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Numerical solution for one dimensional thermal problems using the finite element method

Hisham Bin Md. Basir
UiTM Kota Kinabalu Campus, Locked Bag 71, 88997 Kota Kinabalu.

Abstract

Finite Element is a useful and effective numerical method for developing mathematical models to simulate problems related to stress analysis, heat transfer, electromagnetism and fluid flow. It uses a complex system of points, called nodes, which form a grid, or mesh, across the studied model. Every node assigned with an equation determines how the parameter will react to a certain load conditions. This paper discusses the formulation of that equation and the related solution based on the Galerkin's Weighted Residual Method.

Keywords: Finite Element; Heat Transfer; Mathematical Model; Thermal

1. Introduction

Many researches have been done on the application of Finite Element Method for solving thermal problems. In literature, Yang Liu and Nhan Phan-Thien (1997) have studied the coupled conduction-convection problem for an underground circular duct containing an insulating cable using the operator-splitting time-stepping Finite Element Method. The method solves for the temperature distribution in the solid domain and the flow field in the fluid domain simultaneously, and automatically satisfies the continuity of the temperature and the heat flux across the interfaces.

In another study by Ahmed and Chandra (1997), the Finite Element Method has been used to predict hot tears, hot cracks, residual stresses and distortion in castings. This study has successfully demonstrated the effects of creep below solidus on stresses (transient and residual) and distortions. It has also made a modification to the hot tear prediction.

The Finite Element Method has also been successfully employed to study variety of engineering problems such as sound radiation (Dar-Ming Chiang and Wen-Hwa Chen, 2000), transverse vibration of geometrically segmented beams (Chaudhari and Maiti, 1999), design analysis (Pinfold and Chapman, 1999), stress analysis (Ferguson, 1997), electric field analysis (Tabor, 1998), threaded connection (Zadoks and Kokatam, 1999), calculation of transient temperature during welding (Little, et. al., 1998), fire resistance (Istas, 1997; Badri, 1997) and many more.

e-mail: **samaiza@tm.net.my**

In literature, several technical papers have been published on the effectiveness and the accuracy of the Finite Element Method. Bathe (1998a and 1998b) discusses the effects of using the automatic mesher on the accuracy of the Finite Element result. He suggests analysts to prescribe reasonable element sizes for the mesh generator in crucial areas prior to the execution of the so-called automatic meshing task. Only then, the result of the Finite Element analysis will be more accurate.

David and Loui (1997) analysed the simplifying of the general procedure developed for eliminating computer coding errors for Finite Element applications by making use of volume-weighted residuals method. Based on this method, they modified the previous procedure by Shih (1985) and developed an alternative method that can even be used without carrying out any additional discretization steps and with minimum additional coding. The overall method has been shown to be effective and versatile in affirming the integrity of finite element codes that deal with real-life problems.

This paper shows the general derivative of Finite Element Method for solving thermal problems that have been used in the above research. The formulation derived at the end of this paper is indeed very useful to assist scientists and engineers to solve any thermal problems.

2. Modes of heat transfer

In order to derive the finite element formulation of thermal problems, fundamental knowledge on heat transfer equation is necessary. There are three modes of heat transfer, i.e. conduction, convection and radiation. Hence, there are three different equations of heat transfer.

2.1 Heat Conduction Equation

Heat conduction was defined as the transfer of thermal energy from the more energetic particles of a medium to the adjacent less energetic ones as a result of interactions between the particles. Unlike temperature, heat transfer has direction as well as magnitude, and thus it is a vector quantity.

According to the Fourier's Law, the rate of heat flow, Q_x by conduction through an area A in the positive x-direction is:

$$Q_x = -kA\frac{\partial T}{\partial x}, \; Watt \; or \; \frac{J}{Sec} \tag{1}$$

where

k = Thermal conductivity, $\frac{Watt}{m \cdot {}^{\circ}C}$

$\frac{\partial T}{\partial x}$ = Area which perpendicular to the direction of the heat flow, m^2

= The temperature gradient in the direction of the heat flow, $\frac{{}^{\circ}C}{m}$

2.2 Heat convection equation

Convection is the mode of energy transfer between solid surfaces and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion. The faster the fluid motion, the greater the convection heat transfers. In the absence of any bulk fluid motion, heat transfer between a solid surface and the adjacent fluid is by pure conduction. The presence of bulk motion of the fluid enhances the heat transfer between the solid surface and the fluid, but it also complicates the determination of heat transfer rates.

Convection has two modes, called as forced convection and natural convection. Convection is called forced convection if the fluid is forced to flow over the surface by external means such as a fan, pump or the wind. In contrast, convection is called natural (or free) convection if buoyancy forces that are induced by density differences due to the variation of temperature in the fluid cause the fluid motion.

