end{pmatrix}$$

According to Eq. (5),

$$\sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q =$$

$$\begin{pmatrix} \sqrt{\frac{\sum_{i=1}^n \sum_{j=1}^n (t_i^+)^p (t_j^+)^q}{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(1-t_i^+))^p (1+(\tau-1)\left(1-t_j^+\right))^q - (\tau-1) \sum_{i=1}^n \sum_{j=1}^n (t_i^+)^p (t_j^+)^q}}, \\ \frac{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)i_i^+)^p (1+(\tau-1)i_j^+)^q - \sum_{i=1}^n \sum_{j=1}^n (1-i_i^+)^p (1-i_j^+)^q}{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)i_i^+)^p (1+(\tau-1)i_j^+)^q + (\tau-1) \sum_{i=1}^n \sum_{j=1}^n (1-i_i^+)^p (1-i_j^+)^q}, \\ \frac{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)\left(-f_i^+\right)^2)^p (1+(\tau-1)\left(-f_j^+\right)^2)^q - \sum_{i=1}^n \sum_{j=1}^n (1-\left(-f_i^+\right)^2)^p (1-\left(-f_j^+\right)^2)^q}{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)\left(-f_i^+\right)^2)^p (1+(\tau-1)\left(-f_j^+\right)^2)^q + (\tau-1) \sum_{i=1}^n \sum_{j=1}^n (1-\left(-f_i^+\right)^2)^p (1-\left(-f_j^+\right)^2)^q}, \\ -\frac{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(t_i^-)^2)^p (1+(\tau-1)(t_j^-)^2)^q - \sum_{i=1}^n \sum_{j=1}^n (1-\left(t_i^-\right)^2)^p (1-\left(t_j^-\right)^2)^q}{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(t_i^-)^2)^p (1+(\tau-1)(t_j^-)^2)^q + (\tau-1) \sum_{i=1}^n \sum_{j=1}^n (1-\left(t_i^-\right)^2)^p (1-\left(t_j^-\right)^2)^q}, \\ \frac{-\tau \sum_{i=1}^n \sum_{j=1}^n |i_i^-|^p |i_j^-|^q}{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(1+i_i^-))^p (1+(\tau-1)\left(1+i_j^-\right))^q + (\tau-1) \sum_{i=1}^n \sum_{j=1}^n |i_i^-|^p |i_j^-|^q}, \\ -\sqrt{\frac{\tau \sum_{i=1}^n \sum_{j=1}^n |f_i^-|^p |f_j^-|^q}{\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(1+f_i^-))^p (1+(\tau-1)\left(1+f_j^-\right))^q + (\tau-1) \sum_{i=1}^n \sum_{j=1}^n \left|f_i^-\right|^p \left|f_j^-\right|^q}} \end{pmatrix}$$

Based on Eq. (7),

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$$\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i)^p\otimes(A_j)^q\right)=$$