According to the Newton's Law of Cooling, the rate of heat flow through convection can be represented mathematically as:

$$Q_{convection} = hA(T_s - T_\infty),\ Watt\ or\ \frac{J}{Sec} \tag{2}$$

where

h	=	Convection heat transfer coefficient, $\frac{Watt}{m.^\circ C}$
A	=	Surface area that exposed to convection, m^2
T_s	=	Temperature of the surface that exposed to convection, $^\circ C$
$T\infty$	=	Temperature of the fluid sufficiently far from the surface, $^\circ C$

2.3 Heat radiation equation

Radiation is the energy transferred in the form of electromagnetic waves (or photons) as a result of the changes in the electronic configurations of the atoms or molecules. It doesn't need any intervening medium to transfer heat. Compared to the other two modes, radiation is fastest (at the speed of light) and it suffers no attenuation in a vacuum.

Based on the Stefan-Boltzman Law, the rate of heat radiated through an area A can be expressed mathematically as:

$$Q_{radiation} = \varepsilon\sigma A\left(T_s^4 - T_{surr}^4\right),\ Watt\ \text{or}\ \frac{J}{Sec} \tag{3}$$

where

ε	=	Emmisivity of the surface that expose to radiation, dimensionless
σ	=	Stefan-Boltzman constant, i.e. $5.67 \times 10^{-8}\ \frac{Watt}{m^2 \cdot K^4}$
A	=	Surface area that exposed to radiation, m^2
T_s	=	Temperature of the surface that exposed to radiation, $^\circ C$
T_{surr}	=	Temperature of the surrounding, $^\circ C$

Note that the above equation is appropriate for the case, which the emmisivity and the surface area of the surrounding surface do not have any effect on the net radiation heat transfer.

2.4 The complete heat transfer partial differential equation (PDE)

To derive the complete heat transfer PDE for any case, four major things have to be identified first, i.e. (i) the heat transfer modes that may take place, (ii) either the problems is categorized as transient or steady, (iii) the dimension of the problem, and (iv) the involvement of heat generation, if any.

A case is considered as steady-state problem if the temperature involved or heat flux remains unchanged with respect to time. While, the transient or unsteady-state problem is for the case where the temperature is time dependent. Therefore, in the case of steady-state problems, the heat flux or temperature remains unchanged with time at any location of the heat transfer medium, although both quantities may vary from one location to another.

The dimension of any problem is determined based on two factors, i.e. (i) the relative magnitudes of heat transfer rates in different directions, and (ii) the level of accuracy needed. A case is considered as one-dimensional if the temperature in the medium varies in only one direction, means that heat is transferred in only one direction, while heat transfer in other direction is negligible. Thus, the two-dimensional and three-dimensional problems are for cases which heat is transferred in two directions and three directions respectively. In terms of accuracy, three-dimensional analysis will gives the most accurate result, followed with two-dimensional and one-dimensional.

The heat generation term has to be considered and included in the overall heat transfer equation when the medium which heat is conducted involve the conversion of electrical, nuclear or chemical energy into heat (or thermal) energy. Since heat generation is a volumetric phenomenon, it occurs throughout the body of a medium.

Therefore, the rate of heat generation in a medium is usually specified per unit volume and is denoted by $\dot{q}$, whose unit is $\frac{Watt}{m^3}$.

As for an example, the object that is subjected to a heat source at one of the ends of the body receives heat and conducts it to the other end, which has lower temperature. Since the surrounding temperature is relatively lower too, some of the heat is convected and radiated out from the body to the surrounding. By using the principle of energy balance, this phenomenon can be expressed mathematically as follows:

$$\left(Q_{conduction}\right) - \left(Q_{convection}\right) - \left(Q_{radiation}\right) + \left(Q_{generated}\right) = \begin{pmatrix} Rate\ of\ Change \\ in\ Internal\ Energy \end{pmatrix} \quad (4)$$

Looking at the volume element inside the body, the sum of $Q_{conduction}$ and $Q_{generated}$ with the complete term of rate of change in internal energy can be derived as follows:

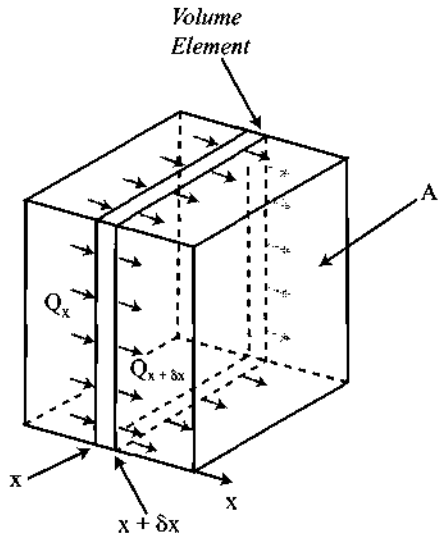

Figure 1: One-dimensional Heat Conduction Through A Volume Element.

$$Q_x - Q_{x+\delta x} + \dot{q} = \frac{\delta E}{\delta t} \tag{5}$$

whereby

$$\delta E = E_{t+\delta t} - E_t = mC\left(T_{t+\delta t} - T_t\right) = \rho CA\delta x\left(T_{t+\delta t} - T_t\right)$$

$$\dot{q} = \dot{q} A\delta x$$

Substituting the definition of δE and $\dot{q}$ into (5), yields

$$Q_x - Q_{x+\delta x} + \dot{q} A\delta x = \rho CA\delta x \frac{T_{t+\delta t} - T_t}{\delta t} \tag{6}$$

Dividing the above equation by δx:

$$\frac{Q_x - Q_{x+\delta x}}{\delta x} + \dot{q} A = \rho CA \frac{T_{t+\delta t} - T_t}{\delta t} \tag{7}$$

Taking the limit as $\delta x \to 0$ and $\delta t \to 0$, (Eqn. 7) becomes:

$$\lim_{\delta x \to 0} \frac{Q_x - Q_{x+\delta x}}{\delta x} + \dot{q} A = \rho CA \lim_{\delta t \to 0} \frac{T_{t+\delta t} - T_t}{\delta t}$$

$$\frac{\partial Q_x}{\partial x} + \dot{q} A = \rho CA \frac{\partial T}{\partial t} \tag{8}$$

Substituting $Q_x = -kA\frac{\partial T}{\partial x}$ (from the Fourier's Law), yields

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{q} A = \rho CA \frac{\partial T}{\partial t} \tag{9}$$

For each volume element that heat is conducted and generated through it, some of the heat is emitted out by convection and radiation (4). As described in the earlier section, equation for heat convection and heat radiation are:

$$Q_{convection} = hA(T_s - T_\infty)$$

$$Q_{radiation} = \varepsilon\sigma A(T_s^4 - T_{surr}^4)$$

In the context of this discussion, the area A included in both of the above equations is the total area for the volume element. Since the depth of the volume element (δx) is relatively very small, the total for area A is very close to the perimeter of the body. Thus, to be more practical, both of the above equations are modified as follows,

$$Q_{convection} = hP(T_s - T_\infty)$$

$Q_{radiation} = \varepsilon\sigma P(T_s^4 - T_{surr}^4)$, whereby P is the perimeter of the body.

Substituting the above equation into (9), the complete PDE for the case in this particular example can be written as:

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) - hP(T_s - T_\infty) - \varepsilon\sigma P(T_s^4 - T_{surr}^4) + \dot{q}A = \rho CA\frac{\partial T}{\partial t} \qquad (10)$$

3. Finite element formulation for one-dimensional thermal problem

The Finite Element formulation begins with meshing the object studied into numbers of element. Size for each element, thus number of the elements too, is depends upon the level of result accuracy desired. That is, the result obtained will much accurate if the size of the element is very small. For example, the studied object can be divided into several elements as follows:

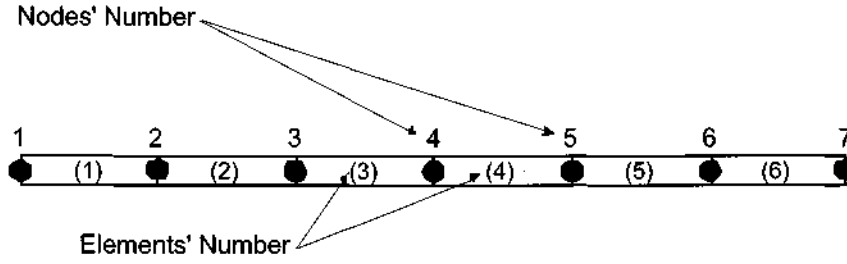

Figure 2: Meshing an Object for One-dimensional Analysis

Every element is assigned with one unique number, and each end of the element is called node. Those nodes must also be assigned with one unique number, as shown below:

Table 1: Element Data

Element Number	Node-i	Node-j
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	7

The developed mathematical model is expected to be able to calculate the value of the particular parameter studied at each node of the object. Since the case discussed in this paper is related to thermal problems, the parameter studied is the temperature, T. This means that, the final result of the formulation is the value of T at each node exists.