$$\left(\begin{array}{c}
\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i^+)^p(t_j^+)^q\right)^{\frac{2}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-t_i^+))^p(1+(\tau-1)\left(1-t_j^+\right))^q\right)^{\frac{2}{n(n+1)}}-(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i^+)^p(t_j^+)^q\right)^{\frac{2}{n(n+1)}}}},\\
\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)i_i^+)^p(1+(\tau-1)i_j^+)^q\right)^{\frac{2}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-i_i^+)^p(1-i_j^+)^q\right)^{\frac{2}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)i_i^+)^p(1+(\tau-1)i_j^+)^q\right)^{\frac{2}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-i_i^+)^p(1-i_j^+)^q\right)^{\frac{2}{n(n+1)}}},\\
\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)\left(-f_i^+\right)^2)^p(1+(\tau-1)\left(-f_j^+\right)^2)^q\right)^{\frac{2}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(-f_i^+\right)^2)^p(1-\left(-f_j^+\right)^2)^q\right)^{\frac{2}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)\left(-f_i^+\right)^2)^p(1+(\tau-1)\left(-f_j^+\right)^2)^q\right)^{\frac{2}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(-f_i^+\right)^2)^p(1-\left(-f_j^+\right)^2)^q\right)^{\frac{2}{n(n+1)}}},\\
-\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^-)^2)^p(1+(\tau-1)(t_j^-)^2)^q\right)^{\frac{2}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^-)^2)^p(1-\left(t_j^-\right)^2)^q\right)^{\frac{2}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^-)^2)^p(1+(\tau-1)(t_j^-)^2)^q\right)^{\frac{2}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(t_i^-\right)^2)^p(1-\left(t_j^-\right)^2)^q\right)^{\frac{2}{n(n+1)}}},\\
\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^p|i_j^-|^q\right)^{\frac{2}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^-))^p(1+(\tau-1)\left(1+i_j^-\right))^q\right)^{\frac{2}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^p|i_j^-|^q\right)^{\frac{2}{n(n+1)}}},\\
-\sqrt{\frac{\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^-|^p|f_j^-|^q\right)^{\frac{2}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+f_i^-))^p(1+(\tau-1)\left(1+f_j^-\right))^q\right)^{\frac{2}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|f_i^-\right|^p\left|f_j^-\right|^q\right)^{\frac{2}{n(n+1)}}}}
\end{array}\right)$$

By referring to Eq. (8),

$$\left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i)^p\otimes(A_j)^q\right)\right]^{\frac{1}{p+q}}=$$

$$\left(\begin{array}{c}
\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i^+)^p(t_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-t_i^+))^p(1+(\tau-1)\left(1-t_j^+\right))^q\right)^{\frac{2(p+q)}{n(n+1)}}-(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(t_i^+)^p(t_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}},\\
\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)i_i^+)^p(1+(\tau-1)i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-i_i^+)^p(1-i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)i_i^+)^p(1+(\tau-1)i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-i_i^+)^p(1-i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}},\\
\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)\left(-f_i^+\right)^2)^p(1+(\tau-1)\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(-f_i^+\right)^2)^p(1-\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)\left(-f_i^+\right)^2)^p(1+(\tau-1)\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(-f_i^+\right)^2)^p(1-\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}},\\
-\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^-)^2)^p(1+(\tau-1)(t_j^-)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(t_i^-\right)^2)^p(1-\left(t_j^-\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^-)^2)^p(1+(\tau-1)(t_j^-)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\left(t_i^-\right)^2)^p(1-\left(t_j^-\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}},\\
\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^p|i_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^-))^p(1+(\tau-1)\left(1+i_j^-\right))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^p|i_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}},\\
\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^-|^p|f_j^-|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+f_i^-))^p(1+(\tau-1)\left(1+f_j^-\right))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|f_i^-\right|^p\left|f_j^-\right|^q\right)^{\frac{2(p+q)}{n(n+1)}}}
\end{array}\right)$$

Eq. (10) has shown proven.

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Theorem 2. (Idempotency property) $A = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2,\dots,n)$ be a set of BPNSs. If $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle$ for all i, then $BPNS - GHHM^{p,q}(A_1, A_2, \dots, A_n) = A$.

Proof: Since $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2,\dots,n)$, then,

$$\begin{aligned} BPNS - GHHM^{p,q}(A_1, A_2, \dots, A_n) &= \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^n \sum_{j=1}^n (A_i)^p \otimes (A_j)^q\right)\right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^n \sum_{j=1}^n A^p \otimes A^q\right)\right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^n \sum_{j=1}^n A^{p+q}\right)\right]^{\frac{1}{p+q}} \\ &= \left[\frac{2}{n(n+1)}\left(\frac{n(n+1)}{2}\right)\left(A^{p+q}\right)\right]^{\frac{1}{p+q}} \\ &= A \end{aligned}$$