3.1 The shape function

From the meshing, as shown in Figure 2, every element is just like a straight line, which has two nodes at the end, i.e. usually denoted as node-i for the left end and node-j for the right end. This type of linear element is called simplex element. The general picture of each element is redrawn below:

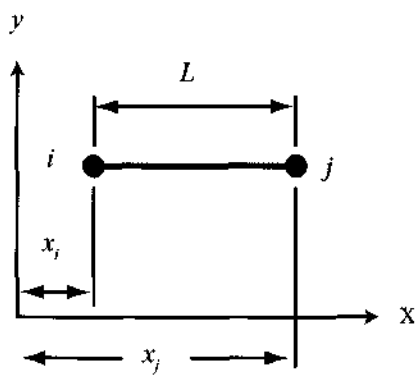

Figure 3: One-dimensional Simplex Element Redrawn From Figure 2

Hence,

$$T_i = \alpha_1 + \alpha_2 x_i \tag{12a}$$

$$T_j = \alpha_1 + \alpha_2 x_j \tag{12b}$$

By solving (12a) and (12b) simultaneously, we have

$$\alpha_1 = \frac{T_i x_j - T_j x_i}{x_j - x_i} = \frac{T_i x_j - T_j x_i}{L} \tag{13a}$$

$$\alpha_2 = \frac{T_j - T_i}{x_j - x_i} = \frac{T_j - T_i}{L} \tag{13b}$$

Substituting (13a) and (13b) into (11):

$$T(x) = \left(\frac{T_i x_j - T_j x_i}{L}\right) + \left(\frac{T_j - T_i}{L}\right) x$$

Rearranging the above equation, we get the final equation for the approximate polynomial solution, as follows:

$$T(x) = \left(\frac{x_j - x}{L}\right) T_i + \left(\frac{x - x_i}{L}\right) T_j \tag{14}$$

The coefficient of T_i and T_j in above equation, is defined as shape function for node-i and node-j respectively:

$$N_i(x) = \frac{x_j - x}{L} = \text{The shape function for Node-}\boldsymbol{i} \tag{15a}$$

$$N_j(x) = \frac{x - x_i}{L} = \text{The shape function for Node-}\boldsymbol{j} \tag{15b}$$

In matrix form, after substituting (15a) and (15b) into (14), the final equation for the approximate polynomial solution can be expressed as:

$$T(x) = [N(x)]\{T\} \tag{16}$$

where

$$\{T\} = \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

Analyzing the definition of the shape function (15a and 15b), the properties of the shape function can be derived as follows:

At node-*i*, **which** $x = x_i$,

$$N_i(x) = \frac{x_j - x_i}{L} = \frac{L}{L} = 1$$

$$N_j(x) = \frac{x_i - x_i}{L} = \frac{0}{L} = 0$$

while at node-*j*, **in which** $x = x_j$,

$$N_i(x) = \frac{x_j - x_j}{L} = \frac{0}{L} = 0$$

$$N_j(x) = \frac{x_j - x_i}{L} = \frac{L}{L} = 1$$

These properties can be expressed graphically as below:

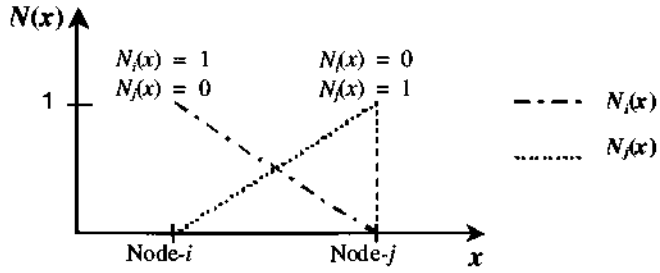

Figure 4: Properties of Shape Function

3.2 Length coordinate

The Length Coordinate is introduced to modify the shape function in such a way that it can be expressed without dimension.

At node-i, where $x = x_i$

$$N_i(x) = \frac{x_j - x_i}{L} = \frac{L}{L} = 1 \tag{17a}$$

$$N_i(x) = \frac{x_i - x_i}{L} = \frac{O}{L} = 1 \tag{17b}$$

while at node-j, where $x = x_j$

$$N_i(x) = \frac{x_j - x_i}{L} = \frac{L}{L} = 0 \tag{17c}$$

$$N_i(x) = \frac{x_j - x_i}{L} = \frac{L}{L} = 1 \tag{17d}$$

Consider the following two figures:

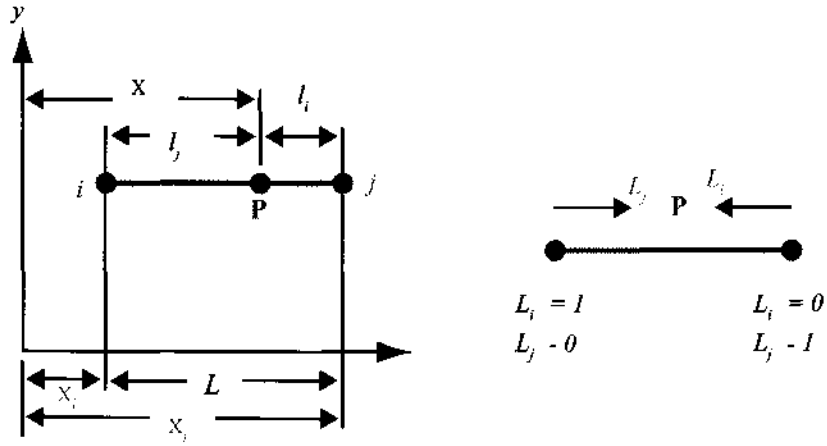

Figure 5: The Derivation of Length Coordinate

Note that the point P in the above figures is not a node. It is only a notation to represent all the possible point that exists within the element between node-i and node-j. The existence of the point P is in conjunction with the definition of integration, whereby it will take care of the limit

of δL when integrating $\int_{x_i}^{x_j} f(x)\delta L$

Length coordinate is defined on the basis to represent the location of P by using the ratio of the distance between P from the neighboring node-i and the distance between P from the neighboring node-j with the length of the element.

Thus, from the Figure 5 above, location of P can be represented by the following references:

$$L_i = \frac{length\ P-j}{length\ i-j} \tag{18a}$$

$$L_j = \frac{length\ i-P}{length\ i-j} \tag{18b}$$

where

$$0 \le L_i, L_j \le 1$$

Both equations (18a and 18b), represent the basic definition of Length Coordinate.

Studying Figure 5, we can derive the relationship between Length Coordinate and Cartesian coordinate:

$$L_i = \frac{x_j - x}{x_j - x_i} = \frac{l_i}{L} \tag{19a}$$

$$L_j = \frac{x - x_i}{x_j - x_i} = \frac{l_j}{L} \tag{19b}$$

By comparing (18a and 18b) and (15a and 15b), we can conclude that:

$$N_i(x) = L_i \tag{20a}$$

$$N_j(x) = L_j \tag{20b}$$

Means that, shape function that has dimension, can now be expressed as length coordinate which dimensionless. Therefore, any integration involving the shape function in the future can now be performed easily.

3.3 Galerkin's method

By substituting the approximate polynomial solution in (16) into the given PDE in (10), it gives a non-zero value called Element Residual, $R^{(e)}$.

$$\frac{\partial}{\partial x}\left(kA\frac{\partial\left[N(x)\right]\{T\}}{\partial x}\right) - hP\left(\left[N(x)\right]\{T\} - T_\infty\right) - \varepsilon\sigma P\left(\left[N(x)\right]\{T\}^4 - T_{surr}^4\right) + \dot{q}A$$

$$-\rho CA\frac{\partial\left[N(x)\right]\{T\}}{\partial t} = 0 = R^{(e)} \tag{21}$$

Although the energy balance principle suggests that the residual is suppose to be zero, but due to the limited fineness of the meshing, the residual is happened not equal to zero.

If $R^{(e)}$ can be minimized to zero, whereby it approaches the ideal condition as described in the principle of Energy Balance the result of analysis will be more accurate. To serve this purpose, Galerkin suggested the solution by using the Weighted-Residual Method, i.e. by integrating the product of the residual, $R^{(e)}$ with the transpose matrix of Shape Function, $[N]^T$ as follows:

$$\int_v [N]^T \{R^{(e)}\}\, dV = 0$$

Since this formulation is for one-dimensional analysis, i.e. only in x-direction, the Galerkin Weighted-Residual Equation above becomes,

$$\int_{l^{(e)}} [N]^T \left\{ R^{(e)} \right\} dx = 0 \tag{22}$$

Substituting (21) into (22), we have

$$\int_{l^{(e)}} |N|^T \left\{ \frac{\partial}{\partial x} \left(kA \frac{\partial [N(x)]\{T\}}{\partial x} \right) - hP([N(x)]\{T\} - T_\infty) - \varepsilon\sigma P([N(x)]\{T\}^4 - T_{surr}^4) + \dot{q} A \right.$$

$$\left. -\rho CA \frac{\partial [N(x)]\{T\}}{\partial t} \right\} dx = 0 \tag{23}$$