$$\Rightarrow BPNS - GHHM^{p,q}(A_1, A_2, \dots, A_n) = A.$$

Theorem 3. (Monotonicity property) For any two set of BPNSs, $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle$ and $A_i^* = \langle t_i^{+*}, i_i^{+*}, f_i^{+*}, t_i^{-*}, i_i^{-*}, f_i^{-*} \rangle$, such that $A_i \le A_i^*$ for all $i = 1,2,\dots,n$. *If* $t^+ \le t^{+*}, i^+ \ge i^{+*}, f^+ \ge f^{+*}$ and $t^- \ge t^{-*}, i^- \le i^{-*}, f^- \le f^{-*}$, then,

$$BPNS - GHHM^{p,q}(A_1, A_2, \dots, A_n) \le BPNS - GHHM^{p,q}(A_1^*, A_2^*, \dots, A_n^*).$$

Proof: If $t^+ \le t^{+*}, i^+ \ge i^{+*}, f^+ \ge f^{+*}$ and $t^- \ge t^{-*}, i^- \le i^{-*}, f^- \le f^{-*}$, then,

$$\frac{\left(\sum_{i=1}^n \sum_{j=1}^n (t_i^+)^p (t_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(1-t_i^+))^p (1+(\tau-1)\left(1-t_j^+\right))^q\right)^{\frac{2(p+q)}{n(n+1)}} - (\tau-1)\left(\sum_{i=1}^n \sum_{j=1}^n (t_i^+)^p (t_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}} \le$$

$$\frac{\left(\sum_{i=1}^n \sum_{j=1}^n (t_i^{+*})^p (t_j^{+*})^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(1-t_i^{+*}))^p (1+(\tau-1)\left(1-t_j^{+*}\right))^q\right)^{\frac{2(p+q)}{n(n+1)}} - (\tau-1)\left(\sum_{i=1}^n \sum_{j=1}^n (t_i^{+*})^p (t_j^{+*})^q\right)^{\frac{2(p+q)}{n(n+1)}}},$$

Similarly,

$$\frac{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)i_i^+)^p (1+(\tau-1)i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}} - \left(\sum_{i=1}^n \sum_{j=1}^n (1-i_i^+)^p (1-i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)i_i^+)^p (1+(\tau-1)i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^n \sum_{j=1}^n (1-i_i^+)^p (1-i_j^+)^q\right)^{\frac{2(p+q)}{n(n+1)}}} \ge$$

$$\frac{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)i_i^{+*})^p (1+(\tau-1)i_j^{+*})^q\right)^{\frac{2(p+q)}{n(n+1)}} - \left(\sum_{i=1}^n \sum_{j=1}^n (1-i_i^{+*})^p (1-i_j^{+*})^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)i_i^{+*})^p (1+(\tau-1)i_j^{+*})^q\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^n \sum_{j=1}^n (1-i_i^{+*})^p (1-i_j^{+*})^q\right)^{\frac{2(p+q)}{n(n+1)}}},$$

$$\sqrt{\frac{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(-f_i^+)^2)^p (1+(\tau-1)\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}} - \left(\sum_{i=1}^n \sum_{j=1}^n (1-(-f_i^+)^2)^p (1-\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^n \sum_{j=1}^n (1+(\tau-1)(-f_i^+)^2)^p (1+(\tau-1)\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^n \sum_{j=1}^n (1-(-f_i^+)^2)^p (1-\left(-f_j^+\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}} \ge$$