Using the Green's Theorem, $\int u\,dv = uv + \int v\,du$, the first term of the above equation can be simplified to become:

$$\{R\}^e = |N|^T kA \frac{\partial T}{\partial x}\Big|_{x=x_j}^{x=x_i} - \int_{l^e} kA \left(\frac{\partial |N|^T}{\partial x} \frac{\partial |N|}{\partial x} \right) dx \{T\}^e$$

$$- \int_{l^e} [N]^T hP[N]\{T\}^e\,dx + \int_{l^e} [N]^T hPT_\infty\,dx - \int_{l^e} [N]^T \varepsilon\sigma P\left[[N]\{T\}\right][N]\{T\}^e\,dx$$

$$+ \int_{l^e} |N|^T \varepsilon\sigma PT_{surr}^3 \cdot T_{surr}\,dx + \int_{l^e} [N]^T \dot{q} A dx - \int_{l^e} [N]^T \rho CA [N] \frac{\partial \{T\}^e}{\partial t} dx = 0 \tag{24}$$

The first term in the above equation is representing the boundary condition.

$$|N|^T kA \frac{\partial T}{\partial x}\Big|_{x=x_j}^{x=x_i} \equiv \text{Boundary Condition} \tag{25}$$

Physically, the typical boundary condition can be analyzed as follows:

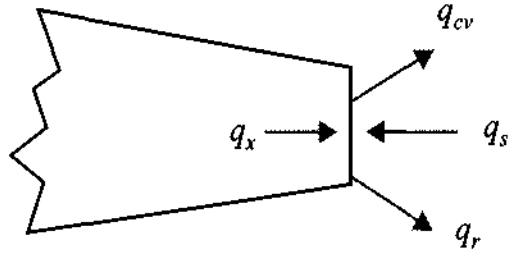

Figure 6: Heat Transfer at Typical Boundary Condition

As shown in Figure 6 above, the Energy Balance equation can be expressed as:

$$q_x + q_s = q_{cv} + q_r$$

where q_s = heat flux from the heat source
q_{cv} = heat flux by convection
q_r = heat flux by radiation

Since there are two boundaries that exist in the one-dimensional analysis, i.e. at the first element node-i is on the boundary, while at the last element node-j is on the boundary, analysis of boundary condition (25) has to consider both of them.

When node-i is on the boundary, i.e. node i = 1, $q_s = q_i$, thus

$$\begin{aligned}
[N]^T kA\left.\frac{\partial T}{\partial x}\right|_{x=x_i} &= [N]^T\left(-q_x A\right)\Big|_{x=x_i} \\
&= [N]^T A_i\left(-q_{cv} - q_r + q_s\right)\Big|_{x=x_i} \\
&= [N]^T A_i\left\{-h_i\left(T_s - [N]\{T\}^e\right)\right. \\
&\quad \left. - \varepsilon_i\sigma\left(T_s^4 - \left([N]\{T\}^e\right)^3 [N]\{T\}^e\right) + q_s\right\}\begin{bmatrix}1\\0\end{bmatrix}
\end{aligned}$$

where

$$q_x + q_s = = q_{cv} + q_r$$

Simplifying the above equation, we have

$$\begin{aligned}
[N]^T kA\left.\frac{\partial T}{\partial x}\right|_{x=x_i} &= -[N]^T h_i A_i T_s + [N]^T h_i A_i [N]\{T\}^e - [N]^T \varepsilon_i\sigma_i A_i T_s^4 \\
&\quad + [N]^T \varepsilon_i\sigma_i A_i \left([N]\{T\}^e\right)^2 [N]\{T\}^e + [N]^T A_i q_s\Big|_{x=x_i}
\end{aligned} \tag{26a}$$

When node-j is on the boundary, i.e. node j = end-node, $q_s = q_j$, thus

$$\begin{aligned}
[N]^T kA\left.\frac{\partial T}{\partial x}\right|_{x=x_j} &= [N]^T\left(-q_x A\right)\Big|_{x=x_j} \\
&= [N]^T A_j\left(-q_{cv} - q_r + q_s\right)\Big|_{x=x_j} \\
&= [N]^T A_j\left\{h_j\left(T_s - [N]\{T\}^e\right)\right. \\
&\quad \left. - \varepsilon_j\sigma\left(T_s^4 - \left([N]\{T\}^e\right)^2 [N]\{T\}^{e^4}\right) + q_s\right\}\begin{bmatrix}0\\1\end{bmatrix}
\end{aligned} \tag{26b}$$

3.4 Simplifying the galerkin weighted-residual equation

At this point, the original Galerkin Weighted-Residual equation (22) is written as the combination of equations (23) and (25), whereby in the simplified form, it can be written as follows,

$$[K]^e\{T\}^e+[C]^e\left\{\dot{T}\right\}^e=\{f\}^e \tag{27}$$

where

$[K]^e$ ≡ Element Stiffness Matrix

$[C]^e$ ≡ Capacitance Matrix

$\{f\}^e$ ≡ Nodal Force Vector

$\{T\}^e$ ≡ Unknown nodal temperature vector of the element

$\left\{\dot{T}\right\}$ ≡ Rate of change of temperature with respect to time

To perform this simplification, all the terms related to the unknown nodal temperature is collected into the Element Stiffness Matrix, while all the terms related to the known nodal temperature is collected into the Nodal Force Vector. The left term, which is related to the rate of change of temperature with respect to time, is taken into the Capacitance Matrix. Hence, after the simplification we have,

$$[K]^e = [K]^e_x + [K]^e_{cv_L} + [K]^e_{r_L} + [K]^e_{cv_R} + [K]^e_{r_R} \tag{28}$$

$$\{f\}^e = \{f\}^e_{cv_L} + \{f\}^e_{r_L} + \{f\}^e_q + \{f\}^e_{cv_R} + \{f\}^e_{r_R} + \{f\}^e_{q_R} \tag{29}$$

$$[C]^e = \int_{l^e} \rho cA[N]^T[N]dx \tag{30}$$

3.5 Integration of element stiffness matrix

The integration for all members in the Element Stiffness Matrix is shown below:

(i) $$[K]^e_x = \int_{l^e} kA\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}dx$$

$$= \int_{x_i}^{x_j} kA\begin{bmatrix}\frac{-1}{L}\\ \frac{1}{L}\end{bmatrix}\begin{bmatrix}\frac{-1}{L} & \frac{1}{L}\end{bmatrix}dx$$

$$= kA\begin{bmatrix}1/L^2 & -1/L^2\\ -1/L^2 & 1/L^2\end{bmatrix}L = \frac{kA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

(ii) $$[K]^e_{cv_L} = \int_{l^e} [N]^T hP[N]dx$$

$$= hP\int_{l^e} \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} [N_i \quad N_j]dx$$

$$= \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(iii) $$[K]^e_{r_L} = \int_{l^e} \varepsilon\sigma P[N]^T\left([N]\{T\}^e\right)^3[N]\,dx$$

This term is ignored. It has to b solved analytically.

(iv) $$[K]^e_{cv_R} = [N]^T h_i A_i [N]\Big|_{x=x_i} + [N]^T h_j A_j [N]\Big|_{x=x_j}$$

$$= \begin{bmatrix} h_i A_i & 0 \\ 0 & h_j A_j \end{bmatrix}$$

(v) $$[K]^e_{rR} = [N]^T \varepsilon_i \sigma A_i\left([N]\{T\}^e\right)^3[N]\Big|_{x=x_i} + [N]^T \varepsilon_j \sigma A_j\left([N]\{T\}^e\right)^3[N]\Big|_{x=x_j}$$

$$= \varepsilon\sigma A\begin{bmatrix} T_i^3 & 0 \\ 0 & T_j^3 \end{bmatrix}$$

3.6 Integration of fodal force vector

Following are the integraton for all member of Nodal Force Vector:

(i) $$\{f\}^e_{cv_L} = hPT_\infty\int_{l^e}\begin{Bmatrix} N_i \\ N_j \end{Bmatrix}dx$$

$$= \frac{hPLT_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

(ii) $$\{f\}^e_{r_L} = \int_{l^e} \varepsilon\sigma P[N]^T T^4_{surr}dx$$

$$= \frac{\varepsilon\sigma PLT^4_{surr}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\text{(iii)}\quad \{f\}^e_q = \int_{l^e} |N|^T \dot{q}\, A dx$$

$$= \frac{\dot{q}\, AL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\text{(iv)}\quad \{f\}^e_{rv_h} = |N|^T h_i A_i \{T_\infty\}\Big|_{x=x_i} + |N|^T h_j A_j \{T_\infty\}\Big|_{x=x_j}$$

$$= \begin{bmatrix} h_i A_i \{T_{\infty_i}\} \\ h_j A_j \{T_{\infty_j}\} \end{bmatrix}$$

$$\text{(v)}\quad \{f\}^e_{r_h} = |N|^T \varepsilon_i \sigma A_i \{T\}_s^4\Big|_{x=x_i} + |N|^T \varepsilon_j \sigma A_j \{T\}_s^4\Big|_{x=x_j}$$

$$= \begin{bmatrix} \varepsilon_i \sigma A_i \{T\}_s^4 \Big| \\ \varepsilon_j \sigma A_j \{T\}_s^4 \Big| \end{bmatrix}$$

$$\text{(vi)}\quad \{f\}^e_{q_n} = |N|^T q_i A_i\Big|_{x=x_i} + |N|^T q_j A_j\Big|_{x=x_j}$$

$$= \begin{bmatrix} hA_i T_s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ hA_j T_s \end{bmatrix}$$