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$$\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(-f_i^{+*})^2)^p(1+(\tau-1)\left(-f_j^{+*}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(-f_i^{+*})^2)^p(1-\left(-f_j^{+*}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(-f_i^{+*})^2)^p(1+(\tau-1)\left(-f_j^{+*}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(-f_i^{+*})^2)^p(1-\left(-f_j^{+*}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}},$$

$$-\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^{-})^2)^p(1+(\tau-1)(t_j^{-})^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^{-})^2)^p(1-\left(t_j^{-}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^{-})^2)^p(1+(\tau-1)(t_j^{-})^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^{-})^2)^p(1-\left(t_j^{-}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}} \geq$$

$$-\sqrt{\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^{-*})^2)^p(1+(\tau-1)(t_j^{-*})^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}-\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^{-*})^2)^p(1-\left(t_j^{-*}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^{-*})^2)^p(1+(\tau-1)(t_j^{-*})^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^{-*})^2)^p(1-\left(t_j^{-*}\right)^2)^q\right)^{\frac{2(p+q)}{n(n+1)}}}},$$

$$\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-}|^p|i_j^{-}|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^{-}))^p(1+(\tau-1)\left(1+i_j^{-}\right))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-}|^p|i_j^{-}|^q\right)^{\frac{2(p+q)}{n(n+1)}}} \leq$$

$$\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-*}|^p|i_j^{-*}|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(1+i_i^{-*})\right)^p\left(1+(\tau-1)\left(1+i_j^{-*}\right)\right)^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-*}|^p\left|i_j^{-*}\right|^q\right)^{\frac{2(p+q)}{n(n+1)}}},$$

and

$$\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^{-}|^p|f_j^{-}|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+f_i^{-}))^p(1+(\tau-1)\left(1+f_j^{-}\right))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^{-}|^p\left|f_j^{-}\right|^q\right)^{\frac{2(p+q)}{n(n+1)}}} \leq$$

$$\frac{-\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^{-*}|^p|f_j^{-*}|^q\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+f_i^{-*}))^p(1+(\tau-1)\left(1+f_j^{-*}\right))^q\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^{-*}|^p\left|f_j^{-*}\right|^q\right)^{\frac{2(p+q)}{n(n+1)}}}.$$

The validity of Theorem 3 has been proven.

Theorem 4. (Boundedness property) For a collection of BPNSs, let $A_i = \langle t_i^{+}, i_i^{+}, f_i^{+}, t_i^{-}, i_i^{-}, f_i^{-}\rangle (i = 1,2, \dots, n)$, where $\tau > 0$ and if

$$A^{-} = \langle min(t^{+}), max(i^{+}), max(f^{+}), max(t^{-}), min(i^{-}), min(f^{-})\rangle$$
$$A^{+} = \langle max(t^{+*}), min(i^{+*}), min(f^{+*}), min(t^{-*}), max(i^{-*}), max(f^{-*})\rangle$$

Proof: From the results presented in Theorem 2 and 3, the following can be inferred as:

$$A^{+} = BPNS - GHHM^{p,q}(A^{+}, A^{+}, \dots, A^{+}),$$
$$A^{-} = BPNS - GHHM^{p,q}(A^{-}, A^{-}, \dots, A^{-}).$$

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As a result, $BPNS - GHHM^{p,q}(A^-, A^-, \dots, A^-) \leq BPNS - GHHM^{p,q}(A_1, A_2, \dots, A_n) \leq BPNS - GHHM^{p,q}(A^+, A^+, \dots, A^+)$. Hence, $A^- \leq BPNS - GHHM^{p,q}(A_1, A_2, \dots, A_n) \leq A^+$.

4. CONCLUSIONS

This study introduces the Hamacher t-norm and t-conorm under BPNS information. A new operational law of BPNS based on Hamacher aggregation operators has been proposed, which is then applied to Heronian mean operators, to introduce the BPNS-GHHM operator. The findings of this study can be applied to address complex, real-world problems characterized by uncertain and incomplete information. The proposed model offers a versatile framework that can be implemented across a variety of real-life applications, thereby providing a solid foundation for future research and advancements in this area.

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6. CONFLICT OF INTEREST STATEMENT

The authors declared that there is no conflict of interest in this paper.

7. AUTHORS' CONTRIBUTIONS

All authors contributed equally to this work. All authors read and approved of the final manuscript.

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