3.7 Integration of capacitance matrix

Shown below is the integration of Capacitance Matrix,

$$|C|^e = \int_{l^e} \rho c A |N|^T |N| dx$$

$$= \tilde{n} c A \int_{l^e} \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} \begin{bmatrix} N_i & N_j \end{bmatrix} dx$$

$$= \frac{\rho c A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

3.8 Global matrices

Each node must have only one unique value of parameter T regardless to which element it is located. If node-j = a in element (n) has the value of $T(a)^{(n)}$ = b , the same node-i = a in element $(n+1)$ must have the same value of T, i.e. $T(a)^{(n+1)}$ = b.

Based on this simple idea, equations for all elements in the body must be assembled together and must then be solved simultaneously. This assemblage equation in matrix form, we call it as Global Matrices;

$$|K\|T\} + |C|_g\left\{\dot{T}\right\} = \{f\}_g \tag{31}$$

where

$|K|_g$ = Global Stiffness Matrix, which is the sum of stiffness matrix for all elements.

$\{T\}$ = Unknown temperature for all nodes

$|C|_g$ = Global Capacitance Matrix, which is the sum of capacitance matrix for all elements.

$\left\{\dot{T}\right\}$ = Rate of change of temperature with respect to time.

$\{f\}_g$ = Global Force Vector, which is the sum of matrix force vector for all elements.

Note that the term $\left\{\dot{T}\right\}$ is the rate of change of temperature with respect to time. Borrowing the concept of $\left\{\dot{T}\right\}$ from the Finite Difference Method, there are four different ways to solve this term.

There are; (i) Forward Difference Scheme, (ii) Backward Difference Scheme, (iii) Central Difference Scheme, and (iv) General Difference Scheme. In this paper, we will use the Backward Difference Scheme as the solution.

According to the Backward Difference Scheme,

$$\left\{\dot{T}\right\} = \frac{T(t) - T(t-\Delta t)}{\Delta t} \tag{32}$$

Substituting (Eqn. 32) into (Eqn. 31), yields

$$|K|_g T(t) + |C|_g \frac{T(t) - T(t-\Delta t)}{\Delta t} = \{f\}_g(t)$$

Rearrange the above equation:

$$\left[|C|_g + |K|_g \Delta t\right] T(t) = |C|_g T(t - \Delta t) + \{f\}_g(t)\Delta t$$

Note that the arguments t and $(t - \Delta t)$ on T and f are relative. In other words, nothing is changed if we substitute $(t + \Delta t)$ for t (and t for $t - \Delta t$) to give

$$\left[|C|_g + |K|_g \Delta t\right] T(t + \Delta t) = |C|_g T(t) + \{f\}_g(t + \Delta t)\Delta t$$

Using i to denote t and $(i + 1)$ to denote $(t + Dt)$,

$$\left[|C|_g + |K|_g \Delta t\right] \{T\}_{i+1} = |C|_g \{T\}_i + \{F\}_{g_{i+1}} \Delta t$$

Considering that $\left[|C|_g + |K|_g \Delta t\right]$ is Effective Stiffness Matrix and $\left\{|C|_g \{T\}_i + \{f\}_{g_{i+1}} \Delta t\right\}$ is Effective Nodal Force Vector, the huge assemblage equation can finally be written in the following form,

$$\left[K_{eff}\right] \{T\} = \left\{f_{eff}\right\} \tag{33}$$

Finally, the above equation can be solved simultaneously (for an example using the Gauss elimination Method) to get the temperature for each node at a particular time step desired.

4. Conclusion

In today's world, engineers and scientists are dealing with complex thermal problems. They usually use the two-dimensional numerical solution to get the accurate result. Even in some problems, which needs the highestaccuracy in prediction like thermal problem in aircraft or space shuttle they use the three-dimensional solutions. However, both solutions (two and three-dimensional) were derived from the one-dimensional formulation as presented in this paper.

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Biography

Hisham Md Basir is a mathematics lecturer at Universiti Teknologi MARA, Kota Kinabalu Campus. He is now doing a research on the development of numerical models for soil mechanics and heat transfer for his post- graduate project in engineering mathematics. He has good teaching experience in engineering mathematics, business mathematics and computer science. His research interests are finite element and finite difference modeling, heat transfer and mathematics education.